Astron. Astrophys. 335, 1093–1100 (1998)

ASTRONOMY
AND
ASTROPHYSICS

Diurnal Yarkovsky effect as a source of mobility of meter-sized asteroidal fragments

I. Linear theory

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Received 20 January 1998 / Accepted 20 April 1998

Abstract. A linear theory for the heat conduction in a spherical, solid and rotating body illuminated by solar radiation is developed in detail. The principal aim is to compute the recoil force, due to thermally reemitted radiation, which is commonly known as the "Yarkovsky force". We concentrate on the thermal effect which depends on the rotational period of a body rather than on the period of revolution around the Sun and deal with the general case of an arbitrary obliquity of the spin axis to the orbital plane. This "diurnal" thermal effect is considered to be an important source of mobility for meter-sized stony asteroidal fragments in the main belt. We compare our results with those of previous authors and show that the results of Peterson (1976) are accurate for meter-sized asteroidal bodies (although he used unrealistically long rotation periods).

Key words: celestial mechanics, stellar dynamics – minor planets, asteroids – meteors, meteoroids

1. Introduction

The Earth's vicinity in the solar system is by no means an empty space. A number of small objects of sizes ranging from kilometers down to micrometers, with a typical power-law distribution, cross the Earth orbit (Rabinowitz 1993, 1994; Ceplecha 1992, 1996). It has been recognized long ago that these objects are relatively short-lived, and are being permanently replaced by a population of similar objects from the outer regions of the solar system. Their most prominent sources are the asteroid belt and the short-period comets. Transport mechanisms by means of which such bodies can be delivered to the Earth's neighbourhood have been studied intensively over the last decades. The crucial point has been the recognition of a special role of the mean-motion and secular resonances in the main asteroid belt (e.g. Morbidelli et al. 1994, Gladman et al. 1997, Migliorini et al. 1997). However, several issues remain to be understood in order to fit all observational data. In particular, it has been recently argued that the fast transport by resonant effects (on time scales of a few Myr) must be necessarily preceded by a relatively slower phase (on time scales of $10-100$ Myr). The latter may be dominated by orbital perturbations due to the thermal Yarkovsky effects (Farinella et al. 1998, Hartmann et al. 1998), which lead us to reconsider the importance and develop a precise modelling of these thermal orbital effects.

The present series of papers is devoted to a special type of thermal effects – the so-called diurnal effect, which depends on the rotational period of a body. Farinella et al. (1998) concluded that this effect is most relevant for the delivery of two important classes of bodies: (i) submeter stone bodies which are the most frequent parents of meteorites, and (ii) $20-70$ meter regolith covered bodies which, interestingly, are overabundant among the near-Earth objects (Rabinowitz 1993, 1994). A first treatment of the problem of the orbital perturbations due to the diurnal effect was given by Vokrouhlický and Farinella (1998a) (see also Afonso et al. 1995, but notice discussion below). They considered the interaction of these objects with the ν_6 resonance at the inner edge of the asteroid belt. Our present aim is to develop a complete numerical model allowing to include both gravitational interactions with all the planets and a full treatment of the diurnal Yarkovsky effect in any region of the asteroid belt. A first issue we face in this program is a proper modelling of the instantaneous thermal force. This paper develops such a theory. First, however, let us briefly recall the previous work on the Yarkovsky effect in celestial mechanics.

The modern history of the applications of the thermal recoil force in solar system dynamics begins with a classical paper by Öpik (1951). He is also at the origin of an interesting "mystery" surrounding the effect. Referring to an unpublished pamphlet of an unknown Polish/Russian engineer Yarkovsky, Öpik decided to name this effect *Yarkovsky effect* in his honour. Without performing any precise calculations, Öpik gave some basic estimates for the magnitude of the effect. Interestingly, if one disregards numerical factors of the order of unity, Öpik's results are in full agreement with today's models, at least for the diurnal effect (see below).

Radzievskii (1952) then presented the first detailed, mathematical study of the thermal effects on a sphere. Unfortunately, his work is limited by two unrealistic assumptions: (i) the body's spin axis must be normal to the orbital plane, and (ii) the body is bigger than the depth of a thermal wave. These simplifications made his calculations of little use for a further progress; nevertheless his work remains a pioneering one.

Radzievskii's work was followed by Peterson (1976). Though his analysis was much more detailed, he actually focused on aspects of the thermal effects which have little relevance: for instance, he analyzed in detail the complicated second-order solution of the heat conduction equation in a cylinder. However, from what we presently know in case of the seasonal effect (e.g. Vokrouhlický and Farinella 1998a,b), the nonlinearity effects do not produce any new qualitative feature. For "large" bodies they alter the amplitude of the thermal effects by about 15–20%, whereas in the case of "small' bodies the difference between the linear and non-linear solutions is below the 1% level. Peterson realized the importance of deriving at least some estimates for spherical bodies, and invented a clever (though somewhat cumbersome) method of converting the results for a cylinder to a sphere. His results, like those of Radzievskii, were correct for "large" bodies only, and he had to follow the qualitative arguments of Öpik (1951) to get information for "small" bodies. Yet his work may play an important role in application to meteoritics. He was almost alone in advocating the relevance of non-graviational dynamical mechanisms at that time, and the introduction to his 1976 paper still gives an interesting historic and conceptual discussion. Influenced by Peterson's work, Burns et al. (1979) reserved a special chapter to the thermal effects in their comprehensive study of radiative forces in the solar system.

Despite the progress mentioned above, one major point had been missing in all these studies of the thermal effects (although the first indications of new concepts may be traced back to Burns et al. 1979). Interestingly, a key inspiration for a generalization of the classical Yarkovsky effect came from the investigation of the motion of the artificial satellite LAGEOS, whose orbit is perturbed by thermal effects similar to those affecting the small natural bodies in the solar system (see e.g. Rubincam 1987, Vokrouhlický and Farinella 1998a). The reason is that LAGEOS has an important well-known property: a very fast rotation (e.g. Farinella et al. 1996). Rubincam (1987) thus developed a "LAGEOS-taylored" technique for computing thermal force perturbations on a rapidly spinning body. If, say in 1990, we had tried to compare models of thermal effects acting on natural solar system bodies and models applied to LAGEOS, we would have been surprised by a number of different assumptions and techniques. Yet, in both cases the physics is essentially the same.

