Plasma as fluid

→ starting from Boltzmann (Vlasov) equation, introducing „ad-hoc" force terms

→ separately for electrons and ions

$\alpha = \{e, i\}$ ... species

Double-fluid model

→ electron (e) and ion (i) fluids „separated"

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha u_\alpha) = 0$$

$$m_\alpha n_\alpha \left[\frac{\partial \vec{u}_\alpha}{\partial t} + (\vec{u}_\alpha \cdot \nabla) \vec{u}_\alpha \right] = q_\alpha n_\alpha (\vec{E} + \vec{u}_\alpha \times \vec{B}) - \nabla p_\alpha + \text{other forces (gravit...)}$$

$$p_\alpha = C_\alpha (m_\alpha n_\alpha)^{\gamma_\alpha} \quad \dots \text{ E.O.S.}$$

γ_α ... ratio of specific heats

\+ Maxwell

$$\varepsilon_0 \nabla \cdot \vec{E} = \sum_\alpha n_\alpha q_\alpha$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

coupling here, so fluids are not fact independent

$$\nabla \cdot B = 0$$

$$\frac{1}{\mu} \nabla \times \vec{B} = \sum_\alpha n_\alpha q_\alpha u_\alpha + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

→ sum: equations $2 \cdot (1 + 3 + 1) + 1 + 3 + 1 + 3 = 18$

variables 16

but! $0 = \nabla \cdot (\nabla \times E) = -\nabla \cdot \frac{\partial B}{\partial t} = -\frac{\partial}{\partial t} \nabla \cdot B$

$\Rightarrow \nabla \cdot B = \text{const}$

similarly for $\nabla \times B$ ⇒ scalar Maxwell are not independent („initial conditions")

⇒ only 16 independent eqs.

Note: <u>plasma approximation</u>

on large scales quasi-neutrality

$\Rightarrow n_e = n_i$; do not use $\nabla \cdot E = \sum n_\alpha q_\alpha$

Highly ionised plasma

capturing particle interactions on scales smaller than Debye: „collision" → need to parameterize

eq. $$m_e n_e \frac{d\vec{u}_e}{dt} = -en_e(\vec{E} + \vec{u}_e \times \vec{B}) - \nabla p_e + \vec{P}_{ei}$$

momentum exchange between e and i

momentum conserved = symmetry

$$\vec{P}_{ei} = -\vec{P}_{ie}$$

heuristics

$$P_{ei} = \eta \, e^2 \, n_e \, n_i (\vec{u}_i - \vec{u}_e) \sim \eta e^2 n^2 (\vec{u}_i - \vec{u}_e)$$

efficiency of collisions ⇒ specific resistivity

Coulomb interaction ⇒ charge^2

relative speeds

how often do e^- meet i^+

proportional to electron flux

alternatively

$$P_{ei} = \underbrace{\underbrace{m_e n (\vec{u}_i - \vec{u}_e)}_{\text{momentum}} \nu_{ei}}_{\text{momentum per time of collisions}}$$

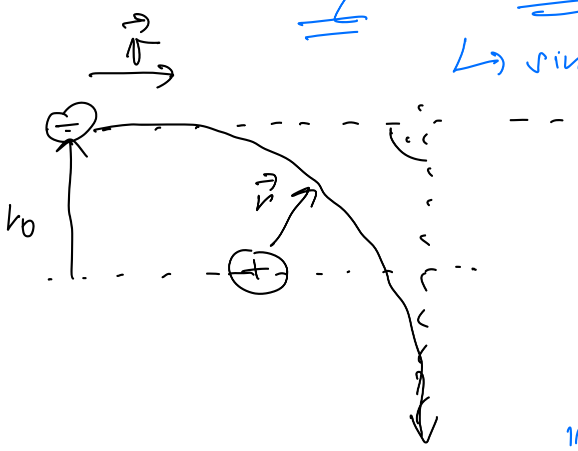
collisional frequency $\sim \frac{1}{\tau_{ei}}$

compare: $$\nu_{ei} = \frac{ne^2}{m_e}\eta = \omega_p^2 \varepsilon_0 \eta$$

$$\omega_p^2 = \frac{ne^2}{m_e \varepsilon_0}$$ plasma frequency

η is a key

↳ simplistic model



Coulomb: $$F = -\frac{e^2}{4\pi\varepsilon_0 r^2}$$

interaction at the closest point takes $\Delta t \sim \frac{r_0}{v}$ ... estimate

