WINDS OF HOT MASSIVE STARS
II Lecture: Basic theory of winds of hot massive stars

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Selected Topics in Astrophysics
Faculty of Mathematics and Physics
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Prague
1. Properties of winds of hot massive stars
2. Line-driven wind theory
3. Wind hydrodynamic equations
4. Radiative force
5. Sobolev approximation
Properties of winds of hot massive stars

- EXTREMELY LUMINOUS
  - spectral types A, B, and O; $L \gtrsim 10^2 \ [L_\odot]$  
  - W-R, LBV, B[e] stars
- HOT - $T_{\text{eff}} \gtrsim 8000 \ \text{[K]}$
- MASSIVE - $M \gtrsim 2 \ [M_\odot]$
- SHORT LIFETIMES
  - ($\sim 10^6 \ \text{yr}$)
- END IN SUPERNOVA EXPLOSION
- HAVE WIND
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Typical parameters for O-type stars and their winds

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SHORT LIFETIMES
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END IN SUPERNOVA
EXPLOSION

HAVE WIND

TYPICAL $\dot{M}$
from $10^{-7}$ to $10^{-4} \, M_{\odot}$

TYPICAL $v_\infty$ - from 200 km s$^{-1}$ (for A-supergiant) to 3 000 km s$^{-1}$ (for early O-stars)

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Properties of winds of hot massive stars

- Hot stars emit their peak radiation in the UV wavelength region

Wien's displacement law

\[ \lambda_{\text{max}} T = b \]

\[ b = 0.29 \text{ cm K}; \quad T = 30000 \text{ K} \implies \lambda_{\text{max}} = 960 \text{ Å} \]
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- The rocket AEROBEE (1965) - it was possible to obtain stellar spectra in the UV region; the beginning of far-UV stellar astronomy (later IUE, COPERNICUS, FUSE)
Properties of winds of hot massive stars

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- Important result from UV observation: basically all hot stars with initial mass larger than 15 M\(_\odot\) show a high velocity outflow.

The outer atmospheres of hot stars have plenty of absorption lines in the ultraviolet, e.g., resonance lines from N V \(\lambda\lambda 1239, 1243 \text{ Å}\), Si IV \(\lambda\lambda 1394, 1403 \text{ Å}\), C IV \(\lambda\lambda 1548, 1551 \text{ Å}\) (see Morton, 1967).
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- Massive hot stars are luminous \( \Rightarrow \) accelerating force: **RADIATIVE FORCE**
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- Massive hot stars are luminous \( \Rightarrow \) accelerating force: RADIATIVE FORCE

- Lucy & Solomon (1970) - winds can be driven by absorption of radiation in spectral lines
Line-driven wind theory

- Initial idea - electromagnetic radiation carries momentum that can be transferred to matter in the process of light scattering

- Milne (1924, 1926) and Johnson (1925, 1926) - material can be ejected from the star by the absorption and scattering of the radiation

- Milne (1926) - Doppler effect is important for the line radiative acceleration. The force acting on selected ions due to absorption of photons can exceed gravity and ions then can leave the surface of the star

- Modern studies of hot stars' winds were stimulated mainly by UV observations

- Pioneering works of Lucy & Solomon (1970) and Castor, Abbott, & Klein (1975, CAK) serve as a basis for present hot star wind theory
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Principle of radiative line-driving

Hot star winds are accelerated via a two-step process:

1. The photons are scattered in lines of ions of heavier elements (e.g., C, N, O, Ne, Si, P, S, Ni, Fe-group elements etc.)
   - physical process: momentum and energy transfer by absorption and scattering

2. The outward accelerated ions transfer their momenta to the bulk plasma of the wind (hydrogen and helium - mostly passive component)
   - physical process: Coulomb collisions
Principle of radiative line-driving

The light scattering in lines of heavier elements
- Photons transfer (part of) their momentum to heavier ions and electrons by line scattering
  - photon is absorbed by an ion

from homepage of Joachim Puls
The light scattering in lines of heavier elements

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  - energy of the photon is “transformed” into excitation energy (photon is destroyed)

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The principle of radiatively driven winds

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  - resulting net-acceleration of the ion due to absorption and emission is the vector-sum of both accelerations

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  - resulting net-acCELERATION of the ion due to absorption and emission is the vector-sum of both accelerations
  - only the outward directed acceleration due to absorption processes survives

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Principle of radiative line-driving

- The light scattering in lines of heavier elements

  - momentum of an ion after absorption of photon
    
    \[ mv'_r = mv_r + \frac{hv}{c} \]

  - increase of velocity
    
    \[ \Delta v_r = \frac{hv}{c} \]

  - momentum of an ion after emission of photon
    
    \[ mv''_r = mv'_r - \frac{hv'}{c} \cos \lambda \]
The light scattering in lines of heavier elements

- frequency of absorbed photon in observer frame
  \[ \nu = \nu_0 \left( 1 + \frac{v_r}{c} \right) \]

- frequency of emitted photon in observer frame
  \[ \nu' = \nu_0 \left( 1 + \frac{v'_r}{c} \right) \]
Principle of radiative line-driving