Rubincam (1995) first tried to reconcile the modelling of thermal effects in the two cases. He applied a LAGEOS-like modelling to the case of asteroidal fragments, after having checked the physical consistency. A similar approach led Farinella et al. (1998) to reconsider the concepts of large vs. small size and slow vs. fast rotation of the fragments, both introduced by Burns et al. (1979). The variety of possible thermal effects was found to be larger than recognized before. Regarding the orbital effects, a new classification based on "diurnal" vs. "seasonal" variants was suggested.

Farinella et al. (1998) demonstrated that the diurnal Yarkovsky effect may play a very important role in the dynamics of stony fragments of asteroids, especially in the applications to meteoritics (i.e., for meter-sized bodies) and in delivering Tunguska-sized cosmic bodies (10 to 100 m across) to the Earth's vicinity. However, their discussion of the diurnal effect was based on Peterson's results, and they did not try to develop a more accurate thermal force model. The development of such model is the primary goal of this series of papers.

In the present paper we give a complete formulation of the diurnal Yarkovsky effect, in which we suitably linearize the boundary emission term. Afonso et al. (1995) also computed explicitly the diurnal Yarkovsky acceleration of a spherical fragment, but only in the assumption that the fragment's spin axis is normal to the orbital plane. Owing to frequent collisions of fragments, resulting in changes of the orientation of their spin axes, the very special configuration considered by Afonso et al. is unrealistic and thus their formulation is incomplete. Since the typical time scale for the change of the spin axis orientation of meter-sized stony fragments is a few Myr only (Farinella et al., 1998), in order to progress in a numerical exploration of the orbital perturbations due to thermal effects on meter-sized objects, we need a formulation which allows for any mutual orientation of the spin axis and the orbital plane. The results which we are going to present and discuss are valid for an arbitrarily oriented spin axis.

2. Theory

2.1. General formulation and linearization

The heat conduction in a solid medium is described by the parabolic equation (often called Fourier equation; e.g. Landau and Lifchitz, 1986)

$$\rho C \frac{\partial T}{\partial t} = K \nabla^2 T \; , \tag{1}$$

yielding a distribution of temperature T throughout the medium at any time t. ∇^2 is the Laplace operator, K is the thermal conductivity, C the specific heat and ρ the density of the material. In principle all these three parameters may be temperature dependent, which would result in a more complex form of the heat conduction equation. However, in this paper we shall neglect these phenomena and adopt average quantities of the physical parameters over the temperature range involved.

Eq. (1) must be supplemented by an appropriate boundary constraint at the surface of the body and by the condition that the temperature is regular inside the body. The boundary condition is provided by the conservation of energy and is given by

$$\epsilon \sigma T^4 + K \left(\mathbf{n} \cdot \frac{\partial T}{\partial \mathbf{r}} \right) = \alpha \, \mathcal{E} \; . \tag{2}$$

The first term on the left-hand side accounts for the energy thermally reradiated by the body (the isotropic Lambert's law is assumed), ϵ is the emissivity and σ the Stefan-Boltzmann constant; the second term gives the energy conducted to the deeper layers of the body, with $\mathbf{n}$ the unit vector normal to the surface of the body. The right-hand side of (2) gives the radiation energy entering the unit surface area of the body per unit time, with α the absorption coefficient and $\mathcal{E}$ the external radiation flux.

A general solution of Eqs. (1) and (2) is fairly complicated even if the geometry of the body is simple (plane-parallel, spherical etc.). The non-linear, fourth-power emission law in the first term of (2) is the main source of difficulties. A standard technique to handle this problem is based on the assumption that the temperature throughout the body does not differ much from some average value. Hereafter we shall adopt this approximation and split the temperature in the following way: $T = T_{\rm av} + \Delta T$. The mean temperature $T_{\rm av}$ will be chosen such that $\Delta T \ll T_{\rm av}$. If the previous condition is fulfilled, we can linearize the emission term as $T^4 \approx T_{\rm av}^4 + 4T_{\rm av}^3 \Delta T + \ldots$. Neglecting higher-order terms in this expansion allows us to treat the problem analytically in proper variables, which follow from symmetries of the body. Assuming spherical the fragments, we shall use spherical coordinates (r, θ, ϕ). The origin $r = 0$ coincides with the center of the body and the colatitude θ is measured from the fragment's spin axis. The origin of the ϕ coordinate, which will be discussed later, depends on an appropriate coordinate frame.

2.2. The most suitable variables

Before embarking on the solution of the problem a careful choice of variables is necessary. Following the previous work (Spencer et al., 1989; Vokrouhlický and Farinella, 1998b,c), we adopt the following set of the non-dimensional quantities:

– the radial coordinate r will be scaled by the thermal length l_s given by

$$l_s = \sqrt{\frac{K}{\rho C \omega}} \ , \tag{3}$$

and shall be denoted by $r' = r/l_s$, ω being the angular velocity of the fragment's rotation;

– the time t will be replaced by a complex variable ζ given by

$$\zeta = \exp(i\omega t) \tag{4}$$

(here $i = \sqrt{-1}$ is the imaginary unit);

– the temperature T will be scaled by an auxiliary value $T_\star$ defined by

$$\epsilon\sigma T_\star^4 = \alpha \mathcal{E}_\star \ , \tag{5}$$

where $\mathcal{E}_\star$ is the solar radiation flux at the position of the fragment. The resulting non-dimensional variable will be denoted by $T' = T/T_\star$, and similarly we define $\Delta T' = \Delta T/T_\star$;

– the energy source term $\mathcal{E}$ in the right hand side of (2) will be scaled by the reference flux $\mathcal{E}_\star$, so we define $\mathcal{E}' = \mathcal{E}/\mathcal{E}_\star$.