„interaction is important only at close distances"

⇒ momentum exchange: $$\Delta(m_e v) \sim |F \Delta t| = \frac{e^2}{4\pi\varepsilon_0 r_0 v}$$

assumption: deflection by $90°$ $\Rightarrow$ "all momentum lost"

$$\Rightarrow \Delta(m_e v) = m_e v = \frac{e^2}{4\pi\varepsilon_0 r_0 v}$$

looking for:

$$r_{90} = \frac{e^2}{4\pi\varepsilon_0 m_e v^2}$$

effective cross-section: $\sigma = \pi r_{90}^2 = \dfrac{e^4}{16\pi\varepsilon_0^2 m_e^2 v^4}$

an collisional frequency: $\nu_{ei} = \sigma n v = \dfrac{ne^4}{16\pi\varepsilon_0^2 m_e^2 v^3}$

now relate electron velocity to temperature, assuming the equilibrium

$m_e v_e^2 \sim k_B T_e$ and previosly: $\nu_{ei} = \dfrac{ne^2}{m_e}\eta$

hence $\eta = \dfrac{\nu_{ei} m_e}{ne^2} = \dfrac{e^2}{16\pi\varepsilon_0^2 m_e v^3} = \dfrac{e^2 m_e^{1/2}}{16\pi\varepsilon_0^2 (k_B T_e)^{3/2}}$

So far - only for a specific $r_0 = r_{90}$

we should account for all, integrating over $\int_0^\infty \frac{1}{r} dr$, but it diverges:

So limits: lower $\to$ perpendicular deflection
upper $\to$ Debye length

$$\Rightarrow \int_{r_{90}}^{\lambda_D} \frac{1}{r} dr = [\ln r]_{r_{90}}^{\lambda_D} = \ln\frac{\lambda_D}{r_{90}} \equiv \underbrace{\ln\Lambda}_{\text{Coulombic logarithm}}$$

Hence $\eta = \dfrac{e^2 m_e^{1/2}}{16\pi\varepsilon_0^2 (k_B T_e)^{3/2}} \ln\Lambda$

! $\eta \neq \eta(n_e)$
$\eta \propto T^{-3/2}$ $\Rightarrow$ $T\nearrow, \eta\searrow$ $\to$ Joule heating not efficient in plasma

for $k_B T_e \sim 100\,eV$ ... $\eta = 5\times10^{-7}\,\Omega m$ $\to$ like copper

Note: $P_{ei} = \eta e^2 n^2 (\vec{w}_i - \vec{w}_e) = e n_e E$; $E = \eta e n (w_i - u_e) = \eta \vec{j}$
Ohm's law

Single fluid model

→ we cannot distinguish between electrons and ions, so a mixture is considered

→ combined variables are closer to be measured

Definition:

plasma approximation

bulk density: $\rho \equiv n_i M + n_e m \sim n(m+M) \sim nM$

usually $m \ll M$

bulk velocity: $\vec{w} \equiv \frac{1}{\rho}(n_i M \vec{u_i} + n_e m \vec{u_e})$

$$\sim \frac{n(M u_i + m u_e)}{n(M+m)} = \frac{M\vec{u_i} + m\vec{u_e}}{M+m}$$

plasma approx.