The light scattering in lines of heavier elements

- velocity of the ion after absorption and re-emission

\[ v''_r = v_r + \frac{h\nu_0}{mc} (1 + \frac{v_r}{c}) - \frac{h\nu_0}{mc} (1 + \frac{v'_r}{c}) \cos \lambda \]

- for \( v \ll c \) and \( h\nu_0 \ll c \)

\[ \Delta v_r = v''_r - v_r = \frac{h\nu_0}{mc} (1 - \cos \lambda) \]

- forward scattering (\( \cos \lambda = 1 \)) \Rightarrow the momentum does not increase

- backward scattering (\( \cos \lambda = -1 \)) \Rightarrow the momentum increases by \( 2h\nu_0/c \)

- re-emission of photons is in random direction
The light scattering in lines of heavier elements

- velocity of the ion after absorption and re-emission

\[ v''_r = v_r + \frac{h\nu_0}{mc} \left( 1 + \frac{v_r}{c} \right) - \frac{h\nu_0}{mc} \left( 1 + \frac{v'_r}{c} \right) \cos \lambda \]

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\[ \Delta v_r = v''_r - v_r = \frac{h\nu_0}{mc} \left( 1 - \cos \lambda \right) \]

- the mean transfer of momentum

\[ \langle m\Delta v \rangle = \frac{h\nu_0}{c} \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} (1 - \cos \lambda) 2\pi \sin \lambda \, d\lambda = \frac{h\nu_0}{c} \]
The light scattering in lines of heavier elements
- line scatterings are of bound-bound type, i.e., line transitions
- the wind acceleration is due to RADIATIVE LINE DRIVING
Principle of radiative line-driving

1. The light scattering in lines of heavier elements
2. Momentum transfer by Coulomb coupling
   - the outward accelerated ions transfer their momenta to the bulk plasma of the wind (basically H and He) via Coulomb collisions

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Principle of radiative line-driving

1. The light scattering in lines of heavier elements
2. Momentum transfer by Coulomb coupling
   - the outward accelerated ions transfer their momenta to the bulk plasma of the wind (basically H and He) via Coulomb collisions
   - the total wind is accelerated outward

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Principle of radiative line-driving

1. The light scattering in lines of heavier elements
2. Momentum transfer by Coulomb coupling
   - Condition for the Coulomb coupling to be efficient
     \[ t_s < t_d \]
     - \( t_s \) [s] - characteristic time for slowing down heavier ions by collisions
     - \( t_d \) [s] - time takes the heavier ions to gain a large drift velocity with respect to H and He
     - first shown by Lucy and Solomon (1970) and improved by Lamers and Morton (1976)

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Principle of radiative line-driving

1. The light scattering in lines of heavier elements
2. Momentum transfer by Coulomb coupling
   - Condition for the Coulomb coupling to be efficient
     \[ t_s = 0.305 \frac{A}{Z^2} \frac{T_e^{3/2}}{n_e(1 - 0.022 \ln n_e)} \]
   - \( A \) - mass of charged particles (in units of \( m_H \))
   - \( Z \) - charge (in units of the electron charge) due to interaction with \( H^+, He^{++} \) and electrons
   - \( n_e \) - the electron density
   - for winds with \( 10^8 \leq n_e \leq 10^{12} \Rightarrow (1 - 0.022 \ln n_e) \approx 0.5 \)
Principle of radiative line-driving

1. The light scattering in lines of heavier elements
2. Momentum transfer by Coulomb coupling
   - Condition for efficient Coulomb coupling
     \[ t_d = \frac{v_{th}}{g_i} \]
     \[ v_{th} = \sqrt{\frac{2k_BT_e}{m_HA_f}} \]
   - \( A_f \) - atomic mass for field particles (\( A_f \approx 1 \) for protons)
   - \( g_i \) - acceleration of the absorbing ions
   - \( T_e \) - temperature of the wind

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Principle of radiative line-driving

1. The light scattering in lines of heavier elements
2. Momentum transfer by Coulomb coupling
   - Momentum transfer from photons to ions

\[
\frac{d(mv)}{dt} = Am_H g_i = \frac{\pi e^2}{m_e} f \frac{F_{v_0}}{c}
\]

\[
F_{v_0} = F_{v_0}^* \left(\frac{R_*}{r}\right)^2
\]

- \((\pi e^2/m_e c)f\) - cross section for absorption
- \(F_{v_0}\) - flux at distance \(r\) from the star at the frequency of the line \(v_0\)
- \(F_{v_0}^* = L_{v_0}^*/4\pi R_*^2\) - flux at surface of the star

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The principle of radiative line-driving

