7 Henyey numerical method for the integration of stellar interior

Currently, the numerical solution of stellar structure equations is usually performed by a method of complete linearisation, which was first suggested by Henyey et al. (1959) and which was later used also for computations of stellar atmospheres.

7.1 Method of complete linearisation

The method of complete linearisation consists of the following steps:

1. a discretisation of the equations;
2. supplying of the boundary conditions in the centre;
3. a construction of the outer boundary conditions, consisting of: (a) an estimate of the luminosity L_* and the effective temperature T_{eff}; (b) a calculation of three models of the photosphere and sub-surface layers; (c) a solution for coefficients of the linear functions $R_1(P_1, T_1)$, $L_1(P_1, T_1)$;
4. a linearisation of the equations;
5. an iterative calculation of a stationary model;
6. performing a time step in case of an evolutionary model.

Discretisation. The very first step is a transition from differential to *difference* equations, in other words a discretisation of the problem. We divide whole stellar interior (i.e. the volume, where matter is in a fully ionised state) to a sufficient number of concentric shells and number them from the surface to the centre by the index $j = 1 \dots N$ (Fig. 7.1), we choose e.g. $N = 200$. This way we determine discrete values M_j of the formerly continuous independent variable M_R.

We replace derivatives of the continuous functions R, P, L_R, T on the left-hand sides of the equations by differences of variables R_j, P_j, L_j, T_j between neighbouring shells j and $j+1$. Instead of we write their arithmetic means between the j-th and $(j+1)$-th shells.[1]For example, for the equation of the hydrostatic equilibrium we have (after all terms were moved to the left)

$$\frac{\mathrm{d}P}{\mathrm{d}M_R} + \frac{GM_R}{4\pi R^4} \simeq \frac{P_j - P_{j+1}}{M_j - M_{j+1}} + \frac{1}{2}\left(\frac{GM_j}{4\pi R_j^4} + \frac{GM_{j+1}}{4\pi R_{j+1}^4}\right) = 0\,. \tag{7.1}$$

We have four stellar structure equations for each pair of shells , so we can arrange a whole set of equations, which we abstractly denote

$$G_{ij} = 0\,, \tag{7.2}$$

where $i = 1 \dots 4$ and $j = 1 \dots N-1$. It is $4(N-1)$ equations for $4N$ unknowns R_j, P_j, L_j, T_j. Clearly, we will have to add some more equations.

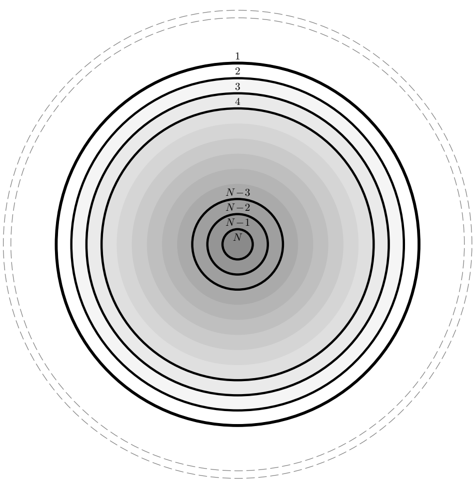

Figure 7.1: Discretisation of a stellar interior to N concentric shells. The sub-surface layers and the photosphere are also plotted as dashed.

Boundary conditions in the centre. A rewrite of the boundary conditions (**??**) and (**??**) is very simple

$$L_N = 0\,, \tag{7.3}$$

$$R_N = 0\,. \tag{7.4}$$

The number of equations then increases to $4N-2$.

[1]Or we can substitute average values of the variables to the right-hand sides, $\frac{1}{2}(R_j + R_{j+1})$, etc.

Outer boundary conditions. The boundary conditions on the surface are more complicated thought, because it is impossible to prescribe a certain value for R_1, P_1, L_1 or T_1, as we would wrongly restrict the solution! Instead, for the given mass M_* of the star we 'guess' three pairs of values for the luminosity and the effective temperature

$$L_*, T_{\mathrm{eff}}{}^{(1,2,3)}, \tag{7.5}$$

in the surroundings of the expected position of the star on the HR diagram. We then compute *three photospheric models* (from $\tau = 0$ to $\tau \simeq 2/3$) and obtain three sets of values

$$\begin{aligned} &R_1, P_1, L_1, T_1^{(1)}, \\ &R_1, P_1, L_1, T_1^{(2)}, \\ &R_1, P_1, L_1, T_1^{(3)}, \end{aligned} \tag{7.6}$$

which we use to compute coefficients $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ of the linear functions of two variables

$$R_1 = \alpha_1 P_1 + \beta_1 T_1 + \gamma_1 , \tag{7.7}$$

$$L_1 = \alpha_2 P_1 + \beta_2 T_1 + \gamma_2 . \tag{7.8}$$

by the least-squares method. These two equations are in turn used as the boundary conditions. Indeed, this is clever — we prescribe only the approximate relations between variables $R_1 = f_1(P_1, T_1)$, $L_1 = f_2(P_1, T_1)$, not the values of variables! Finally, we have $4N$ equations for $4N$ unknowns, but we are not done. The set is *strongly non-linear* and it not possible to solve it with simple methods.

