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THE EFFECTS OF LIMB DARKENING ON MEASUREMENTS OF ANGULAR SIZE WITH AN INTENSITY INTERFEROMETER

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SUMMARY

The correlation curve measured with an intensity interferometer has a shape out to its first minimum which is insensitive to the brightness distribution across a star. Observations are therefore initially interpreted in terms of the angular diameter of an equivalent uniform disc. The necessary correction to obtain the true limb-darkened angular diameter may be made by means of an approximate formula and a limb-darkening coefficient u_λ. More precise corrections, used in analysing the results from Narrabri Observatory, are given for a grid of model stellar atmospheres and these are compared with those given by the approximate formula. Observations of the secondary maximum of the correlation curve for Sirius are reported and compared with predictions based on a model atmosphere.

1. INTRODUCTION

The angular diameters of 32 single stars have been measured with the stellar intensity interferometer at Narrabri Observatory. The results have been reported in a recent paper (**1**) and descriptions of the equipment, observational procedure and reduction of the data have been given in earlier papers (**2, 3**). A significant factor in the interpretation of the data in terms of angular diameters is the effect of limb darkening. In the present paper we give a general discussion of this effect and of how we have taken it into account in analysing the results from Narrabri.

An intensity interferometer measures the correlation between the fluctuations of light intensity at two points separated by a baseline. It has been shown (**4**) that the time average of the normalized correlation $c_N(d)$ varies with baseline d as

$$c_N(d) \propto \Delta_\lambda \Gamma_\lambda{}^2(d) \tag{1}$$

where Δ_λ is the *partial coherence factor* and $\Gamma_\lambda{}^2(d)$ is the *correlation factor*.

The partial coherence factor Δ_λ takes account of the finite size of the detector apertures which, in a practical case, may be so large that they partially resolve the star and reduce the correlation. The correlation factor takes account of the variation of correlation with baseline and in the simple case, where the aperture of the detectors is small ($\Delta_\lambda \simeq 1$), $\Gamma_\lambda{}^2(d)$ is simply proportional to the square of the modulus of the Fourier transform of the apparent angular brightness distribution across the light source reduced to an equivalent strip parallel to the baseline which, for a radially symmetrical source, corresponds to the square of the modulus of the Hankel transform of the apparent angular radial distribution of brightness.

It follows that, if a star were to present a circular disc of uniform brightness,

$$\Gamma_\lambda{}^2(d) = |2J_1(x)/x|^2 \tag{2}$$

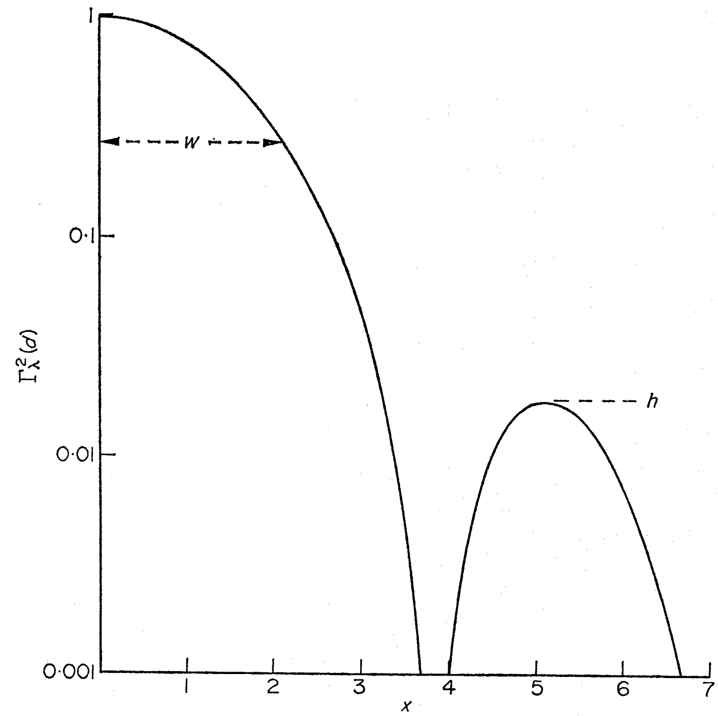

FIG. 1. *The correlation factor* $\Gamma_\lambda{}^2(d)$ *for a uniformly bright circular disc. The abscissa* $x = \pi\theta_{UD}\, d/\lambda_0$*; the curve assumes no partial resolution* ($\Delta_\lambda = 1$).