1. The light scattering in lines of heavier elements

2. Momentum transfer by Coulomb coupling
   - Condition for efficient Coulomb coupling
     \[
     \frac{L_v T_e}{4\pi r^2 n_e} < \frac{Z^2 c}{0.61} \sqrt{2k_B m_H \left( \frac{\pi e^2}{m_e c f} \right)} A_f^{-1/2} = 3.6 \times 10^{-6}
     \]
     \( A_f = 1, \ f = 0.1, \) and \( Z = 3 \)
     \( T_e \approx 0.5 T_{\text{eff}}, \ L_v T_e = 5.26 \times 10^{-12} L_*; \)
     \( n_e = 5.2 \times 10^{23} \text{gcm}^{-3} \)

   \[
   \frac{L_* v}{M} < 5.9 \times 10^{16}
   \]
   - for hot stars this is satisfied
Principle of radiative line-driving

1. The light scattering in lines of heavier elements
2. Momentum transfer by Coulomb coupling
   - hydrogen and helium are mostly passive components of the wind (inefficient for wind driving)

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   - if the transfer of momentum is inefficient, the wind components may decouple (Springmann and Pauldrach, 1992, Krčíčka and Kubát 2000)
Wind hydrodynamic equations

Single-fluid treatment, neglecting viscosity and forces due to electric and magnetic fields

- equations of the continuity

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
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- equations of motion (momentum)

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \mathbf{g}_{\text{ex}}
\]

- \( \mathbf{v} = \mathbf{v}(r, t) \) - velocity field
- \( \rho = \rho(r, t) \) - mass density
- \( p = p(r, t) \) - gas pressure
- \( g_{\text{ex}} \) - external acceleration; \( g_{\text{ex}} = g_{\text{grav}} + g_{\text{rad}} \)
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- energy equation
  - an approximate solution of the energy equation is allowed (see Klein and Castor, 1978)
  - \( T_e \) is approximately constant with radius and slightly less than \( T_{\text{eff}} \), i.e. isothermal wind
Wind hydrodynamic equations

Assumption: stationary and spherically symmetric wind

- equations of the continuity

\[
\frac{1}{r^2} \frac{d}{dr} (\rho v_r r^2) = 0
\]

- after integration \( \Rightarrow \) total outward mass flux, i.e. \( \dot{M} \)

\[
\dot{M} \equiv \frac{dM_*}{dt} = 4\pi \rho(r) v_r(r) r^2 = \text{const.}
\]
Wind hydrodynamic equations

Assumption: stationary and spherically symmetric wind

- equations of motion (momentum)

\[ v_r \frac{dv_r}{dr} = \frac{1}{\rho} \frac{dp}{dr} - g_{\text{grav}} + g_{\text{rad}} \]

- \( g_{\text{grav}} = \frac{GM_*}{r^2} \) (\( G \) - the gravitational constant)
- the gas pressure \( p \) is given by an ideal gas equation of state

\[ p = \frac{\rho k_B T}{\mu m_H} = \rho a^2 \]

- \( a \) - isothermal speed of sound (const.)
- \( k_B \) - Boltzmann’s constant
- \( m_H \) - the mass of a hydrogen atom
- \( \mu \) - the mean molecular weight of gas particles
Wind hydrodynamic equations

Assumption: stationary and spherically symmetric wind

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Wind hydrodynamic equations

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- equations of motion (momentum)

\[ \rho v_r \frac{dv_r}{dr} = -a^2 \frac{d\rho}{dr} - \frac{\rho G M_*}{r^2} + f_{\text{rad}} \]

- \( f_{\text{grav}} = \frac{\rho G M_*}{r^2} \) - gravitational force
- \( f_{\text{rad}} \) - radiative force
- the gas pressure \( p \) is given by an ideal gas equation of state
- \( a \) - isothermal speed of sound (const.)
Radiative force

- $f_{\text{rad}}$ - force due to a radiation field at a point $r$

\[
f_{\text{rad}}(r) = \frac{1}{c} \int_{0}^{\infty} d\nu \oint_{\Omega=4\pi} (\chi(r, \nu) I(r, \nu, k) - \eta(r, \nu)) k \, d\Omega
\]

- $\chi_\nu$ - absorption coefficient
- $\eta_\nu$ - emission coefficient
- $I_\nu$ - radiative intensity
- $k$ - unit vector of the direction of the radiation propagation

For isotropic emissivity, the integral over all angles vanishes as well as the second term, and $\chi(r, \nu)$ can be factored out of angular integration.
Radiative force

- $f_{\text{rad}}$ - force due to a radiation field at a point $r$

\[ f_{\text{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) \, d\nu \oint_{\Omega=4\pi} I(r, \nu, k) \, k \, d\Omega = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) F(r, \nu) \, d\nu \]

- $\chi_{\nu}$ - absorption coefficient
- $I_{\nu}$ - radiative intensity
- $k$ - unit vector of the direction of the radiation propagation
- $F$ - radiation flux

\[ F(r, \nu) = \oint_{\Omega=4\pi} I(r, \nu, k) \, k \, d\Omega \]
Radiative force

- **Total radiative force**

\[ f_{\text{rad}}(r) = f_{\text{cont}}(r) + f_{\text{line}}^{\text{tot}}(r) \]

- \( f_{\text{cont}}(r) \) - force due to continuum opacity
- \( f_{\text{line}}^{\text{tot}}(r) \) - force due to an ensemble of spectral lines

