Lecture notes

NAST008: Cosmic electrodynamics

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Preface

Blah

Korektury: Jan Kára, Lýdia Štofanová, Lucia Mravcová, Alžběta Oplištilová, Michael Vávra, Robert Jurenka.
Obrázky: Alžběta Oplištilová.

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Chapter 1

Basics of statistical physics

Physical system, or, more precisely, mathematical model of physical system is defined by a certain set of mathematical objects. These, together with their values (in a very general sense) define the state of the system. As an example, let's consider a set of N point masses $(m_1, \ldots, m_N)$ which may be characterised by $2N$ vectors of positions, $\boldsymbol{r}_i$, and momenta, $\boldsymbol{p}_i$, in the framework of classical physics. Such a set of $6N$ numbers may be viewed as a vector, $\boldsymbol{q} \equiv (\boldsymbol{r_1}, \boldsymbol{p_1}, \boldsymbol{r_2}, \boldsymbol{p_2}, ..., \boldsymbol{r_N}, \boldsymbol{p_N})$, in $6N$-dimensional *phase space* and it is usually referred to as *state*.

Proper modelling of the state of physical system is valuable by itself, nevertheless, further piece of information is needed if we wish to know how the system would evolve in time. This information is carried by equations of motion (again, in a very general sense). In the following, we will usually consider the evolution of the system to be determined by Hamilton's canonical equations, i.e.,

$$\dot{\boldsymbol{r}}_i = \frac{\partial \mathcal{H}(\boldsymbol{q})}{\partial \boldsymbol{p}_i} \qquad \text{and} \qquad \dot{\boldsymbol{p}}_i = -\frac{\partial \mathcal{H}(\boldsymbol{q})}{\partial \boldsymbol{r}_i}, \quad \forall i = 1 \ldots N, \tag{1.1}$$

with $\mathcal{H}(\boldsymbol{q})$ being a *Hamiltonian* of the system.

1.1 Liouville's theorem

Liouville's theorem states that the density of states in the phase space is constant along any evolutionary trajectory provided it is determined by the Hamilton's canonical equations of motion. Equivalently, we may consider an arbitrary set of states in phase space, Ω_0. If each of the states from that set evolve for the same (yet arbitrary) time interval, t, they define another set, Ω_t at that time, as it is schematically sketched in the left panel of Fig. 1.1. Liouville's theorem implies that volumes of Ω_0 and Ω_t are equal.

As a particular (and probably the simplest possible) example, let's consider a single free point-mass particle which is only allowed to move along the axis x with momentum p_x. Hamiltonian of this system is $\mathcal{H} = p_x^2/2m$, where m is mass of the particle. Solution of the Hamilton equations of motion is then $x(t) = t\, p_x/m \wedge p_x(t) = \text{const}$. This means

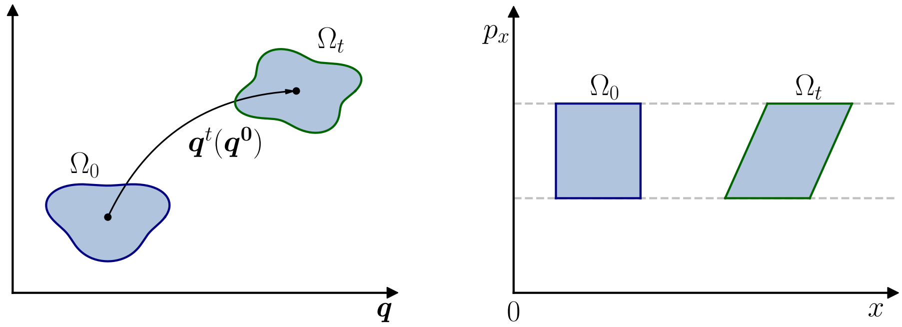

Figure 1.1: Left: Sketch of time evolution of a set of states in the phase space; $\boldsymbol{q}^t(\boldsymbol{q}_0)$ represents phase space trajectory of one (initial) particular state $\boldsymbol{q}_0$. Right: Evolution of particular (rectangular initially) set of states of a free particle with two degrees of freedom.

that states in the phase space move along lines parallel to the x-axis. Different states with the same value of momentum travel equal distance within equal time interval t, however, these distances differ between states with different values of momentum. Hence, the rectangular set Ω_0 as depicted in the right panel of Fig. 1.1 transforms to rhomboid Ω_t with the same length of base and height. Both straightforward geometric intuition or simple integration show that their volumes are the same.