where $x = \pi\theta_{\mathrm{UD}}\, d/\lambda_0$, θ_{UD} is the angular diameter of the uniform disc, λ_0 is the effective wavelength of the light and it is assumed that the fractional optical bandwidth is $\ll 1$. This function is illustrated in Fig. 1.

In the practical case where the star is limb darkened, the correlation at any given baseline depends not only on the angular diameter but also on the limb darkening. We must therefore interpret the observation in terms of models of a limb-darkened star.

2. A SIMPLE REPRESENTATION OF LIMB DARKENING

In the conventional linear representation of limb darkening the distribution of brightness across the star's disc is given by,

$$I_\lambda(\mu) = I_\lambda(1)[1 - u_\lambda(1-\mu)] \tag{3}$$

where $I_\lambda(\mu)$ is the brightness of a point on the disc at a wavelength λ, μ is the cosine of the angle between the normal to the surface at that point and the line of sight from the star to the observer, and u_λ is the limb-darkening coefficient. By taking the Hankel transform of the apparent angular distribution of intensity across the source it can be shown that,

$$\Gamma_\lambda{}^2(d) = (\alpha/2+\beta/3)^{-2}[\alpha J_1(x)/x + \beta(\pi/2)^{1/2}\, J_{3/2}(x)/x^{3/2}]^2 \tag{4}$$

where $\alpha = 1 - u_\lambda$, $\beta = u_\lambda$, $x = \pi\theta_{\mathrm{LD}}\, d/\lambda_0$, θ_{LD} is the true angular diameter of the limb-darkened star, and it is assumed that $\Delta_\lambda = 1$.

A comparison of equations (2) and (4) shows that curves of correlation versus baseline for a limb-darkened star and a uniform disc are almost exactly the same shape for $\theta_{LD}/\theta_{UD} = 1$ but they differ in width (w in Fig. 1) along the x axis. For example, if we consider the case of complete limb darkening ($u_\lambda = 1$) and adjust θ_{LD}/θ_{UD} so that both curves have the same width, or more precisely so that they coincide at $\Gamma_\lambda^2(d) = 0{\cdot}3$ where we have made most of our measurements of angular diameter, then any differences between the two correlation factors are less than 0·5 per cent of $\Gamma_\lambda^2(0)$ over the whole range from zero baseline ($x = 0$) to the first minimum at $x \simeq 4$. However, the difference between the widths of the two curves is considerable and hence the adjusted value of θ_{LD}/θ_{UD} departs significantly from unity. Thus in Table I we show, as a function of u_λ, the ratio θ_{LD}/θ_{UD} of the angular size of a limb-darkened star to that of a uniform disc which gives the same correlation as a function of baseline. The values in Table I were computed directly from equations (2) and (4) but it is worth noting (**5**) that a good approximation is given by,

$$\theta_{LD}/\theta_{UD} = [(1-u_\lambda/3)/(1-7u_\lambda/15)]^{1/2}. \tag{5}$$

For $u_\lambda = 0 \rightarrow 0{\cdot}5$ the error in θ_{LD} due to this approximation is $< 0{\cdot}2$ per cent, for $u_\lambda = 0{\cdot}5 \rightarrow 1$ it is $< 0{\cdot}4$ per cent.