- **Continuum opacity**

  - continuum processes: atomic free-free and bound-free transitions and scattering on free electrons
  - continuum opacity due to free-free and bound-free processes can be neglected in the winds of O and B type stars
  - scattering of free electrons (Thomson scattering) - the main contributor to the continuum opacity
Radiative force

- Radiative force

\[
    f_{\text{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) \, d\nu \oint_{\Omega=4\pi} I(r, \nu, k) \, k \, d\Omega = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) \mathcal{F}(r, \nu) \, d\nu
\]

- Radiative force due to radiation scattering on free electrons

\[
    f_{\text{cont}}(r) = \frac{\chi_{\text{th}}(r)}{c} \int_{\nu=0}^{\infty} d\nu \oint_{\Omega=4\pi} I^c(r, \nu, k) \, k \, d\Omega = \frac{\chi_{\text{th}}(r)}{c} \int_{0}^{\infty} \mathcal{F}(r, \nu) = \frac{n_e(r) \sigma_{\text{Th}} L}{4\pi r^2 c}
\]

- \( I^c \) is "direct" continuum intensity from the stellar surface
- \( \chi_{\text{th}} \) - the Thomson scattering opacity

\[
    \chi_{\text{th}}(r) = n_e(r) \sigma_{\text{Th}}
\]

- \( \sigma_{\text{Th}} = 6.65 \times 10^{-25} \text{ cm}^2 \) - the cross-section for Thomson scattering
- \( n_e \) - the number density of free electrons
- \( L = 4\pi r^2 \int_{0}^{\infty} \mathcal{F}(r, \nu) \)
Radiative force

- Radiative force

\[ f_{\text{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) \, d\nu \int_{\Omega=4\pi} I(r, \nu, k) \, k \, d\Omega = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) \, F(r, \nu) \, d\nu \]

- Radiative force due to radiation scattering on free electrons

\[ f_{\text{cont}}(r) = \frac{\chi_{\text{th}}(r)}{c} \int_{\nu=0}^{\infty} \int_{\Omega=4\pi} I^c(r, \nu, k) \, k \, d\Omega = \frac{\chi_{\text{th}}(r)}{c} \int_{0}^{\infty} F(r, \nu) = \frac{n_e(r) \sigma_{\text{Th}} L}{4\pi r^2 c} \]

- Ratio between the force due to the light scattering on free electrons and the gravitational force - Eddington factor (luminosity-to-mass ratio)

\[ \Gamma_e = \frac{f_{\text{cont}}}{f_{\text{grav}}} = \frac{\sigma_{\text{Th}} n_e(r) L}{\rho(r) 4\pi c G M} \]

- \( \Gamma_e \rightarrow 1 \) - the Eddington limit
Radiative force

- Radiative force

\[
f_{\text{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) \, d\nu \oint_{\Omega=4\pi} I(r, \nu, k) \, k \, d\Omega = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) \mathcal{F}(r, \nu) \, d\nu
\]

- Radiative force due to radiation scattering on free electrons

\[
f_{\text{cont}}(r) = \frac{\chi_{\text{th}}(r)}{c} \int_{\nu=0}^{\infty} d\nu \oint_{\Omega=4\pi} I^c(r, \nu, k) \, k \, d\Omega = \frac{\chi_{\text{th}}(r)}{c} \int_{0}^{\infty} \mathcal{F}(r, \nu) = \frac{n_e(r) \sigma_{\text{Th}} L}{4\pi r^2 c}
\]

- Comparison with the gravity force

\[
\Gamma_e = \frac{f_{\text{cont}}}{f_{\text{grav}}} = \frac{\sigma_{\text{Th}} n_e(r) \rho(r) L}{4\pi c G M}
\]

\[
\Gamma_e = 10^{-5} \left( \frac{L}{L_\odot} \right) \left( \frac{M}{M_\odot} \right)^{-1}
\]
Radiative force

Radiative force

\[ f_{\text{rad}}(r) = \frac{1}{c} \int_0^\infty \chi(r, \nu) d\nu \oint_{\Omega=4\pi} I(r, \nu, k) k d\Omega = \frac{1}{c} \int_0^\infty \chi(r, \nu) F(r, \nu) d\nu \]

Radiative force due to radiation scattering on free electrons

\[ f_{\text{cont}}(r) = \frac{\chi_{\text{th}}(r)}{c} \int_0^\infty d\nu \oint_{\Omega=4\pi} I^c(r, \nu, k) k d\Omega = \frac{\chi_{\text{th}}(r)}{c} \int_0^\infty F(r, \nu) = \frac{n_e(r) \sigma_{\text{Th}} L}{4\pi r^2 c} \]

Radiative force due to the light scattering on free electrons is important, but it never exceeds the gravity force.
Radiative force

- Radiative force

\[ f_{\text{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) \mathcal{F}(r, \nu) \, d\nu \]

- Radiative force due to line transition

\[ \chi(r, \nu) = \frac{\pi e^2}{m_e c} \sum_{\text{lines}} g_i f_{ij} \left( \frac{n_i(r)}{g_i} - \frac{n_j(r)}{g_j} \right) \phi_{ij}(\nu) \]