In the following, we will provide a proof of the Liouville's theorem, i.e., we will show that the volume of the set Ω_t, which equals $\int_{\Omega_t} \mathrm{d}^{6\mathrm{N}}q$, is equal to the volume of Ω_0. At the first step, let us write

$$\int_{\Omega_0} \mathrm{d}^{6\mathrm{N}}q = \int_{\Omega_t} \mathrm{d}^{6\mathrm{N}}q^t \ , \tag{1.2}$$

which can be viewed just as trivial renaming of the integration variables. Next, let us consider the theorem about integration by substitution of variables, which states:

1) Let ϕ be a regular one-to-one mapping from $\mathrm{R_N}$ to $\mathrm{R_N}$ defined on an open set $G \subset \mathrm{R_N}$;
2) let $M \subset \phi(G) \subset \mathrm{R_N}$;
3) let f be a regular mapping $\mathrm{R_N} \to \mathrm{R_1}$ defined almost everywhere on M.

Then,

$$\int_M f(\boldsymbol{y})\mathrm{d}^N\boldsymbol{y} = \int_{\phi^{-1}(M)} f\left(\phi(\boldsymbol{x})\right) \left|\det\frac{\mathrm{D}\phi}{\mathrm{D}x}\right| \mathrm{d}^N\boldsymbol{x} \ . \tag{1.3}$$

For the sake of completeness, we stress out that the explicit form of the determinant matrix is

$$\frac{\mathrm{D}\phi}{\mathrm{D}x} = \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_2}{\partial x_1} & \cdots & \frac{\partial\phi_N}{\partial x_1} \\ \frac{\partial\phi_1}{\partial x_2} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \\ \vdots & & & \ddots \end{bmatrix} . \tag{1.4}$$

We will apply (1.3) to the right-hand side of equation (1.2) with $f \equiv 1$, $\boldsymbol{y} \to \boldsymbol{q}^t$, $\boldsymbol{x} \to \boldsymbol{q}^0$, $\phi(\boldsymbol{x}) \to \boldsymbol{q}^t(q)$. The mapping ϕ^t is such that it maps the initial state $\boldsymbol{q}^0$ into the new state $\boldsymbol{q}^t$ for each t according to the Hamiltonian equations of motion. Note that the mathematical meaning of $\boldsymbol{q}^t$ is ambiguous here. It has a meaning of the coordinates, when it describes a particular state in the phase space. On the other hand, it has a meaning of a function $\boldsymbol{q}^t(\boldsymbol{q}^0;t)$, when it describes the mapping from one state to another according to the equations of motion, t is then a parameter of this function. That is because the Hamiltonian may be viewed also as a recipe for the transformation of coordinates between the states. Finally, $M = \Omega_t$ and $\phi^{-1}(M) = \Omega_0$, and $N \to 6N$, which together yields

$$\int_{\Omega_t} \mathrm{d}^{6\mathrm{N}}\boldsymbol{q} = \int_{\Omega_t} \mathrm{d}^{6\mathrm{N}}\boldsymbol{q}^t = \int_{\Omega_0} \left|\det\frac{\mathrm{D}q^t}{\mathrm{D}q^0}\right| \mathrm{d}^{6N}\boldsymbol{q}^{\boldsymbol{0}} \,. \tag{1.5}$$

Consequently, we want to prove that

$$\int_{\Omega_t} 1\mathrm{d}^{6\mathrm{N}}\boldsymbol{q} = \int_{\Omega_0} 1\mathrm{d}^{6\mathrm{N}}\boldsymbol{q} \,. \tag{1.6}$$