TABLE I

u_λ	θ_{LD}/θ_{UD}
0	1·000
0·1	1·008
0·2	1·016
0·3	1·025
0·4	1·035
0·5	1·046
0·6	1·059
0·7	1·073
0·8	1·088
0·9	1·106
1·0	1·127

The ratio of the true angular diameter θ_{LD} to the equivalent uniform disc θ_{UD} as a function of the limb-darkening coefficient u_λ for the simple limb-darkening law of equation (3). The x coordinates of the two curves have been scaled so that the correlation factors coincide at $\Gamma_\lambda^2(d) = 0{\cdot}3$ (see Section 2).

It follows from this discussion that it is not practicable to distinguish between a limb darkened and a uniform disc by measurements of correlation made with baselines in the range from zero ($x = 0$) out to the first minimum in the correlation factor at $x \simeq 4$. The differences between the shapes of the two curves are too small to measure reliably. For this reason we have listed (**1**), for every star measured at Narrabri, θ_{UD} the angular diameter of the *equivalent uniform disc*; in this way the observations are expressed in a convenient form free from any assumption about limb darkening.

3. A MORE PRECISE REPRESENTATION OF LIMB DARKENING

A more precise estimate of the limb darkening correction for any given model atmospheres requires a closer approximation to the radial distribution of brightness than equation (3). We have therefore fitted to model atmosphere data, by least squares, a limb-darkening equation of the form,

$$I_\lambda(\mu) = I_\lambda(1)[1-u_\lambda(1-\mu)-v_\lambda(1-\mu)^2-w_\lambda(1-\mu)^3]. \tag{6}$$

This follows the limb darkening of model atmospheres closely, the residuals in all cases being less than about 0·3 per cent. The corresponding correlation factor is

$$\Gamma_\lambda{}^2(d) = C^{-2}[\alpha J_1(x)/x+\beta(\pi/2)^{1/2}\, J_{3/2}(x)/x^{3/2}+\gamma 2J_2(x)/x^2 + \delta 3(\pi/2)^{1/2}\, J_{5/2}(x)/x^{5/2}]^2 \tag{7}$$

where

$$C = [\alpha/2+\beta/3+\gamma/4+\delta/5],$$
$$x = \pi\theta_{\mathrm{LD}}\, d/\lambda,$$
$$\alpha = 1-u_\lambda-v_\lambda-w_\lambda,$$
$$\beta = u_\lambda+2v_\lambda+3w_\lambda,$$
$$\gamma = -3w_\lambda-v_\lambda,$$
$$\delta = w_\lambda,$$

and it is assumed that $\Delta_\lambda = 1$.

For the model atmospheres computed by Kurucz, Carbon & Gingerich (**6**) we have found u_λ, v_λ and w_λ in equation (6) ($\lambda = 4500$ Å) for a wide range of T_e and $\log g$. We then computed from equations (2) and (7) the ratios $\theta_{\mathrm{LD}}/\theta_{\mathrm{UD}}$ which match the correlation factors for limb-darkened and uniform discs at the point $\Gamma_\lambda{}^2(d) = 0{\cdot}30$. The results are shown in Table II. For each model we have shown the conventional limb-darkening coefficient u_λ at 4500 Å derived by a least-squares fit of equation (3) to the model and also a value for the intrinsic colour $(B–V)_0$ derived from the tables of T_e versus $(B–V)_0$ given by Webb (**7**).

It is interesting to compare the limb-darkening corrections given by the simple formula in equation (5) with the more precise values computed from equations (6) and (7). Such a comparison, made over the whole range of model atmospheres listed in Table II, shows that any differences in the final values of θ_{LD} do not exceed about 0·5 per cent and for most of the models they are considerably less. It follows that, for most purposes, the simple method of computing limb-darkening corrections from equation (5) is sufficiently precise. It should be noted that the values of u_λ used in equation (5) were found by fitting the simple limb-darkening law of equation (3) to the radial distribution of the model by the method of least squares; we have found that this method yields values for the limb-darkening correction which give the best approximation to more precise values calculated from equations (6) and (7).