Some comments are appropriate here. First, the auxiliary temperature $T_\star$ should not be confused with the mean temperature $T_{\rm av}$. Later on we shall see that the most natural choice for the scaled mean temperature is $T'_{\rm av} = T_{\rm av}/T_\star = 1/\sqrt{2}$ (see also Rubincam 1995, 1998), while $T_\star$ has a meaning of a subsolar temperature. By the way, we remark that Afonso et al. (1994) missed this point and chose $T_{\rm av} = T_\star$. Secondly, the above-mentioned choice of variables reduces the number of parameters in the transformed equations (1) and (2) to a minimum set, particularly the thermal parameter Θ,

$$\Theta = \frac{\Gamma\sqrt{\omega}}{\epsilon\sigma T_\star^3} \ , \tag{6}$$

where $\Gamma = \sqrt{\rho C K}$ is the thermal inertia, and the scaled radius of the body, $R' = R/l_s$. The thermal parameter Θ defined above differs by a factor of the order of unity from similar quantities introduced in the literature. Notably, Peterson's P-variable is given by $P = (\pi^{3/4}/\sqrt{2})\,\Theta$ (Peterson, 1976), while Farinella et al. (1998) define $\Theta_\omega = (\sqrt{2}/\pi)\,\Theta$.

Adopting this new set of variables, the heat conduction Eq. (1) has the following form

$$\begin{aligned} i\zeta \frac{\partial}{\partial\zeta}\ \Delta T'(r';\theta,\phi;\zeta) &= \\ &= \frac{1}{r'^2}\left\{\frac{\partial}{\partial r'}\left(r'^2\frac{\partial}{\partial r'}\right) + \Lambda\left(\theta,\phi\right)\right\}\ \Delta T'(r';\theta,\phi;\zeta)\ , \end{aligned} \tag{7}$$

with the operator $\Lambda(\theta, \phi)$ given by

$$\Lambda\left(\theta,\phi\right) = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right] \ . \tag{8}$$

The linearized boundary condition (2) reads

$$\sqrt{2}\Delta T' + \Theta\left(\frac{\partial\Delta T'}{\partial r'}\right)_{R'} = \Delta\mathcal{E}' \ , \tag{9}$$

where the right-hand term is defined by $\mathcal{E}' = \frac{1}{4} + \Delta\mathcal{E}'$. Here, the first term, $\mathcal{E}'_{\rm av} = 1/4$, is the averaged irradiation of the fragment's surface which determines the averaged temperature mentioned above: $\epsilon\sigma T_{\rm av}^4 \equiv \alpha\mathcal{E}_{\rm av}$. The expansion of the source term $\Delta\mathcal{E}'$ deserves a special care and is related to the reference systems which will be used in the following. Notice that this is a major issue which was not properly solved by Afonso et al. (1995), and we devote the next section to its discussion.

2.3. Radiation source term

The principal reference system in which we shall solve the heat conduction problem is rigidly rotating with the fragment. Its z-axis is aligned with the unit vector $\mathbf{s}$ of the fragment's spin axis and the x-axis is chosen so that the Sun lies in the xz-plane at the instant $t = 0$. The unit position vector of the Sun, $\mathbf{n}_0$, in this system has the components

$$\mathbf{n}_0 = \begin{pmatrix} \frac{1}{2}\zeta\sin\theta_0 \\ \frac{i}{2}\zeta\sin\theta_0 \\ \cos\theta_0 \end{pmatrix} + \mathrm{C.C.} \ , \tag{10}$$

where θ_0 is the solar colatitude and C.C. is a complex conjugate quantity. A crucial point here is the time variability of the quantities in (10). In the case of the diurnal Yarkovsky thermal effect we *assume* that the solar colatitude θ_0 in (10) is constant. This means that the thermal response of the fragment is local or "instantaneous", because the spin period is much shorter than the orbital one. On the contrary, in the case of the seasonal Yarkovsky thermal effect one assumes a variable solar colatitude and averages over ζ (i.e., the rotation of the fragment). These are the two extreme cases of the Yarkovsky thermal effects. More in general one should assume that θ_0 is time-dependent and still the source term $\Delta\mathcal{E}'$ is not averaged over the fragment's rotation. In other words, one should deal with a generic mixture of the diurnal and seasonal Yarkovsky effects. In this paper we adopt the deal with the diurnal case only, and therefore we use the approximation of a constant solar colatitude θ_0.

We can now proceed directly to the evaluation of the radiation source term $\Delta\mathcal{E}'$ in (9). Given a surface element with a unit normal vector $\mathbf{n}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$, we have

$$\begin{aligned}\mathcal{E}' &= \mathbf{n}(\theta,\phi).\mathbf{n}_0(\theta_0,\zeta) \qquad \text{if } (\mathbf{n}.\mathbf{n}_0) > 0, \\ &= 0 \qquad\qquad\qquad\qquad \text{otherwise.}\end{aligned} \tag{11}$$