- \( \phi_{ij}(\nu) \) - line profile; \( \int_{0}^{\infty} \phi_{ij}(\nu) \, d\nu = 1 \)
- \( f_{ij} \) - oscillator strength
- \( n_i(r), n_j(r) \) level occupation number
- \( g_i \) - statistical weight of the level
Radiative force

- Radiative force
  \[ f_{\text{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) \mathcal{F}(r, \nu) \, d\nu \]

- Radiative force due to line transition
  \[ \chi(r, \nu) = \frac{\pi e^2}{m_e c} \sum_{\text{lines}} g_i f_{ij} \left( \frac{n_i(r)}{g_i} - \frac{n_j(r)}{g_j} \right) \phi_{ij}(\nu) \]
  \[ f_{\text{line}}(r) = \frac{\pi e^2}{m_e c^2} \sum_{\text{lines}} \int_{0}^{\infty} g_i f_{ij} \left( \frac{n_i(r)}{g_i} - \frac{n_j(r)}{g_j} \right) \phi_{ij}(\nu) \mathcal{F}(r, \nu) \, d\nu \]

- lines influence on \( \mathcal{F}(r, \nu) \)
- assumption: \( \mathcal{F}(r, \nu) \) constant for frequencies corresponding to a given line, \( \nu \approx \nu_{i,j} \)
Radiative force

\[ f_{\text{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) F(r, \nu) \, d\nu \]

Radiative force due to line transition

maximum force

\[ f_{\text{line}}^{\text{max}}(r) = \frac{\pi e^2}{m_e c^2} \sum_{\text{lines}} g_i f_{ij} \left( \frac{n_i(r)}{g_i} - \frac{n_j(r)}{g_j} \right) F(r, \nu_{i,j}) \]

- \( \nu_{i,j} \) - the line center frequency
- neglect of \( n_j(r) \ll n_i(r) \)
- \( L_{\nu_{i,j}} = 4\pi r^2 F(r, \nu_{i,j}) \)
Radiative force

\[ f_{\text{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) \mathcal{F}(r, \nu) \, d\nu \]

Radiative force due to line transition

- maximum force: comparison with gravity

\[ \frac{f_{\text{line}}^{\text{max}}(r)}{f_{\text{grav}}(r)} = \frac{L e^2}{4 m_e \rho G M c^2} \sum_{\text{lines}} f_{ij} n_i(r) \frac{L_{\nu_{i,j}}}{L} \]

- \( \nu_{i,j} \) - the line center frequency
- neglect of \( n_j(r) \ll n_i(r) \)
- \( L_{\nu_{i,j}} = 4 \pi r^2 \mathcal{F}(r, \nu_{i,j}) \)
Radiative force

\[ f_{\text{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) \mathcal{F}(r, \nu) \, d\nu \]

Radiative force due to line transition

- maximum force: comparison with gravity

\[ \frac{f_{\text{line}}^{\text{max}}(r)}{f_{\text{grav}}(r)} = \Gamma \sum_{\text{lines}} \frac{\sigma_{ij}}{\sigma_{\text{Th}}} \frac{n_i(r)}{n_e(r)} \frac{\nu_{i,j} L_{\nu}(\nu_{i,j})}{L} \]

\[ \sigma_{ij} = \frac{\pi e^2}{\nu_{i,j} m_e c} \]

- hydrogen: mostly ionised in the stellar envelopes \( \Rightarrow n_i(r)/n_e(r) \) very small \( \Rightarrow \) negligible contribution to radiative force
- neutral helium: \( n_i(r)/n_e(r) \) very small \( \Rightarrow \) negligible contribution to radiative force
- ionised helium: very small contribution to the radiative force
Radiative force

- Radiative force

\[ f_{\text{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) F(r, \nu) \, d\nu \]

- Radiative force due to line transition
  - maximum force: comparison with gravity

\[ \frac{f_{\text{line}}^{\text{max}}(r)}{f_{\text{grav}}(r)} = \Gamma \sum_{\text{lines}} \frac{\sigma_{ij}}{\sigma_{\text{Th}}} \frac{n_i(r)}{n_e(r)} \frac{\nu_{i,j}}{L} \frac{L_{\nu}(\nu_{i,j})}{L} \]

\[ \sigma_{ij} = \frac{\pi e^2 f_{ij}}{\nu_{i,j} m_e c} \]

- heavier elements (Fe, C, N, O, . . .): large number of lines, \( \sigma_{ij}/\sigma_{\text{Th}} \approx 10^7 \Rightarrow f_{\text{line}}^{\text{max}} / f_{\text{grav}} \) up to \( 10^3 \)
- radiative force may be larger than gravity (for many O stars \( f_{\text{line}}^{\text{max}} / f_{\text{grav}} \approx 2000 \), Abbott 1982, Gayley 1995) \( \Rightarrow \) stellar wind
Radiative force

- Radiative force
  \[
  f_{rad}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r, \nu) F(r, \nu) \, d\nu
  \]