Let us further denote

$$J^t \equiv \left|\det\frac{\mathrm{D}q^t}{\mathrm{D}q^0}\right| \,, \tag{1.7}$$

which is the Jacobian of the transform $\boldsymbol{q} \to \boldsymbol{q}^t$. Apparently, the Liouville's theorem is valid provided $J^t = 1$. In practise, we will show that $\mathrm{d}J^t/\mathrm{d}t = 0$ which, together with trivial identity $J^0 \equiv J^{t=0} = 1$, implies the required property of the Jacobian. Let us denote element of the Jacobi matrix $J^t_{ij} \equiv \partial q^t_i/\partial q^0_j$. It's time derivative may be expressed as

$$\frac{\mathrm{d}J^t_{ij}}{\mathrm{d}t} = \frac{\partial}{\partial q^0_j}\frac{\mathrm{d}q^t_i}{\mathrm{d}t} = \frac{\partial \dot{q}^t_i}{\partial q^0_j} = \sum_{k=1}^{6N} \frac{\partial \dot{q}^t_i}{\partial q^t_k}\frac{\partial q^t_k}{\partial q^0_j} = \sum_{k=1}^{6N} \frac{\partial \dot{q}^t_i}{\partial q^t_k} J^t_{kj} \,, \tag{1.8}$$

where we denote $\dot{q}^t_i \equiv \mathrm{d}q^t_i/\mathrm{d}t$. Here we used

$$J^t_{kj} = \frac{\partial q^t_k}{\partial q^0_j}. \tag{1.9}$$

In equation (1.8) we have exchanged order of derivatives with respect to t and q_j which is possible provided these two are continuous. The time derivative of the Jacobian J^t can be written with the help of eq. (1.8):

$$\frac{\mathrm{d}J^t}{\mathrm{d}t} = \sum_{i,j} \frac{\partial J^t}{\partial J^t_{ij}} \frac{\mathrm{d}J^t_{ij}}{\mathrm{d}t} = \sum_{i,j,k} \frac{\partial J^t}{\partial J^t_{ij}} J^t_{kj} \frac{\partial \dot{q}^t_i}{\partial q^t_k} \,. \tag{1.10}$$

Equation (1.10) can be simplified due to identity (see Box 1.1)

$$\sum_j \frac{\partial J^t}{\partial J^t_{ij}} J^t_{kj} = \delta_{ik} J^t \tag{1.11}$$

to

$$\frac{\mathrm{d}J^t}{\mathrm{d}t} = J^t \sum_{i=1}^{6N} \frac{\partial \dot{q}_i^t}{\partial q_i^t} \,. \tag{1.12}$$

Finally, let us rewrite (1.12) in terms of canonical coordinates and momenta and consider Hamilton's equations to evaluate $\dot{r}_i^t$ and $\dot{p}_i^t$:

$$\frac{\mathrm{d}J^t}{\mathrm{d}t} = J^t \sum_{i=1}^{3\mathrm{N}} \left(\frac{\partial \dot{r}_i^t}{\partial r_i^t} + \frac{\partial \dot{p}_i^t}{\partial p_i^t} \right) = J^t \sum_{i=1}^{3\mathrm{N}} \left(\frac{\partial^2 \mathcal{H}}{\partial r_i^t \partial p_i^t} - \frac{\partial^2 \mathcal{H}}{\partial p_i^t \partial r_i^t} \right) \tag{1.13}$$

Hence we assume that we may switch the order of partial derivatives of $\mathcal{H}$, the value of the expression above vanishes. Hence $J^t = 1$ for each t and the Liouville's theorem holds.

(Add example of non-Hamiltonian system (in which case different initial states may converge to common final state).)

(Psat carku po now, kterym zacina veta? Spravne jsou snad obe varianty, ale neni to v textu sjednoceno.)

1.2 Statistical description of a physical system

The probability density is constant along each state evolutionary track through the phase space, i.e. each particular state keeps in time its probability of being realised (found/measured). This, together with the assumptions required for the Liouville's theorem to be valid (i.e. Hamiltonian nature of the system) allows to formulate a Liouville's equation for the temporal evolution of the distribution function D_{N}.

Let's assume an arbitrary *fixed* subset, Ω, of the phase space. Probability to find the system in some state in Ω is $\mathcal{P}(\Omega) = \int_\Omega D_{\mathrm{N}} \mathrm{d}\omega$. This property implies the *normalisation* of the distribution function bearing the usual form

$$\int_{-\infty}^{+\infty} D_{\mathrm{N}} \mathrm{d}\omega = \mathcal{C}, \tag{1.18}$$

where $\mathcal{C}$ is a scalar value. Taking into account the previous paragraph, the natural selection dictates $\mathcal{C} = 1$. The probabilistic interpration states that if the subset Ω covers the whole phase space, the corresponding probability obtained from the distribution function should be 1 (the certainty).