Obviously, it would be too cumbersome to work analytically with such a piecewise-defined source function. Moreover, as we shall see below, we do not need the whole information about $\mathcal{E}'$ for computing the thermal force. It is suitable to represent (11) in a spherical harmonics expansion

$$\begin{aligned}\mathcal{E}' &= \mathbf{n}(\theta,\phi).\mathbf{n}_0(\theta_0,\zeta) \\ &= \sum_{n\geq 0}\sum_{k=-n}^{n} a_{nk}(\theta_0,\zeta)\,Y_{nk}(\theta,\phi)\ ,\end{aligned} \tag{12}$$

from which only the monopole ($n = 0$) and dipole ($n = 1$) parts will be relevant. One can prove the relation $a_{nk}(\theta_0,\zeta) = b_{nk}(\theta_0)\,\zeta^k$, a particular case of which is

$$b_{00} = a_{00} = \frac{\sqrt{\pi}}{2} \tag{13}$$

for the monopole coefficient, and

$$b_{10}(\theta_0) = \sqrt{\frac{\pi}{3}}\cos\theta_0\ , \tag{14}$$

$$b_{1\pm 1}(\theta_0) = \mp\sqrt{\frac{\pi}{6}}\sin\theta_0 \tag{15}$$

for the dipole coefficients. The best technique to obtain Eqs. (14) and (15) is to compute them for the special case $\theta_0 = 0$ (the solar direction along the fragment's spin axis) by simple quadratures. Then the general case is determined by a transformation of the right-hand side of (12) to a new system, with a given solar direction. The formulae for the transformation of the spherical functions Y_{nk} using Wigner's matrixes are discussed in the standard textbooks of Wigner (1959) and Edmonds (1974). In the astronomical context we refer, for instance, to Šidlichovský (1983).

Recalling that $Y_{00} = 1/(2\sqrt{\pi})$, we observe that the monopole term in (12) represents exactly the averaged irradiation of the fragment. The important terms for determination of the thermal force, which contribute to the $\Delta\mathcal{E}'$ source, are those of the dipole part.

2.4. Regular solution satisfying the boundary condition

After having discussed in detail the source term, we are ready to find a solution of the problem (7) with the surface condition (9). First we determine a general form of the solution which is regular throughout the body.

The linearity of the heat conduction Eq. (7) allows us a convenient separation of the variables. Thanks to the spherical system of coordinates we observe that an expansion in spherical harmonics is the best choice. Thus, we write

$$\Delta T'(r';\theta,\phi;\zeta) = \sum_{n\geq 1}\sum_{k=-n}^{n} t'_{nk}(r';\zeta)\,Y_{nk}(\theta,\phi)\ . \tag{16}$$

Considering the properties of the coefficients of the source terms expansion (12), we realize that $t'_{nk}(r';\zeta) = \tau'_{nk}(r')\,\zeta^k$, where the radial functions satisfy the system of decoupled ordinary differential equations

$$\left\{\frac{d}{dr'}\left(r'^2\frac{d}{dr'}\right) - \left[n\,(n+1) + ikr'^2\right]\right\}\tau'_{nk}(r') = 0\ . \tag{17}$$

Their general solution, which is regular inside the body (i.e. at $r' = 0$), reads

$$\tau'_{nk}(r') = c_n\, r'^n \qquad \text{for}\ \ k = 0\ , \tag{18}$$

$$\tau'_{nk}(r') = c_{nk}\, j_n(\sqrt{-ik}r') \qquad \text{for}\ \ k \neq 0\ , \tag{19}$$

where $j_n(z)$ are the spherical Bessel functions of order n (note the complex argument z). The constants c_n and c_{nk} are to be determined by fulfilling the boundary constraint (9).

Substituting (18) and (19) into (9) we find

$$c_n = \frac{b_{n0}}{\sqrt{2}R'^n}\frac{1}{1+n\lambda}\ , \tag{20}$$

$$c_{nk} = \frac{b_{nk}}{\sqrt{2}\,j_n\left(\sqrt{-ik}R'\right)}\frac{1}{\left[1+\lambda\,\frac{z}{j_n(z)}\frac{d}{dz}j_n(z)\right]_{z=\sqrt{-ik}R'}}\ , \tag{21}$$

where $\lambda \equiv \Theta/\sqrt{2}R'$. As noted above, a special attention has to be paid to the dipole part, $n = 1$. For this purpose we introduce an auxiliary function $\psi(z)$, defined by

$$1 + \psi(z) = \frac{z}{j_1(z)}\,\frac{d}{dz}j_1(z)\ . \tag{22}$$

One can easily verify that $[\psi(z)/z]$ is nonsingular at $z = 0$.

2.5. Complete linear solution

Putting together all the previous results we find that the general solution for the surface temperature of an arbitrary element defined by spherical angles θ and ϕ is given by

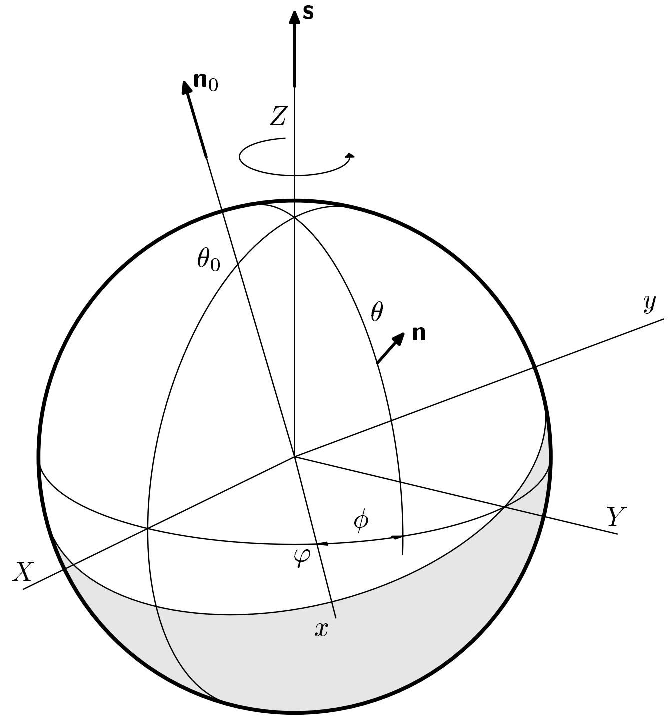

Fig. 1. Coordinate systems and variables introduced in the text. The xyz-system is rigidly rotating with the fragment, while the XYZ-system is fixed with respect to the Sun. The unit vectors are directed as follows: (i) $\mathbf{s}$ along the spin axis of the fragment, (ii) $\mathbf{n}_0$ to the local position of the Sun, and (iii) $\mathbf{n}$ normal to the considered surface element. The shaded region is not illuminated by sunlight.