- Radiative force due to line transition
  \[
  \chi(r, \nu) = \frac{\pi e^2}{m_e c} \sum_{\text{lines}} g_i f_{ij} \left( \frac{n_i(r)}{g_i} - \frac{n_j(r)}{g_j} \right) \phi_{ij}(\nu)
  \]
  \[
  f_{\text{line}}(r) = \frac{\pi e^2}{m_e c^2} \sum_{\text{lines}} \int_{0}^{\infty} g_i f_{ij} \left( \frac{n_i(r)}{g_i} - \frac{n_j(r)}{g_j} \right) \phi_{ij}(\nu) F(r, \nu) \, d\nu
  \]

- the main problem: the line opacity (lines may be optically thick) ⇒
- necessary to solve the radiative transfer equation
Sobolev approximation

Sobolev (1947) developed approach for treating line scattering in a rapidly accelerating flow

This approximation is valid only if the velocity gradient is sufficiently large

Due to the Doppler shift, the geometrical size in which a line can absorb photons with the fixed frequency is so small that $\chi_L$ and $\rho$ change very little

The profile function can be approximated with a $\delta$-function that is sharply peaked around the central line frequency

“Sobolev length”

$$L_S \equiv \frac{v_{th}}{d\nu/dr} \ll H \equiv \frac{\rho}{d\rho/dr} \approx \frac{v}{d\nu/dr}$$

- $H$ - a typical flow variation scale
- $\rho/(d\rho/dr)$ and $v/(d\nu/dr)$ - the density and velocity scale length
- simplification of the calculation of $f_{line}$ possible
The radiative transfer equation

Assumptions: spherical symmetry, stationary (time-independent) flow

\[ \mu \frac{\partial}{\partial r} I(r, \mu, \nu) + \frac{1 - \mu^2}{r} \frac{\partial}{\partial \mu} I(r, \mu, \nu) = \eta(r, \mu, \nu) - \chi(r, \mu, \nu) I(r, \mu, \nu) \]

- frame of static observer
- \( \mu = \cos \theta \)
- \( I(r, \mu, \nu) \) - specific intensity
- \( \chi(r, \mu, \nu) \) - absorption (extinction) coefficient
- \( \eta(r, \mu, \nu) \) - emissivity (emission coefficient)
- problem: \( \chi(r, \mu, \nu) \) and \( \eta(r, \mu, \nu) \) depend on \( \mu \) due to the Doppler effect
- solution: use comoving-frame (CMF)
CMF radiative transfer equation

Assumptions: spherical symmetry, stationary (time-independent) flow

\[ \mu \frac{\partial}{\partial r} I(r, \mu, \nu) + \frac{1 - \mu^2}{r} \frac{\partial}{\partial \mu} I(r, \mu, \nu) - \frac{\nu v(r)}{c r} \left( 1 - \mu^2 + \frac{\mu^2 r}{v(r)} \frac{dv(r)}{dr} \right) \frac{\partial}{\partial \nu} I(r, \mu, \nu) = \eta(r, \nu) - \chi(r, \nu) I(r, \mu, \nu) \]

- \( \chi(r, \mu, \nu) \) and \( \eta(r, \mu, \nu) \) do not depend on \( \mu \)
- neglected aberration, advection (unimportant for \( v \ll c \))
CMF radiative transfer equation

The Sobolev transfer equation (Castor 2004)

\[
\mu \frac{\partial}{\partial r} I(r, \mu, \nu) + \frac{1 - \mu^2}{r} \frac{\partial}{\partial \mu} I(r, \mu, \nu) - \frac{v \nu(r)}{cr} \left(1 - \mu^2 + \frac{\mu^2 r}{v(r)} \frac{dv(r)}{dr}\right) \frac{\partial}{\partial \nu} I(r, \mu, \nu) = \eta(r, \nu) - \chi(r, \nu) I(r, \mu, \nu)
\]

possible when

\[
\frac{v \nu(r)}{cr} \frac{\partial}{\partial \nu} I(r, \mu, \nu) \gg \frac{\partial}{\partial r} I(r, \mu, \nu)
\]
CMF radiative transfer equation

Solution of the transfer equation for one line

\[-\frac{\nu \nu(r)}{c r} \left( 1 - \mu^2 + \frac{\mu^2 r}{v(r)} \frac{dv(r)}{dr} \right) \frac{\partial}{\partial \nu} I(r, \mu, \nu) = \eta(r, \nu) - \chi(r, \nu) I(r, \mu, \nu)\]

- line absorption and emission coefficients

\[\chi(r, \nu) = \frac{\pi e^2}{m_e c} g_i f_{ij} \left( \frac{n_i(r)}{g_i} - \frac{n_j(r)}{g_j} \right) \phi_{ij}(\nu) = \chi_L(r) \phi_{ij}(\nu)\]

\[\eta(r, \nu) = \frac{2\hbar \nu^3}{c^2} \frac{\pi e^2}{m_e c} g_i f_{ij} \frac{n_j(r)}{g_j} \phi_{ij}(\nu) = \chi_L(r) S_L(r) \phi_{ij}(\nu)\]

\[\chi_L(r) = \frac{\pi e^2}{m_e c} g_i f_{ij} \left( \frac{n_i(r)}{g_i} - \frac{n_j(r)}{g_j} \right)\]
CMF radiative transfer equation