single-charged ions

bulk current density: $\vec{j} \equiv e(n_i \vec{u_i} - n_e \vec{u_e}) \sim ne(\vec{w_i} - \vec{w_e})$

bulk pressure: $p = p_i + p_e$

bulk charge density: $\rho_e = n_i q_i + n_e q_e = e(n_i - n_e)$

! new variables are linear combinations of the old ones: $(n_e, n_i) \to (\rho, \rho_e)$; $(\vec{u_e}, \vec{u_i}) \to (\vec{w}, \vec{j})$

one exception: pressure

Linear combinations of fluid equations

$$\left.\begin{aligned} &\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i \vec{w_i}) = 0 \quad \Big|\cdot M \\ &\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e u_e) = 0 \quad \Big|\cdot m \end{aligned}\right\} +$$

$$\frac{\partial}{\partial t}(M n_i + m n_e) + \nabla\cdot\left[M n_i \vec{u_i} + m n_e \vec{u_e}\right] = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho w) = 0$$

$$\left.\begin{aligned} &\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i \vec{u_i}) = 0 \quad \Big|\cdot e \\ &\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e \vec{u_e}) = 0 \quad \Big|\cdot e \end{aligned}\right\} -$$

$$\frac{\partial}{\partial t}(e n_i - e n_e) + \nabla\cdot(e n_i u_i - e n_e u_e) = 0$$

$$\frac{\partial \rho_e}{\partial t} + \nabla\cdot j = 0$$

$$Mn \frac{\partial u_i}{\partial t} = en(E + u_i \times B) - \nabla p_i + Mng + P_{ie}$$

$$mn \frac{\partial u_e}{\partial t} = -en(E + u_e \times B) - \nabla p_e + mng + P_{ei}$$

} +

convective derivative neglected

"ad-hoc" gravity

collisions, but $P_{ie} = -P_{ei}$

$$\underbrace{n(M+m)}_{\rho} \frac{\partial}{\partial t}\underbrace{\left(\frac{Mu_i + mu_e}{M+m}\right)}_{\vec{w}} = \underbrace{en(\vec{u}_i - \vec{u}_e)}_{\vec{j}} \times B - \nabla p + \underbrace{n(M+m)}_{\rho}\vec{g}$$

$$\rho \frac{\partial \vec{w}}{\partial t} = \vec{j} \times \vec{B} - \nabla p + \rho \vec{g}$$

(generalised to $\rho\left(\frac{\partial w}{\partial t} + (w \cdot \nabla) w\right) = \dots$)

note: $\vec{j} \not\parallel \vec{w}$ → do not have to be parallel

different combination → multiple by mass of the other species:

$$mMn \frac{\partial u_i}{\partial t} = emn(E + u_i \times B) - m\nabla p_i + mMng + mP_{ie}$$

$$mMn \frac{\partial u_e}{\partial t} = -eMn(E + u_e \times B) - M\nabla p_e + mMng + MP_{ei}$$

} −

$$nmM \frac{\partial}{\partial t}(u_i - u_e) = en(M+m)E + en(mu_i + Mu_e) \times B - m\nabla p_i + M\nabla p_e - (M+m)P_{ei} \quad \Big| : en(M+m) = e\rho$$

$$mu_i + Mu_e = Mu_i + mu_e + M(u_e - u_i) + m\underbrace{(u_i - u_e)}_{} =$$

add and remove mu_e and Mu_i $\qquad u_i - u_e = \frac{j}{ne}$

$$= \frac{\rho}{n}\vec{w} - (M-m)\frac{\vec{j}}{ne}$$

also $\vec{P}_{ei} = \eta e^2 n^2 (\vec{u}_i - \vec{u}_e) = \eta e n \vec{j}$

$$\vec{E} + \vec{w} \times \vec{B} - \eta \vec{j} = \frac{1}{e\rho}\left[\frac{Mmn}{e}\frac{\partial}{\partial t}\frac{\vec{j}}{n} + (M-m)\vec{j} \times \vec{B} + m\nabla p_i - M\nabla p_e\right]$$