Liouville's theorem states that the fluid of states is incompressible, hence, the only way how the probability $P(\Omega)$ can change is due to the flow of the states through the boundary of Ω, i.e.:

$$\frac{\partial}{\partial t} \int_\Omega D_{\mathrm{N}} \,\mathrm{d}\omega = - \oint_{\partial\Omega} D_{\mathrm{N}} \,\dot{\boldsymbol{q}} \cdot \mathrm{d}\boldsymbol{\Sigma} \tag{1.19}$$

where $\mathrm{d}\boldsymbol{\Sigma}$ is an (outward) oriented element of the surface $\partial\Omega$. The minus sign means that the probability decreases when the state flows parallel to $\mathrm{d}\boldsymbol{\Sigma}$, i.e. out from Ω.

Let $B = A^{-1}$ be inverse matrix of A. Then the elements b_{ij} may be computed by using the adjoint (or adjugate[1]) matrix

$$B = \frac{1}{\det A} \mathrm{adj} A, \tag{1.14}$$

defined by

$$(\mathrm{adj} A)^{ij} \equiv (-1)^{i+j} \det(A^{ji}). \tag{1.15}$$

Then the inverse elements are given by

$$b_{ij} = \frac{(-1)^{i+j}}{a} \det\left(A^{ji}\right),$$

where $a \equiv \det A$ and $'$ indicates the matrix transpose operator. The notation A^{ij} represents the submatrix obtained from matrix A by excluding ith row and jth column. The determinant a of matrix A may be computed using the expansion as

$$a = \sum_i (-1)^{i+j} a_{ij} \det A^{ij}.$$

Then

$$\frac{\partial a}{\partial a_{ij}} = (-1)^{i+j} \det A^{ij} = a b^{ji}. \tag{1.16}$$

Hence

$$\sum_j \frac{\partial a}{\partial a_{ij}} a_{kj} = \sum_j a b^{ji} a_{kj} = a \sum_j b^{ji} a_{kj} = a (BA)_{ki} = a \delta_{ik}, \tag{1.17}$$

Q.E.D.

Box 1.1: Useful algebraic identity to prove the Liouville's theorem.

Provided some physically plausible mathematical conditions are fulfilled (provide an exact formulation of these conditions), we may exchange derivative and integration operators on the left-hand side of eq. (1.19); on the right-hand side, we apply Gauss theorem, yielding

$$\int_\Omega \frac{\partial D_\mathrm{N}}{\partial t}\,\mathrm{d}\omega = -\int_\Omega \nabla\cdot(D_\mathrm{N}\dot{\boldsymbol{q}})\,\mathrm{d}\omega\ . \tag{1.20}$$

Integral equality (1.20) holds for an arbitrary set Ω which implies equality of integrands. Therefore, we obtain a continuity equation for the distribution function,

$$\frac{\partial D_\mathrm{N}}{\partial t} = -\nabla\cdot(D_\mathrm{N}\,\dot{\boldsymbol{q}})\ . \tag{1.21}$$

This equation represents the "continuity equation" for the "fluid of states" and implies that the state keeps its D_N when evolving in time according to the Hamiltonian equations. The relation can be further rewritten as

$$\frac{\partial D_\mathrm{N}}{\partial t} = -\sum_{i=1}^{6N}\left[\frac{\partial D_\mathrm{N}}{\partial q_i}\dot{q}_i + D_\mathrm{N}\frac{\partial \dot{q}_i}{\partial q_i}\right] = -\sum_{i=1}^{3N}\left[\frac{\partial D_\mathrm{N}}{\partial r_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial D_\mathrm{N}}{\partial p_i}\frac{\partial \mathcal{H}}{\partial r_i} + D_\mathrm{N}\left(\frac{\partial \mathcal{H}}{\partial r_i\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i\partial r_i}\right)\right]\ . \tag{1.22}$$

The terms in () brackets cancel out provided $\mathcal{H}$ is continuously differentiable with respect to r_i and p_i. Using the Poisson brackets notation, the *Liouville equation* is usually written as:

$$\frac{\partial D_\mathrm{N}}{\partial t} = \{\mathcal{H}, D_\mathrm{N}\}_\mathrm{P}\ . \tag{1.23}$$

(Mention the analogous von Neumann equation from the quantum theory.)