Linearisation. We can solve the whole set of equations in such a way that we simply estimate or 'guess' the values of variables R_j, P_j, L_j, T_j (then the equations (7.2) would not hold), but we replace all them with "the guesses plus small corrections" in Eqs. (7.2)

$$\begin{aligned} R_j &\to R_j + \Delta R_j , \\ P_j &\to P_j + \Delta P_j , \\ L_j &\to L_j + \Delta L_j , \\ T_j &\to T_j + \Delta T_j , \end{aligned} \tag{7.9}$$

and we linearise the whole set as[2]

$$G_{ij} + \mathrm{d}G_{ij} = 0 , \tag{7.10}$$

[2] It is not sufficient to linearise individual variables, e.g. $(R_j + \Delta R_j)^2 \doteq R_j^2 + 2R_j \Delta R_j$, because there are products and quotients in the equations.

where

$$\begin{aligned}\mathrm{d}G_{ij} &= \frac{\partial G_{ij}}{\partial R_j}\Delta R_j + \frac{\partial G_{ij}}{\partial P_j}\Delta P_j + \frac{\partial G_{ij}}{\partial L_j}\Delta L_j + \frac{\partial G_{ij}}{\partial T_j}\Delta T_j + \\ &+ \frac{\partial G_{ij}}{\partial R_{j+1}}\Delta R_{j+1} + \frac{\partial G_{ij}}{\partial P_{j+1}}\Delta P_{j+1} + \frac{\partial G_{ij}}{\partial L_{j+1}}\Delta L_{j+1} + \frac{\partial G_{ij}}{\partial T_{j+1}}\Delta T_{j+1}\,. \quad (7.11)\end{aligned}$$

We compute the partial derivates easily from the original equations, e.g.

$$\frac{\partial G_{23}}{\partial R_3} = \frac{\partial}{\partial R_3}\left[\frac{P_3 - P_4}{M_3 - M_4} + \frac{1}{2}\left(\frac{GM_3}{4\pi R_3^4} + \frac{GM_4}{4\pi R_4^4}\right)\right] = -\frac{GM_3}{2\pi R_3^5}\,. \quad (7.12)$$

Iterations. The set (7.10) of $4N$ linear equations with $4N$ unknowns $\Delta R_j, \Delta P_j, \Delta L_j$ and ΔT_j can be solved easily (e.g. by the Gauss elimination method or better methods optimised for solutions of band matrices). The obtained corrections of original estimates are used to derive more precise estimates

$$\begin{aligned} R_j^{(2)} &= R_j^{(1)} + \Delta R_j^{(1)}\,, \\ P_j^{(2)} &= P_j^{(1)} + \Delta P_j^{(1)}\,, \\ L_j^{(2)} &= L_j^{(1)} + \Delta L_j^{(1)}\,, \\ T_j^{(2)} &= T_j^{(1)} + \Delta T_j^{(1)}\,, \end{aligned} \quad (7.13)$$

and the set (7.10) is solved again for $\Delta R_j^{(2)}, \Delta P_j^{(2)}, \Delta L_j^{(2)}, \Delta T_j^{(2)}$. We repeat the iterations until a required precision is reached (i.e. $\Delta R_j, \Delta P_j, \Delta L_j, \Delta T_j$ are small). If our model would result in values L_*, T_{eff} outside the range of our original estimates (7.5), we have to return to point 3, of course.

Time step. If we compute a stellar evolution, i.e. a temporal sequence of stationary models, we choose the time step Δt between two consecutive models and compute new chemical composition according to a discretised form of the relation (**??**)

$$Y_j(t + \Delta t) = Y_j(t) + \sum_i \alpha_i \epsilon_i(\rho_j, T_j, X_j, Y_j, Z_j)\,\Delta t\,, \quad (7.14)$$

whereas in convective zones we perform an averaging according to the relation (**??**), we only replace integrals with sums.

7.2 Limits of discretisation

It is instructive to try, how the discretisation can be pushed to the extreme. Let us take for example the equation of hydrostatic equilibrium in the form

$$\frac{\mathrm{d}P}{\mathrm{d}R} = -\frac{GM_R\rho}{R^2}\,. \quad (7.15)$$

If we replace the derivative on the left-hand side with a difference between the centre and the surface (we divide the star to mere two shells), we obtain

$$\frac{P_{\mathrm{c}} - 0}{0 - R_*} = -\left(\frac{GM_R}{R^2}\right)_{\mathrm{mean}} \rho_{\mathrm{mean}} = -\frac{1}{2}\frac{GM_*}{R_*^2}\frac{M_*}{\frac{4}{3}\pi R_*^3}\,, \tag{7.16}$$

with respect to the fact that the gravitational acceleration in the centre is After rearrangement, we obtain an estimate of the central pressure in the star

$$P_{\mathrm{c}} = \frac{3GM_*^2}{8\pi R_*^4}\,. \tag{7.17}$$

If we substitute the observed values M_* and R_* for the Sun, we obtain the pressure approximately $1{,}34 \cdot 10^{15}$ [cgs], while the value inferred from the precise model of the Sun is $2{,}269 \cdot 10^{17}$ [cgs]. It is a rather substantial difference. There is a comparison of the model and the estimates for several masses of stars in Table 7.1. It is interesting thought that the relation between the logarithm of calculated and estimated pressure is almost precisely linear. We see that the estimates demonstrate *a decrease of the central pressure with the increasing star mass*, in accord with realistic models. Using a sport terminology: a certain estimate that the pressure in the interior is very high is all right, even with this crude discretisation, and we even correctly qualitatively estimate how the pressure depends on mass.

Table 7.1: A comparison of the calculated central pressure and estimates.

star mass ($M_\odot$)	$\log P_{\mathrm{c}}$ (model) [cgs]	$\log P_{\mathrm{c}}$ (estimate) [cgs]
1	17,356	15,128
7	16,609	14,709
25	16,275	14,518