$$\Delta T'(R';\theta,\phi;\zeta) = \frac{1}{\sqrt{2}\,(1+\lambda)}\Bigg[b_{10}\,(\theta_0)\;Y_{10}\,(\theta,\phi) + \frac{b_{11}(\theta_0)\,\zeta}{1+\frac{\lambda}{1+\lambda}\psi(\sqrt{-i}R')}\,Y_{11}\,(\theta,\phi) + \mathrm{C.C.} + \text{second and higher order terms}\Bigg] . \quad (23)$$

In the previous formula we have not given in explicit form the quadrupole and higher multipole terms (although the required formulae are given above, these terms will not play any role in computing the resulting thermal force). It turns out to be convenient to transform the previous solution (23) into a new reference system. This is a nonrotating frame, having the Z-axis aligned with the fragment's spin vector as before. The X-axis is now oriented so that the solar position lies always in the XZ-plane. Spherical coordinates in the new, nonrotating system will be denoted by ϑ and φ. Their relationship to the previous coordinates θ and ϕ is given by a trivial rotation: $\vartheta = \theta$ and $\varphi = \phi + \omega\, t$ (see Fig. 1).

Transforming the surface temperature distribution (23) into the new coordinates we obtain

$$\Delta T'(R';\vartheta,\varphi) = \frac{1}{\sqrt{2}\,(1+\lambda)}\Bigg[b_{10}\,(\theta_0)\;Y_{10}\,(\vartheta,\varphi) + \frac{b_{11}(\theta_0)}{1+\frac{\lambda}{1+\lambda}\psi(\sqrt{-i}R')}\,Y_{11}\,(\vartheta,\varphi) + \mathrm{C.C.} + \text{second and higher order terms}\Bigg] . \quad (24)$$

Note a slight, but important difference between formulae (23) and (24): the latter does not depend on time through ζ. This is clearly due to the fact that in the source-oriented coordinate system the surface temperature has to be stationary.

Next we introduce the auxiliary real functions $A(x)$, $B(x)$, $C(x)$, $D(x)$ and $E(x)$, plus a phase $\delta(x)$, by

$$\frac{1}{1+\frac{\lambda}{1+\lambda}\psi(z)} = \frac{A(x)+iB(x)}{C(x)+iD(x)} = E(x)\exp\left[i\delta\,(x)\right] , \quad (25)$$

with $z = \sqrt{-i}R'$ and $x = \sqrt{2}R'$. After some algebra one finds explicitly

$$A(x) = -(x+2) - e^x\left[(x-2)\cos x - x\sin x\right] , \quad (26)$$

$$B(x) = -x - e^x\left[x\cos x + (x-2)\sin x\right] , \quad (27)$$

$$C(x) = A(x) + \frac{\lambda}{1+\lambda}\times \left\{3\,(x+2) + e^x\left[3\,(x-2)\cos x + x\,(x-3)\sin x\right]\right\} , \quad (28)$$

$$D(x) = B(x) + \frac{\lambda}{1+\lambda}\times \left\{x\,(x+3) - e^x\left[x\,(x-3)\cos x - 3\,(x-2)\sin x\right]\right\} , \quad (29)$$

with simple relations for $E(x)$ and $\delta(x)$.

2.6. Thermal force and related quantities

Having determined the temperature distribution on the fragment's surface, we can easily compute the thermal recoil force. Assuming Lambert's isotropic thermal emission as before, one finds the following components of the thermal force per unit of fragment mass, projected onto the X, Y and Z axes of the nonrotating system introduced above:

$$f_X + i f_Y = -\frac{4\alpha}{9}\,\Phi\,\frac{\sin\theta_0}{1+\lambda}\,E_{R'}\exp\left(-i\delta_{R'}\right) , \quad (30)$$

$$f_Z = -\frac{4\alpha}{9}\,\Phi\,\frac{\cos\theta_0}{1+\lambda} , \quad (31)$$

where $E_{R'} \equiv E(\sqrt{2}R')$ and $\delta_{R'} = \delta(\sqrt{2}R')$. The factor $\Phi \equiv (\pi R^2 \mathcal{E}_\star/mc)$, where m is the mass of the fragment and c the velocity of light, is the usual radiation pressure parameter on a spherical body (see e.g. Burns et al. 1979). We note that the remaining factors in (30) and (31) yield the efficiency of the thermal recoil force when the latter is compared to the direct solar radiation pressure.

The physical meaning of the thermal parameter Θ has been discussed by Farinella et al. (1998), who interpreted it as the ratio of the thermal relaxation time, required for the accumulation of the absorbed energy and its reemission, to the rotation period. As a consequence, the limit $\Theta \to 0$, and therefore $\lambda \to 0$, corresponds to the case of instantaneous reemission of the thermal energy. This is normally interpreted as a simple diffusion of

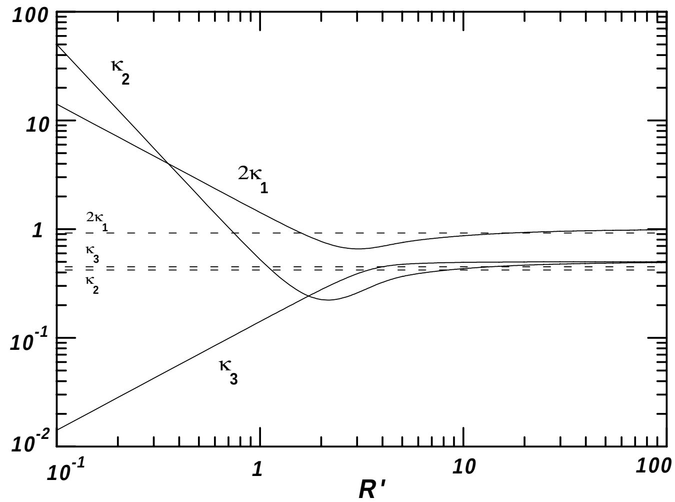

Fig. 2. Coefficients κ_1, κ_2 and κ_3 vs. the scaled radius R' of the body. The values derived by Peterson (1976), given by the dashed lines, match reasonably well our results in the limit of "large" bodies ($R' \to \infty$).