1.3 Reduced distribution functions

Distribution function, D_N holds maximum information of a particular (usually very complex) physical system. This maximum information may often be more than what we need. In some circumstances, it is convenient to define reduced distribution functions (for details see Box 1.2):

$$f(\boldsymbol{r},\boldsymbol{p}) \equiv \int D_\mathrm{N}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{r}_2,\boldsymbol{p}_2,\ldots,\boldsymbol{r}_\mathrm{N},\boldsymbol{p}_\mathrm{N})\,\mathrm{d}\tau_{\mathrm{N}-1}\ , \tag{1.24}$$

where

$$\mathrm{d}\tau_{\mathrm{N}-1} \equiv \frac{\mathrm{d}^3\boldsymbol{r}_2\mathrm{d}^3\boldsymbol{p}_2\ldots\mathrm{d}^3\boldsymbol{r}_\mathrm{N}\mathrm{d}^3\boldsymbol{p}_\mathrm{N}}{(\mathrm{N}-1)!} \tag{1.25}$$

and the integration is taken over the whole definition space of variables $\boldsymbol{r}_2\ldots\boldsymbol{p}_\mathrm{N}$. Definition (1.24) generally gives different results when different pairs $(\boldsymbol{r}_i,\boldsymbol{p}_i)$ are excluded from integration (which means that the result should keep some mark according to the excluded particle). The definition (1.24), however, is usually considered for specific systems of *undistinguishable* particles which imply that D_N as well as $\mathcal{H}$ are symmetric with respect to permutations of individual particles (exchange of pairs of $(\boldsymbol{r}_i,\boldsymbol{p}_i)$ and

Let us comment on different normalisations of the reduced distribution functions. The distribution function D_{N} holds maximum information of a particular physical system, which is often not necessary. For instance, we may only be interested in a probability density of finding particle carrying tag 1 at $\boldsymbol{r}_1$ with the momentum $\boldsymbol{p}_1$, where we do not need any information about the remaining particles. Then we may integrate D_{N} over all other particles yielding

$$f^1(\boldsymbol{r}_1, \boldsymbol{p}_1) = \int D_{\mathrm{N}}(\boldsymbol{r}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{r}_N, \boldsymbol{p}_N) \mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 \ldots \mathrm{d}^3\boldsymbol{r}_N \mathrm{d}^3\boldsymbol{p}_N.$$

Function f^1 is in a good sense a one-particle distribution function. A similar reduction may be written for the particle carrying tag 2:

$$f^2(\boldsymbol{r}_2, \boldsymbol{p}_2) = \int D_{\mathrm{N}}(\boldsymbol{r}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{r}_N, \boldsymbol{p}_N) \mathrm{d}^3\boldsymbol{r}_1 \mathrm{d}^3\boldsymbol{p}_1 \mathrm{d}^3\boldsymbol{r}_3 \mathrm{d}^3\boldsymbol{p}_3 \ldots \mathrm{d}^3\boldsymbol{r}_N \mathrm{d}^3\boldsymbol{p}_N$$

and so forth. As long as the particles are distinguishable, functions f^1, f^2, ... may in principle be different. Note that f^1... functions are each normalised to unity.
In the above given examples, we may always formally rename the considered particle so that its position and momentum is formally written without the subscript, hence $(\boldsymbol{r}, \boldsymbol{p})$. Still, generally, $f^1(\boldsymbol{r}, \boldsymbol{p}) \neq f^2(\boldsymbol{r}, \boldsymbol{p})$. They are identical only in the case of the undistinguishable particles.
Now, in the case we are interested in a probability density $f(\boldsymbol{r}, \boldsymbol{p})$ of finding *any* undistinguishable particle at $(\boldsymbol{r}, \boldsymbol{p})$ we must sum the probabilities of finding particle 1, particle 2,... at $(\boldsymbol{r}, \boldsymbol{p})$. Hence, for example,

$$f(\boldsymbol{r}, \boldsymbol{p}) = \sum_{i=1}^{N} f^i(\boldsymbol{r}, \boldsymbol{p}) = N f^1(\boldsymbol{r}, \boldsymbol{p}) = N f^2(\boldsymbol{r}, \boldsymbol{p}) = \ldots.$$