→ take $m \ll M$

$$\rightarrow \frac{Mmn}{e}\frac{\partial}{\partial t}\frac{j}{n} = MnB\frac{m}{eB}\frac{\partial}{\partial t}\frac{j}{n} = MnB\omega_c^{-1}\frac{\partial}{\partial t}\frac{j}{n}$$

if $\omega_c \gg \left(\frac{\partial}{\partial t}\frac{j}{n}\right)^{-1}$

then $MnB\omega_c^{-1}\frac{\partial}{\partial t}\frac{j}{n} \ll (M-m)\, j\times B$

$$\Rightarrow \vec{E} + \vec{w}\times\vec{B} - \eta\vec{j} = \frac{1}{en}\Big(\underbrace{\underbrace{\vec{j}\times\vec{B}}_{\text{Hall's current}} - \nabla p_e}_{\text{often neglected}}\Big)$$

generalised Ohm's law

Double-fluid model

$$\rho\frac{\partial\vec{w}}{\partial t} = \vec{j}\times\vec{B} - \nabla p + \rho\vec{g}$$

$$\vec{E} + \vec{w}\times\vec{B} = \eta\vec{j}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{w} = 0$$

$$\frac{\partial\rho_e}{\partial t} + \nabla\cdot\vec{j} = 0$$

+ Maxwell's equations → should be solvable, 16 eq, 16 variables

→ almost equivalent to double-fluid model, but some information is lost in the process (pressure neglectances)

Approximations to plasma fluid

Force-free and ideal MHD

force-free → plasma does "not feel" the Lorentz force

$f = \rho_e E + j\times B = 0$

↑ "effective" or "bulk" acting on mixture

→ or Lorentz force small and negligible compared to other forces

→ or EM field induces forces that cancel external forces for E small: $\quad j \times B = 0$ ... often used in astrophysics

for $\nabla \times B = \mu_0 j$

$$(\nabla \times B) \times B = 0 \quad \Rightarrow \quad \nabla \times B \parallel B$$

$$\Rightarrow \nabla \times \vec{B} = \alpha \vec{B} \quad \text{... linear force-free}$$

may change in space

<u>ideal MHD</u> – in a co-moving frame E disappears

$$\vec{E}' = \vec{E} + \vec{w} \times \vec{B} = 0$$

using gen. Ohm's law: $\quad \vec{E}' = \eta \vec{j}$, $\quad \vec{E}' = 0$ for $j = 0$ or $\eta = 0$

if $E \cdot B = 0$ – fields degenerated $\rightarrow$ force-free and ideal MHD are the same

– they coincide if $\vec{j} = \rho_e \vec{w}$

Frozen field

consequence of the ideal MHD

highly conductive plasmas, strong plasma–field coupling

start from Ampère's law, taking a curl

$$\nabla \times (\nabla \times B) = \nabla \underbrace{\nabla \cdot B}_{\nabla \cdot B = 0} - \Delta B = \nabla \times \left(\mu_0 j + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \right)$$

Ohm's law and $\sigma = \frac{1}{\eta}$ conductivity

$$\sigma \mu_0 \nabla \times (E + w \times B) = \underbrace{- \varepsilon_0 \mu_0 \nabla \times \frac{\partial E}{\partial t} - \Delta B}_{\text{if this is finite}}$$

if this is finite, for $\eta \to 0$ $\sigma \to \infty$

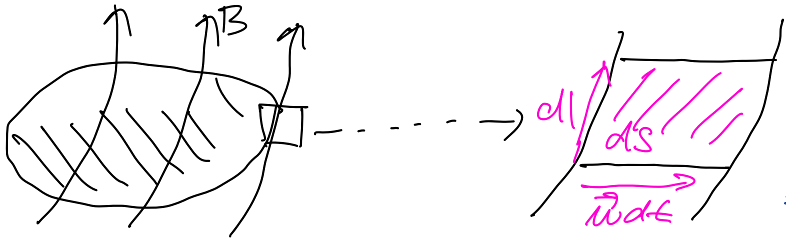
$$\Rightarrow \nabla \times (E + w \times B) \to 0$$