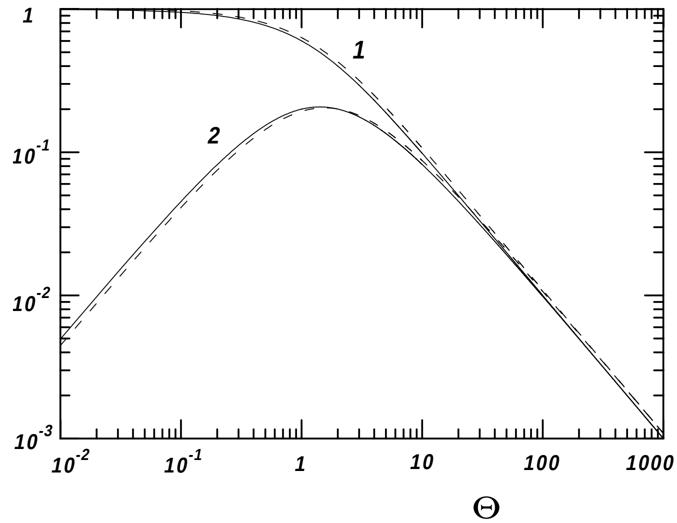

Fig. 3. Normalized equatorial components of the thermal force vs. the thermal parameter Θ. Curves 1 and 2 show $9f_X/(4\alpha\Phi)$ and $9f_Y/(4\alpha\Phi)$, respectively. The radius of the body is $R' = 500$, corresponding to about 1.5 m for a stony fragment. Peterson's (1976) solution (dashed curves) is shown for comparison.

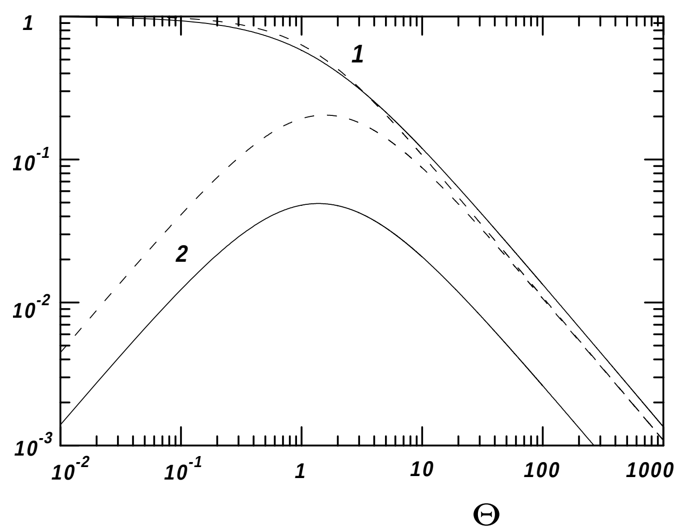

Fig. 4. The same as Fig. 3, but for a $R' = 1$ body, corresponding to about $10\ \mu$m for a basalt particle. A significant discrepancy with respect to Peterson's results is apparent now, mainly in the transverse force component f_Y, as discussed in the text.

light on a macroscopic sphere (no diffraction effects; see, for instance, Milani et al., 1987; Vokrouhlický et al. 1993). We can easily check that $E_{R'} \cos\delta_{R'} \to 1$ when $\lambda \to 0$, which results into

$$\mathbf{f}(\lambda = 0) = -\frac{4\alpha}{9}\,\Phi\,\mathbf{n}_0\ , \tag{32}$$

and obviously $f_Y(\lambda = 0) = 0$. The force (32) is just opposite to the local solar direction, as might have been expected, and its magnitude is just $4\alpha/9$ of the direct radiation pressure.

One can see that the equatorial thermal force components (f_X, f_Y) in (30) are rational functions of λ (and consequently of the thermal parameter Θ). A little algebra yields

$$f_X = -\frac{4\alpha}{9}\,\Phi\,\sin\theta_0\,\frac{1+\kappa_1\Theta}{1+2\kappa_1\Theta+\kappa_2\Theta^2}\ , \tag{33}$$

$$f_Y = -\frac{4\alpha}{9}\,\Phi\,\sin\theta_0\,\frac{\kappa_3\Theta}{1+2\kappa_1\Theta+\kappa_2\Theta^2}\ , \tag{34}$$

where the coefficients κ_1, κ_2 and κ_3 are functions of R'. Note that this result is in agreement with Peterson's (1976) Padé approximant representation [Eq. (26) in his paper]. However, Peterson obtained the values for these coefficients numerically, in the limit of large bodies ($R' \approx \infty$) only. One can easily check that there is a rather good agreement between our results and those of Peterson, by taking the corresponding limit in our formula (34). Fig. 2 shows the dependence on R' of the κ coefficient (solid curves) together with the limit values at large R' obtained by Peterson (dashed lines). Note also a significant increase of the quadratic coefficient κ_2 in the denominator at small R', as well as a significant decrease of the numerator coefficient κ_3 for the transverse force component. Both these results have to do with the important fact that thermal effects are inhibited for small bodies, due to more efficient heat conduction across them. In other words, the temperature differences in the body decay very rapidly with decreasing sizes.

Figs. 3 and 4 show the dependence of the "equatorial" force components f_X and f_Y on the thermal parameter Θ. The former figure has been derived for a large body ($R' = 500$), which according to Farinella et al. (1998) corresponds to a stony fragment about 1.5 m in radius. We note again a good agreement with Peterson's (1976) approximation shown by dashed lines. As the body becomes smaller (Fig. 4, corresponding to $R' = 1$), we can remark a significant difference between the results of the two approaches for the f_Y component. This is consistent with the previous remark concerning the behaviour of κ_2 and κ_3 for small R'.