Solution of the transfer equation for one line

\[- \frac{\nu v(r)}{c r} \left(1 - \mu^2 + \frac{\mu^2 r}{v(r)} \frac{dv(r)}{dr}\right) \frac{\partial}{\partial \nu} I(r, \mu, \nu) = \chi_L(r) \phi_{ij}(\nu)(S_L(r) - I(r, \mu, \nu))\]

- introduce a new variable
  \[y = \int_{\nu}^{\infty} \phi_{ij}(\nu') d\nu'\]

- where
  - \(y = 0\): the incoming side of the line
  - \(y = 1\): the outgoing side of the line
CMF radiative transfer equation

Solution of the transfer equation for one line

\[-\frac{\nu v(r)}{c r} \left( 1 - \mu^2 + \frac{\mu^2 r}{v(r)} \frac{dv(r)}{dr} \right) \frac{\partial}{\partial y} I(r, \mu, y) = \chi_L(r) \phi_{ij}(\nu) (S_L(r) - I(r, \mu, y))\]

assumptions:
- variables do not significantly vary with \( r \) within the “resonance zone” ⇒
- fixed \( r \), \( \frac{\partial}{\partial y} \rightarrow \frac{d}{dy} \)
- \( \nu \rightarrow \nu_0 \)

integration possible
CMF radiative transfer equation

Solution of the transfer equation for one line

\[ I(y) = I_c(\mu) e^{-\tau(\mu)y} + S_L 1 - e^{-\tau(\mu)y} \]

- the Sobolev optical depth in spherical symmetry

\[ \tau(\mu) = \frac{\chi_L(r)cr}{\nu_0 v(r) \left( 1 - \mu^2 + \frac{\mu^2 r}{v(r)} \frac{dv(r)}{dr} \right)} \]

- the boundary condition is \( I(y = 0) = I_c(\mu) \)
- \( \tau \) is given by the slope \( \Rightarrow \tau \sim \left( \frac{dv}{dr} \right)^{-1} \)
Radiative force

the radial component; force per unit of volume

\[ f_{\text{rad}}(r) = \frac{1}{c} \int_0^\infty \chi(r, \nu) F(r, \nu) \, d\nu \]

\[ f_{\text{rad}}(r) = \frac{1}{c} \int_0^\infty \chi(r, \nu) \, d\nu \iint I(r, \nu, \mathbf{k}) \mathbf{k} \, d\Omega \]

\[ f_{\text{rad}}(r) = \frac{2\pi}{c} \int_0^\infty \chi_L(r) \phi_{ij}(\nu) \, d\nu \int_{-1}^1 \mu I(r, \mu, \nu) \, d\mu \]

\[ f_{\text{rad}}(r) = \frac{2\pi \chi_L(r)}{c} \int_0^1 \int_{-1}^1 \mu I(r, \mu, \nu) \, d\mu \, dy \]
Radiative force

the radial component; force per unit of volume

\[ f_{\text{rad}}(r) = \frac{2\pi \chi L(r)}{c} \int_0^1 dy \int_{-1}^1 \left[ I_c(\mu) e^{-\tau(\mu)y} + S_L \left(1 - e^{-\tau(\mu)y}\right)\right] \mu \, d\mu \]

where the Sobolev optical depth is

\[ \tau(\mu) = \frac{\chi L(r) cr}{\nu_0 v(r) \left(1 - \mu^2 + \frac{\mu^2 r}{v(r)} \frac{dv(r)}{dr}\right)} \]

no net contribution of the emission to the radiative force \((S_L \text{ is isotropic in the CMF})\)

\[ f_{\text{rad}}(r) = \frac{2\pi \chi L(r)}{c} \int_0^1 dy \int_{-1}^1 \mu I_c(\mu) e^{-\tau(\mu)y} \, d\mu \]
Radiative force

the radial component; force per unit of volume

\[ f_{\text{rad}}(r) = \frac{2\pi \chi L(r)}{c} \int_{-1}^{1} \mu I_c(\mu) \frac{1 - e^{-\tau(\mu)y}}{\tau(\mu)} d\mu \]

inserting

\[ \tau(\mu) = \frac{\chi L(r)c r}{v_0 v(r) \left(1 - \mu^2 + \frac{\mu^2 r \frac{dv(r)}{dr}}{v(r)}\right)} \]

\[ f_{\text{rad}}(r) = \frac{2\pi v_0 v(r)}{r c^2} \int_{-1}^{1} \mu I_c(\mu) \left[1 + \mu^2 \sigma(r)\right] \left\{1 - \exp \left[-\frac{\chi L(r)c r}{v_0 v(r) \left(1 + \mu^2 \sigma(r) \right)} \right]\right\} d\mu \]

\[ \sigma(r) = \frac{r}{v(r)} \frac{dv(r)}{dr} - 1 \]