Faraday's law

$$- \frac{\partial B}{\partial t} + \nabla \times (w \times B) = 0 \Rightarrow \nabla \times (\vec{w} \times \vec{B}) = \frac{\partial \vec{B}}{\partial t} \quad \Leftarrow \underline{\text{frozen field}}$$

! curl being zero does not mean $E + w \times B = 0$!

interpretation: magnetic flux through closed surface does not change with time



$\vec{dS} = \vec{w}dt \times \vec{dl}$

$\Rightarrow \frac{\vec{dS}}{dt} = \vec{w} \times \vec{dl}$

$d\phi = B \cdot dS$

$$\frac{d\phi}{dt} = \frac{d}{dt}\int B \cdot dS = \int B \cdot \frac{dS}{dt} + \int \frac{\partial B}{\partial t} \cdot dS =$$

Stokes theorem

$$= \oint B \cdot (w \times dl) + \int \frac{\partial B}{\partial t} \cdot dS = \underbrace{-\oint (w \times B) \cdot dl}_{} + \int \frac{\partial B}{\partial t} \cdot dS =$$

$$= -\int \nabla \times (w \times B) \cdot dS + \int \frac{\partial B}{\partial t} \cdot dS = -\int \left[\nabla \times (w \times B) - \frac{\partial B}{\partial t}\right] \cdot dS = 0$$

Q.E.D.

Drifts in plasma fluid

Drifts perpendicular to B

$$\underbrace{mn\left(\frac{\partial w}{\partial t} + (w \cdot \nabla) w\right)}_{\text{large scale, slow changes}} = \underbrace{qn(E + w \times B) - \nabla p}_{\text{small scales, fast changes}}$$

$\Rightarrow$ RHS neglected

$$0 = qn(E + w \times B) - \nabla p \quad / \times B$$

$$0 = qn\left(E \times B + (w \times B) \times B\right) - \nabla p \times B =$$

$$= qn\left(E \times B - wB^2 + Bw \cdot B\right) - \nabla p \times B \quad \Big| \text{ perp. component}$$

$$0 = qn\left(E \times B - w_\perp B^2\right) - \nabla p \times B$$

$$w_\perp = \frac{E \times B}{B^2} - \underbrace{\frac{\nabla p \times B}{qnB^2}}_{\text{diamagnetic drift}} = v_E + v_D$$

$v_D \perp \nabla p \Rightarrow (w \cdot \nabla) w$ negligible if $E = 0$

$\nabla \phi \parallel \nabla p \vee E = -\nabla \phi \Rightarrow (w \cdot \nabla) w$ still negligible

otherwise $(w \cdot \nabla) w$ term complicates things

Drifts parallel to B

$$mn\left(\frac{\partial u_z}{\partial t} + (u\cdot\nabla)u_z\right) = qnE_z - \frac{\partial p}{\partial z}$$

E.O.S. $\quad \frac{\nabla p}{p} = \gamma \frac{\nabla n}{n}$; $\gamma = 1 \to$ isothermic

$(u\cdot\nabla)u_z \to 0$, u_z homogeneous

$$\Rightarrow \nabla p = k_B T \nabla n$$

For electrons: $q \to -e$, $m \to 0$, $T \to T_e$, $E_z = -\frac{\partial \phi}{\partial z}$

then $\quad e\frac{\partial \phi}{\partial z} = \frac{k_B T_e}{n}\frac{\partial n}{\partial z}$

hence $\quad e\phi = k_B T_e \ln n + C$

$$\Rightarrow n = n_0 \exp\left[\frac{e\phi}{k_B T_e}\right]$$

Boltzmann relation hold also for fluids

$\Rightarrow$ ions creating concentrations $\to$ electrons shield following the Boltzmann relation