Having computed the force components we can also estimate some of the orbital effects resulting from the thermal perturbations. In particular the long-term variations of the semimajor axis are essential because of the original motivation of this work: the relevance of the Yarkovsky effect as a source of mobility for fragments in the asteroidal belt. To the zero order in the eccentricity e, we easily obtain

$$\frac{da}{dt} = -\frac{8\alpha}{9}\frac{\Phi}{n}\frac{E_{R'}\sin\delta_{R'}}{1+\lambda}\cos\gamma + \mathcal{O}\left(e\right) \; . \quad (35)$$

Note the obliquity dependent factor $\cos\gamma = \mathbf{s}\cdot\mathbf{N}$, where $\mathbf{N}$ is the unit vector normal to the mean orbit; n is the fragment's orbital mean motion around the Sun. We can also remark that the Z-component of the thermal force (31) does not contribute to the formula (35) but, obviously, yields short-periodic perturbations of the fragment's motion. The spin axis of the fragment is likely to undergo random changes due to collisions with other fragments, and if these are frequent enough we conclude that the long-term change of the semimajor axis vanishes because $\langle\cos\gamma\rangle = 0$ (as already conluded by Burns et al. 1979 and Farinella et al. 1998).

Similarly, we can obtain the long term perturbation of the inclination

$$\frac{dI}{dt} = \frac{2\alpha}{9}\frac{\Phi}{na}\frac{s_P\left(1-E_{R'}\cos\delta_{R'}\right)\cos\gamma + s_Q E_{R'}\sin\delta_{R'}}{1+\lambda} + \mathcal{O}\left(e\right) \; , \quad (36)$$

where $s_P = \mathbf{s}\cdot\mathbf{P}$ and $s_Q = \mathbf{s}\cdot\mathbf{Q}$ are the projections of the spin vector $\mathbf{s}$ onto the position vector $\mathbf{P}$ of the mean pericenter and the vector $\mathbf{Q} = \mathbf{N}\times\mathbf{P}$. Unlike the case of the semimajor axis, the Z thermal force component (31) does contribute to the long term perturbation of the inclination. Taking again the mean value of (36) over random orientations of the spin axis $\mathbf{s}$, we conclude that the diurnal Yarkovsky effect leaves the mean inclination unchanged over a long span of time.

Finally, we observe that the eccentricity's long term perturbation is proportional to the first power of the eccentricity itself: $(de/dt) \propto e$. Although we might compute this term explicitly, we leave a discussion of the analytically determined mean orbital effects to a subsequent study. Indeed, the main objective of this paper was the derivation of the instantaneous thermal force components (30) and (31).

2.7. Comparison with other approaches and discussion

First, let us comment on the work of Afonso et al. (1995) whose results have been applied to assess the diurnal thermal effects on the orbits of spherical bodies. It may appear that our solution is only a slight generalization for the case of an arbitrary orientation of the fragment's spin axis. However, this generalization is essential for carrying out further numerical work. Moreover, we can see that the Afonso et al. solution is not really correct. For instance, these authors did not notice that the axially symmetric ($m = 0$) dipole mode of the temperature distribution is not time-dependent. As a result, the last term in their formula (31) is not correct, as it should not depend on time. Secondly,

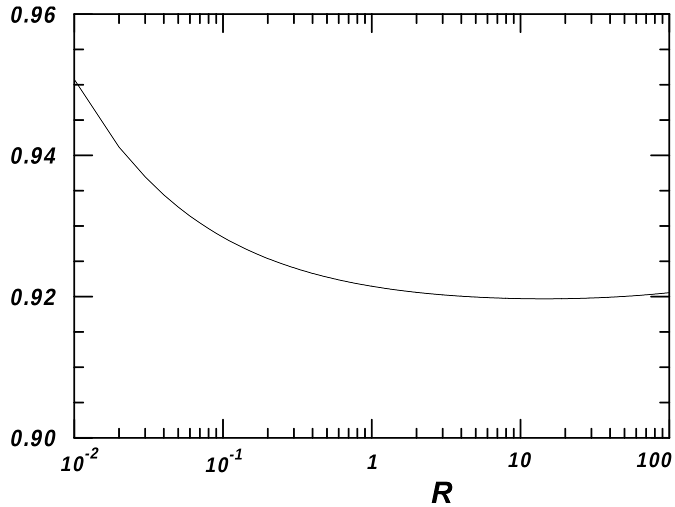

Fig. 5. Ratio of the maximum semimajor axis drift rate da/dt (for $\gamma = 0$, circular orbits) obtained by our Eq. (35) to that of Peterson Eq. (37), as a function of the radius of the body. Thermal properties appropriate for stony fragments are assumed. A fairly good agreement throughout the relevant size range is observed despite the entirely different approaches.

they neglected the difference between the averaged temperature ($T_{\rm av}$) and the subsolar temperature ($T_\star$). Thus a factor of two is missing in their solution. This fact can be easily verified in the limit of the instantaneous rediffusion of sunlight by the fragment's surface. Their formula (37) yields a factor $2/9$, instead of a correct value $4/9$ derived here [see Eq. (32) above]. As a consequence, their discussion of the reasons for a factor two difference with respect to Peterson's solution (see p. 790) is also incorrect.

A comparison with the work of Peterson (1976) is more interesting (note that Farinella et al. 1998 used these results for estimating the orbital drift rate due to the diurnal effect). Peterson's formula for the mean semimajor axis rate, when rewritten in our variables, reads

$$\frac{da}{dt} = \frac{\alpha\Phi}{n}\, f_P(\Theta)\, \cos\gamma \; , \quad (37)$$

with the function f_P given by

$$f_P(\Theta) = 0.4\,\frac{\Theta}{1 + 0.914\,\Theta + 0.413\,\Theta^2} \; . \quad (38)$$

Interestingly, Peterson's result (37) matches quite well our general formula (35), at least in the important case of meter-sized stony fragments — see Fig. 5 — despite the fact that Peterson used an entirely different mathematical technique. This agrees with our previous finding – that Peterson's results are fairly accurate in the case of large bodies. Note that the thermal penetration depth corresponding to the rotation frequency is about $l_s \approx 2.5\,\sqrt{R}$ mm if R is given in meters (this formula follows from the assumption the $\omega \propto R^{-1}$, according to Farinella et al. 1998). The depth l_s is even smaller if the surface is covered by a thin layer of regolith or any porous material which significantly decreases the thermal conductivity K (Rubincam 1995;

Farinella et al. 1998). Therefore asteroid fragments larger than a centimeter can be safely considered "large", from the point of view of the diurnal effect.