In the first list of 15 stars observed at Narrabri (**3**) the limb-darkening corrections were derived from equation (5). But in the final list of 32 stars (**1**) the limb darkening corrections were taken from the data in Table II.

4. AN EXPERIMENTAL TEST OF LIMB DARKENING

4.1 *Introduction*

We have assumed in the previous discussion that the limb darkening of stars can be predicted satisfactorily from model atmospheres. In principle the law of limb darkening could be found from measurements of correlation versus base-

TABLE II

Limb-darkening corrections at 4500 Å for the model atmospheres of Kurucz, Carbon & Gingerich (**6**)

(1) T_e (K)	(2) log g	(3) $(B-V)_0$	(4) u_λ	(5) θ_{LD}/θ_{UD}
50 000	5	−0·32	0·19	1·0131
40 000	5	−0·32	0·22	1·0155
30 000	4	−0·30	0·35	1·0267
25 000	4	−0·26	0·37	1·0288
20 000	4	−0·23	0·39	1·0310
18 000	4	−0·20	0·41	1·0327
16 000	4	−0·17	0·43	1·0348
14 000	4	−0·13	0·46	1·0376
12 000	4	−0·09	0·50	1·0417
10 000	4	−0·02	0·55	1·0483
9 000	4	+0·05	0·62	1·0572
8 500	4	+0·12	0·67	1·0640
8 000	4	+0·19	0·70	1·0693
7 000	4	+0·36	0·74	1·0769
6 500	4	+0·46	0·74	1·0778
6 000	4	+0·58	0·76	1·0817
5 000	4	—	0·85	1·0970
4 000	4	—	1·00	1·1341
20 000	3	−0·23	0·44	1·0358
18 000	3	−0·20	0·44	1·0362
16 000	3	−0·17	0·45	1·0373
14 000	3	−0·13	0·47	1·0396
12 000	3	−0·09	0·51	1·0433
10 000	3	−0·02	0·55	1·0485
9 000	3	+0·05	0·61	1·0554
8 500	3	+0·12	0·65	1·0603
8 000	3	+0·19	0·71	1·0696
7 000	3	+0·36	0·75	1·0789
6 500	3	+0·46	0·76	1·0808
6 000	3	+0·58	0·78	1·0855
5 000	3	—	0·85	1·0977
4 000	3	—	0·99	1·1326
12 000	2	−0·04	0·54	1·0472
10 000	2	+0·01	0·56	1·0502
9 000	2	+0·05	0·60	1·0547
8 000	2	+0·11	0·68	1·0655
7 000	2	+0·24	0·77	1·0814
6 000	2	+0·62	0·79	1·0873
5 000	2	—	—	1·0989
4 000	2	—	—	1·1311

Column 1 Effective temperature of model atmosphere (**6**).
Column 2 Surface gravity of model atmosphere (**6**).
Column 3 Intrinsic colour from relationship given by Webb (**7**)
Column 4 Limb-darkening coefficient at $\lambda = 4500$ Å derived by least-squares fit of equation (3) to the model.
Column 5 Ratio of θ_{LD} (true angular diameter) to θ_{UD} (equivalent uniform disc) computed from equations (2), (6) and (7).

line because, as we have noted in Section 1, the correlation factor $\Gamma_\lambda^2(d)$ is proportional to the square of the modulus of the Hankel transform of the apparent angular radial distribution of brightness. However, we have seen that, for baselines in the range $x = 0$ to $x \simeq 4$, limb darkening has a significant effect on the scale of this function but not on its shape. Thus we can only check the predicted effect of limb darkening if we have prior and fairly precise knowledge of the true angular diameter of a star.

However, for very bright stars, it is possible to extend the measurements of correlation to the second minimum in $\Gamma_\lambda{}^2(d)$ at $x \simeq 7$, thereby finding the height h of the first secondary maximum. We shall now discuss what information can be gained in this way.