Note that f is now normalised to N. This is not convenient for the interpretation. Hence we may naturally re-normalise D_{N} to obtain f which is normalised to unity.
When integrating from D_{N} we get

$$\begin{aligned} f(\boldsymbol{r}, \boldsymbol{p}) &= N \int D_{\mathrm{N}}(\boldsymbol{r}, \boldsymbol{p}, \ldots, \boldsymbol{r}_N, \boldsymbol{p}_N) \mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 \ldots \mathrm{d}^3\boldsymbol{r}_N \mathrm{d}^3\boldsymbol{p}_N = \\ &= N \int \frac{D'_{\mathrm{N}}}{N!} \mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 \ldots \mathrm{d}^3\boldsymbol{r}_N \mathrm{d}^3\boldsymbol{p}_N = \int D'_{\mathrm{N}} \frac{\mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 \ldots \mathrm{d}^3\boldsymbol{r}_N \mathrm{d}^3\boldsymbol{p}_N}{(N-1)!}, \end{aligned}$$

where D'_{N} has a different normalisation (to $N!$) and we naturally see the definition of the new volume element $\mathrm{d}\tau_{N-1}$. In the following, we drop the $'$ for simplicity.

Box 1.2: On the normalisation of the distribution functions.

$(\boldsymbol{r}_j, \boldsymbol{p}_j))$. For these systems, integration (1.24) gives identical functions regardless of over which $\mathrm{N}-1$ pairs of coordinates and momenta it is performed. Function $f(\boldsymbol{r}, \boldsymbol{p})$ is usually called single-particle distribution function and it gives a probability density of finding *any* particle at position $\boldsymbol{r}$ with momentum $\boldsymbol{p}$. Note that the single-particle distribution function really holds considerably reduced information with respect to D_N. We could add that to the meaning of $f(\boldsymbol{r}, \boldsymbol{p})$ that "...and we have no information about positions and momenta of the other particles." Single-particle distribution function holds a sufficient amount of information to calculate mean values of physical quantities which may be written in a form:

$$A(\boldsymbol{r}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{r}_\mathrm{N}, \boldsymbol{p}_\mathrm{N}) = \sum_{i=1}^{\mathrm{N}} a(\boldsymbol{r}_i, \boldsymbol{p}_i) \;, \tag{1.26}$$

i.e. it is a sum of N identical functions. The mean value is then

$$\langle A \rangle \equiv \int A\, D_\mathrm{N}\, \mathrm{d}\tau_\mathrm{N} = \int a(\boldsymbol{r}, \boldsymbol{p})\, f(\boldsymbol{r}, \boldsymbol{p})\, \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{p} \;. \tag{1.27}$$

Similarly to (1.24) we may define "two-particle distribution function,"

$$f_2(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2) \equiv \int D_\mathrm{N}\, \mathrm{d}\tau_{\mathrm{N}-2} \;, \tag{1.28}$$

which gives the probability density of finding *some* particle at position $\boldsymbol{r}_1$ with momentum $\boldsymbol{p}_1$ and *any other* particle at position $\boldsymbol{r}_2$ with momentum $\boldsymbol{p}_2$. Mean values of physical quantities which depend on combinations of positions and momenta of two particles,

$$B(\boldsymbol{r}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{r}_\mathrm{N}, \boldsymbol{p}_\mathrm{N}) = \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} b(\boldsymbol{r}_i, \boldsymbol{p}_i, \boldsymbol{r}_j, \boldsymbol{p}_j) \;, \tag{1.29}$$

can be calculated through integration of the two-particle distribution function:

$$\langle B \rangle = \int b(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2)\, f_2(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2)\, \mathrm{d}^3\boldsymbol{r}_1\, \mathrm{d}^3\boldsymbol{p}_1 \mathrm{d}^3\boldsymbol{r}_2\, \mathrm{d}^3\boldsymbol{p}_2 \;. \tag{1.30}$$