3. Conclusions

The main results of this paper may be summarized as follows:

– We have discussed in detail a linear theory for the diurnal Yarkovsky thermal effect on spherical rotating bodies (presumably asteroidal fragments). Although this is not the first paper on this issue, we believe it gives, for the first time, a clear and unambiguous solution free from the limitations and errors of previous studies.
– The main parameters of the theory are: (i) the radius of the fragment R scaled by the penetration depth of the thermal wave l_s ($R' = R/l_s$), and (ii) the ratio between the thermal parameter Θ and the scaled radius R' of the body. In the limit of large bodies, $R' \to \infty$, and the present solution matches the results of Peterson (1976).
– Given these two parameters, we derive instantaneous thermal force acting on the body for an arbitrary orientation of the spin axis.

Although the analytic estimates of Sect. 2.6 are given for circular orbits, we remark that our solution (30) and (31) for the thermal force is by no means limited to low-eccentricity orbits. This is due to the simpler character of the diurnal Yarkovsky effect, when compared with the seasonal effect. As explained above, the diurnal effect is purely local, since at a given time it depends on the instantaneous dynamical variables only. On the other hand, the seasonal Yarkovsky force depends non-locally on the dynamical state at a number of previous instants. This complicates the analytic expansions for eccentric orbits (see, e.g., Vokrouhlický and Farinella 1998c). In our case, the long-term analytic estimates can be generalized to the case of arbitrary eccentricities provided one pays attention to: (i) the expansion of the true anomaly terms in $\sin\theta_0$ and $\cos\theta_0$, (ii) the simple luminosity decrease relationship $\mathcal{E}_\star \propto (a/r)^2$ in Φ, and (iii) the orbital dependence of the thermal parameter Θ given by

$$\Theta = \Theta_0 a^{3/2} \left(\frac{r}{a}\right)^{3/2} . \qquad (39)$$

Here, Θ_0 is the thermal parameter at 1 AU from the Sun, whereas a is the semimajor axis in AU and r the distance from the Sun. We leave this task for a subsequent study.

As far as theoretical modelling is concerned, we note that a careful understanding of the mixture between the diurnal and seasonal Yarkovsky effects remains a major challenge for the future. In the optics of this paper, one should assume that the solar colatitude θ_0 in (12) – (15) is not constant but changes on the timescale of one revolution around the Sun. The final Fourier expansion in the time-like variable ζ would be, however, more complicated than in the case discussed here.

Acknowledgements. The author thanks M. Brož for checking some calculations, J. Bičák, P. Farinella and L.A. Lebofsky for discussions and language corrections. Partial support from the Czech Grant Agency under contract No. 205/96/K119 is also acknowledged.

References

Afonso G., R.S. Gomes, M.A. Florczak, 1995, Planet. Space Sci. 43, 787
Burns J.A., P.L. Lamy, S. Soter, 1979, Icarus 40, 1
Ceplecha Z., 1992, A&A 263, 361
Ceplecha Z., 1996, A&A 311, 329
Edmonds A.R., 1974, Angular momentum in quantum mechanics, Princeton Univ. Press, Princeton
Farinella P., D. Vokrouhlický, F. Barlier, 1996, J. Geophys. Res. 101, 17861
Farinella P., D. Vokrouhlický, W.K. Hartmann, 1998, Icarus 132, 378
Gladman B.J., F. Migliorini, A. Morbidelli et al., 1997, Science 277, 197
Hartmann W.K., P. Farinella, D. Vokrouhlický et al., 1998, Meteoritics Planet. Sci., in press
Landau L.D., E. Lifchitz, 1986, *Hydrodynamics*, Nauka, Moscow (in russian)
Migliorini F., A. Morbidelli, V. Zappalá et al., 1997, Meteoritics Planet. Sci. 32, 903
Milani A., A.M. Nobili, P. Farinella, 1987, Non-gravitational Perturbations and Satellite Geodesy, A. Hilger, Bristol
Morbidelli A., R. Gonczi, Ch. Froeschlé, P. Farinella, 1994, AA 282, 955
Peterson C., 1976, Icarus 29, 91
Öpik E. J., 1951, Proc. Roy. Irish Acad. 54, 165
Rabinowitz D.L., 1993, ApJ 407, 412
Rabinowitz D.L., 1994, Icarus 111, 364
Radzievskii V.V., 1952, Astron. Zh. 29, 162 (in Russian)
Rubincam D.P., 1987, J. Geophys. Res. 92, 1287
Rubincam D.P., 1995, J. Geophys. Res. 100, 1585
Rubincam D.P., 1998, J. Geophys. Res. 103, 1725
Spencer J.R., L.A. Lebofsky, M.V. Sykes, 1989, Icarus 78, 337
Šidlichovský M., 1983, Bull. Astron. Inst. Czechosl. 34, 65
Vokrouhlický D., P. Farinella, 1997, Planet. Space Sci. 45, 419
Vokrouhlický D., P. Farinella, 1998a, A&A, in press.
Vokrouhlický D., P. Farinella, 1998b, Astron. J., submitted.
Vokrouhlický D., P. Farinella, 1998c, Icarus, submitted.
Vokrouhlický D., P. Farinella, D. Lucchesi, 1993, Celest. Mech. 57, 225
Wigner E.P., 1959, Group Theory, Academic Press, New York