Sobolev (1957), Castor (1974), Rybicki & Hummer (1978)
Radiative force

- Optically thin line

\[
f_{\text{rad}}(r) = \frac{2\pi \nu_0 v(r)}{rc^2} \int_{-1}^{1} \mu I_c(\mu) \left[ 1 + \mu^2 \sigma(r) \right] \left\{ 1 - \exp \left[ -\frac{\chi_L(r) cr}{\nu_0 v(r)(1 + \mu^2 \sigma(r))} \right] \right\} d\mu
\]

- Optically thin line

\[
\frac{\chi_L(r) cr}{\nu_0 v(r)(1 + \mu^2 \sigma(r))} \ll 1
\]

\[
f_{\text{rad}}(r) \sim 1 - \exp \left[ -\frac{\chi_L(r) cr}{\nu_0 v(r)(1 + \mu^2 \sigma(r))} \right] \approx \frac{\chi_L(r) cr}{\nu_0 v(r)(1 + \mu^2 \sigma(r))}
\]

\[
f_{\text{rad}}(r) = \frac{2\pi}{c} \int_{-1}^{1} \mu I_c(\mu) \chi_L(r) d\mu
\]
Radiative force

- Optically thin line

\[
f_{\text{rad}}(r) = \frac{2\pi v_0 v(r)}{rc^2} \int_{-1}^{1} \mu I_c(\mu) \left[ 1 + \mu^2 \sigma(r) \right] \left\{ 1 - \exp \left[ -\frac{\chi_L(r) cr}{v_0 v(r)(1 + \mu^2 \sigma(r))} \right] \right\} d\mu
\]

\[
f_{\text{rad}}(r) = \frac{2\pi}{c} \int_{-1}^{1} \mu I_c(\mu) \chi_L(r) d\mu
\]

\[
f_{\text{rad}}(r) = \frac{1}{c} \chi_L(r) F(r)
\]

- optically thin radiative force proportional to the radiative flux \(F(r)\)
- optically thin radiative force proportional to the normalised line opacity \(\chi_L(r)\) (or to the density)
- the same result as for the static medium
Radiative force

- Optically thick line

\[
f_{\text{rad}}(r) = \frac{2\pi \nu_0 v(r)}{r c^2} \int_{-1}^{1} \mu I_c(\mu) \left[ 1 + \mu^2 \sigma(r) \right] \left\{ 1 - \exp \left[ -\frac{\chi_L(r) cr}{\nu_0 v(r) (1 + \mu^2 \sigma(r))} \right] \right\} d\mu
\]

\[
\frac{\chi_L(r) cr}{\nu_0 v(r) (1 + \mu^2 \sigma(r))} \gg 1
\]

\[
f_{\text{rad}}(r) \sim 1 - \exp \left[ -\frac{\chi_L(r) cr}{\nu_0 v(r) (1 + \mu^2 \sigma(r))} \right] \approx 1
\]

\[
f_{\text{rad}}(r) = \frac{2\pi \nu_0 v(r)}{r c^2} \int_{-1}^{1} \mu I_c(\mu) \left[ 1 + \mu^2 \sigma(r) \right] d\mu
\]
Radiative force

- **Optically thick line**

\[ f_{\text{rad}}(r) = \frac{2\pi \nu_0 v(r)}{rc^2} \int_{-1}^{1} \mu I_c(\mu) \left[ 1 + \mu^2 \sigma(r) \right] d\mu \]

- **neglect of the limb darkening:**
  \[ \mu_* = \sqrt{1 - \frac{R_*}{r^2}} \]

\[ I_c(\mu) = \begin{cases} 
  I_c = \text{const.} & \mu \geq \mu_*, \\
  0, & \mu < \mu_* 
\end{cases} \]

- \( \mathcal{F} = 2\pi \int_{\mu_*}^{1} \mu I_c \, d\mu = \pi \frac{R_*}{r^2} I_c \)

\[ f_{\text{rad}}(r) = \frac{\nu_0 v(r) \mathcal{F}(r)}{rc^2} \left[ 1 + \sigma(r) \left(1 - \frac{1}{2} \frac{R_*}{r^2}\right) \right] \]
Radiative force

- Optically thick line

\[
 f_{\text{rad}}(r) = \frac{\nu_0 v(r) \mathcal{F}(r)}{r c^2} \left[ 1 + \sigma(r) \left( 1 - \frac{1}{2} \frac{R_*}{r^2} \right) \right]
\]

\[
 \sigma(r) = \frac{r}{v(r)} \frac{dv(r)}{dr} - 1
\]

- Large distance from the star: \( r \gg R_* \)

\[
 f_{\text{rad}}(r) = \frac{\nu_0 \mathcal{F}(r)}{c^2} \frac{dv(r)}{dr}
\]

- Optically thick radiative force proportional to the radiative flux \( \mathcal{F}(r) \)
- Optically thick radiative force proportional to \( dv(r)/dr \)
- Optically thick radiative force does not depend on the level populations (opacity) or the density