If we compare $\Gamma_\lambda{}^2(d)$ for a uniform disc (Fig. 1) with that for a limb-darkened star obeying the simple law given in equation (3), we find that for the same angular size the two curves differ not only in width w but also in the height of the secondary maximum h (Fig. 1). The variation of h with u_λ is illustrated in Fig. 2. Thus the angular diameter of a star and the limb-darkening coefficient u_λ can be found on the assumption that the limb darkening follows the simple law.

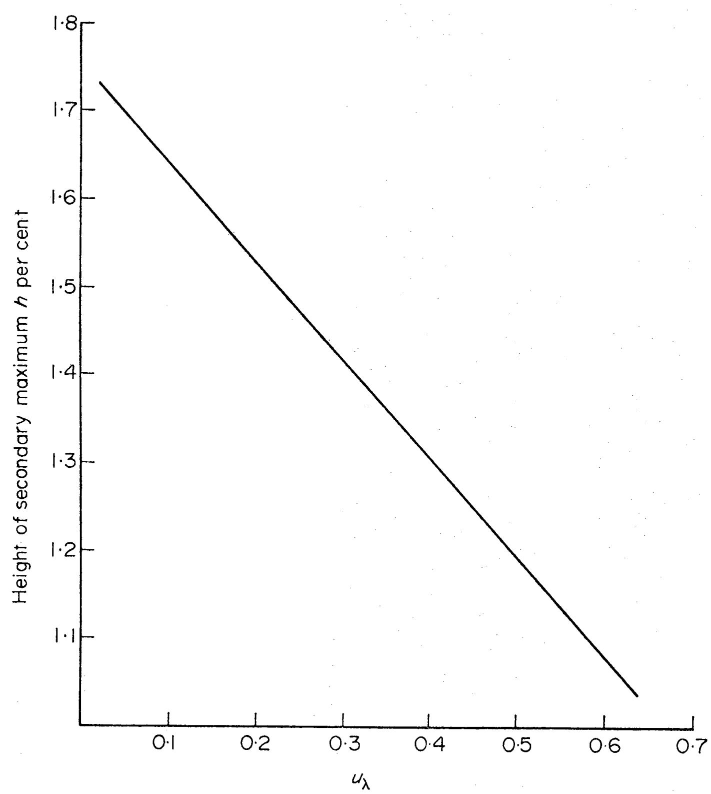

FIG. 2. *Height of secondary maximum h of correlation factor, expressed as a percentage of zero-baseline correlation (see Fig. 1) as a function of the limb-darkening coefficient* u_λ.

However, if we examine the curves of $\Gamma_\lambda{}^2(d)$ for a range of model atmospheres we find that the values of u_λ found in this way are of little significance. They differ substantially from the values found by fitting the simple law directly to the models and are not a useful measure of limb darkening. The explanation is that the height of the first secondary maximum is significantly affected by higher order terms in the radial distribution of the model (v_λ, w_λ in equation (6)), and the angular detail corresponding to these terms is not recorded in measurements made with short baselines. As an example one can match $\Gamma_\lambda{}^2(d)$ for a model atmosphere with

$T_e = 10\,000$ K and $\log g = 4$ (**6**) by a limb-darkened star which follows the simple law in equation (3) with $u_\lambda = 0{\cdot}44$; on the other hand a direct least-squares fit of this simple law to the model gives $u_\lambda = 0{\cdot}55$. Furthermore, one can match the same curve to a whole family of limb-darkening laws with a range of values of θ_{LD}, u_λ, v_λ and w_λ. For all practical purposes all these curves, including that for the simple limb-darkening law, are indistinguishable in shape; for example, if we match them all at $\Gamma_\lambda{}^2(d) = 0{\cdot}30$ and at the peak of the secondary maximum then, over a wide range of parameters, any differences between the curves are less than 1 part in 10^3 of $\Gamma_\lambda{}^2(0)$ for all values of x.

It follows that, from measurements which include only the first secondary maximum in the correlation factor, it is not practicable to distinguish between different limb-darkening laws. To make a useful distinction it would be necessary to extend the measurements to longer baselines; but, in practice, this cannot be done with an intensity interferometer because the signal-to-noise ratio is too low.

4.2 *Observations of Sirius*

Although it is impracticable to measure limb-darkening laws by this method with an intensity interferometer it is possible, for very bright stars, to test whether the height of the first secondary maximum of the correlation factor is correctly predicted by model atmospheres. This is not an attractive experiment with the interferometer at Narrabri because there are only two stars (α CMa, α Car) for which the signal-to-noise ratio approaches a sufficiently high value and, even for them, the exposure times are inconveniently long. Nevertheless, to our knowledge, the secondary maximum in the transform of a star has not been observed previously and we judged it to be worthwhile to try to measure its height for Sirius.

The measurements were made at Narrabri over the periods 1969 November–1970 January and 1970 November to 1971 January using the standard observing procedure described in previous papers (**2, 3**). Sirius was observed for a total of 203 hr with five different baselines in the range 9·56 m to 35·94 m at a wavelength of 4430 Å and with an optical bandwidth of ± 50 Å; this range of baselines corresponds approximately to $x = 2 \rightarrow 7{\cdot}5$ in Fig. 1, and embraces the first secondary maximum. The measured values of correlation were normalized for variations of light flux and correlator gain following the standard procedure described in (**3**). They were also corrected for small systematic changes in normalized correlation with elevation, due to mechanical distortion of the reflectors, and for changes in the parameters of the phototubes, due to excessive light flux from Sirius, which are discussed in detail by Lake (**8**). The final values are shown in Table III and Fig. 3 and are expressed as fractions of the zero-baseline correlation expected from a point source. The associated rms uncertainties σ include the effects of random fluctuations and zero drift in the correlator output as described in (**3**); in addition there is a small ($\pm 0{\cdot}0006$) uncertainty in the zero level of the scale of correlation which allows for the possibility that the zero level of the correlator might be systematically displaced by some unknown source of spurious correlation (**3**).

In Table III and Fig. 3 we also show the corresponding theoretical values for a model atmosphere (**6**) with $T_e = 10\,000$ K, $\log g = 4$ and $\lambda = 4500$ Å. The radial distribution of brightness of this model was approximated by a limb-darkening function of the form given in equation (6) which, to eliminate any possible effects due to approximation, was extended to include terms of order 7 in μ. The results shown in Table III were computed for the values of $\theta._{\mathrm{D}}$ and zero baseline correlation which give the best fit to the observations. They take into account partial resolution of the star by the reflectors, following equation (A12) in (**4**), and give the partial resolution factor as $\Delta_\lambda = 0{\cdot}900$.

A comparison of the model and the observations shows that there are no serious discrepancies. Thus the observed height of the first secondary maximum is about $1{\cdot}3 \pm 0{\cdot}1$ per cent compared with a theoretical value of 1·1 per cent. The largest discrepancy is at 20·67 m where the observed correlation is about 3 σ above the theoretical curve. In assessing the significance of these differences it must be

TABLE III

Correlation versus baseline for Sirius

Baseline (m)	Exposure (h)	Correlation Observed	Correlation Theoretical*
9·56	41·3	$0{\cdot}371 \pm 0{\cdot}007$	0·377
13·07	29·4	$0{\cdot}160 \pm 0{\cdot}002$	0·159
20·67	37·2	$0{\cdot}0107 \pm 0{\cdot}0015$	0·0061
25·89	57·5	$0{\cdot}0118 \pm 0{\cdot}0011$	0·0100
35·94	37·9	$0{\cdot}0001 \pm 0{\cdot}0013$	0·0009

The uncertainty in the zero level for all observed values is $\pm 0{\cdot}0006$

* Calculated for a model atmosphere (**6**) with T_{e} = 10 000 K, $\log g = 4$, λ = 4500 Å.

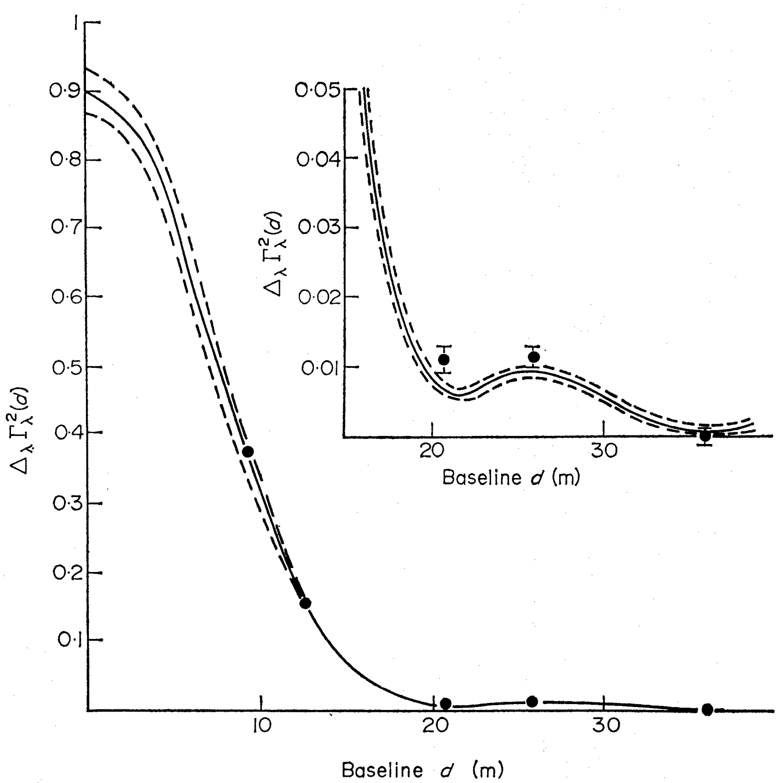

FIG. 3. *The variation of correlation $\Delta_\lambda\Gamma_\lambda$ (d) with baseline d for Sirius. The points show the observed values; the full line is a theoretical curve, based on a model stellar atmosphere (T_e = 10 000K, $\log g = 4$), with zero-baseline correlation and angular size adjusted to give the best fit to the observations. The broken lines represent the rms uncertainty in the theoretical curves.*

remembered that they represent an error of only about 1 part in 500 of the zero baseline correlation and, at such very low levels of correlation, it is possible that there are small systematic errors which we have failed to appreciate. It would therefore be premature to interpret these discrepancies in terms of real differences between the star and the model atmosphere and we must be satisfied with the tentative conclusion that the observations and the predictions of the model atmosphere are not inconsistent.

Throughout this discussion we have assumed that the star is radially symmetrical and in an attempt to test this assumption we have examined the observed correlation at each baseline as a function of the position angle of the star. Two baselines show a small asymmetry about the transit of the star across the meridian; at 13·07 and 20·67 m the average correlations recorded after transit are 2 σ and 1·5 σ greater than before transit. This asymmetry is barely significant but, if it is real, it might be due to some minor systematic effect in the equipment which we have failed to trace.

On the other hand it might be due to a genuine asymmetry in the brightness distribution of Sirius A; it should be noted that the effect of Sirius B on these measurements is negligible. There are well-known anomalies in the spectrum of Sirius A (**9, 10, 11, 12**) and in early accounts of its colour (**13, 14**); as one speculative example, we cannot exclude the possibility that the brightness distribution of Sirius A is affected by a circumstellar belt of dust. It would be interesting to pursue this question with a more sensitive interferometer.

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