An important example of a physical quantity which often takes the form of (1.26) or (1.29) or combination of both is the total energy of the system which is defined by Hamiltonian. Let's introduce its specific (i.e. not general!) form:

$$\mathcal{H} = \sum_{i=1}^{\mathrm{N}} \frac{\boldsymbol{p}_i^2}{2m} + \sum_{i=1}^{\mathrm{N}} V(\boldsymbol{r}_i) + \sum_{i=1}^{\mathrm{N}} \sum_{\substack{j=1 \\ j\neq i}}^{\mathrm{N}} w(\boldsymbol{r}_i, \boldsymbol{r}_j) \;. \tag{1.31}$$

Here, the first sum corresponds to a system of free particles; first plus second sum defines a system of non-interacting particles (of equal mass, m, which is an inevitable consequence of their indistinguishability) in an external potential $V(\boldsymbol{r})$. The third term is a possible form of the interaction term. The mean value of the energy of the statistically described system determined by Hamiltonian (1.31) is

$$\langle \mathcal{H} \rangle = \int h(\boldsymbol{r}, \boldsymbol{p})\, f(\boldsymbol{r}, \boldsymbol{p})\, \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{p} + \int w(\boldsymbol{r}_1, \boldsymbol{r}_2)\, f_2(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2)\, \mathrm{d}^3\boldsymbol{r}_1\, \mathrm{d}^3\boldsymbol{p}_1\, \mathrm{d}^3\boldsymbol{r}_2\, \mathrm{d}^3\boldsymbol{p}_2\;, \tag{1.32}$$

where we denote

$$h(\boldsymbol{r}, \boldsymbol{p}) \equiv \frac{p^2}{2m} + V(\boldsymbol{r})\;. \tag{1.33}$$

In the following, we will use a particular example of the interaction term which describes mutual gravitational interaction among particles:

$$w(\boldsymbol{r}_i, \boldsymbol{r}_j) = -\frac{1}{2} \frac{G m^2}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \tag{1.34}$$

with G representing the gravitational constant and the factor of $1/2$ is to respect the symmetry of the gravity force.

1.4 Boltzmann equation

Similarly to the Liouville's equation (1.23) which defines evolution of the N-particle distribution function, D_{N}, we may derive a partial differential equation for evolution of the single-particle distribution function, $f(\boldsymbol{r}, \boldsymbol{p})$. The procedure lies in the integration of (1.23) over the whole definition space of positions and momenta of all, but one particles. More precisely, let's define $\boldsymbol{r} \equiv \boldsymbol{r}_1$ and $\boldsymbol{p} \equiv \boldsymbol{p}_1$ and perform the integration over all coordinates and momenta except for $\boldsymbol{r}$ and $\boldsymbol{p}$:

$$\int \frac{\partial D_{\mathrm{N}}}{\partial t}\, \mathrm{d}\tau_{\mathrm{N}-1} = \int \sum_{i=1}^{\mathrm{N}} \left(\frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}_i} \cdot \frac{\partial h}{\partial \boldsymbol{r}_i} - \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{r}_i} \cdot \frac{\partial h}{\partial \boldsymbol{p}_i} \right) \mathrm{d}\tau_{\mathrm{N}-1} + \int \sum_{i,j} \frac{\partial w(\boldsymbol{r}_i, \boldsymbol{r}_j)}{\partial \boldsymbol{r}_i} \cdot \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}_i}\, \mathrm{d}\tau_{\mathrm{N}-1}\;. \tag{1.35}$$

Note that when inserting (1.31) into (1.22), the first term of (1.35) should contain double sum and the second term even a triple sum. However, according to (1.22) a derivative of the Hamiltonian is taken with respect to the generalised coordinates, which only keeps certain terms corresponding to the considered r_i or p_i from the sum in (1.31), thereby effectively removing one summation from both terms of (1.35).

Let's first consider the left-hand side of equation (1.35). Provided the conditions of the theorem of the derivative of integral with respect to the parameter are fulfilled (put them down explicitly here?), we may write

$$\int \frac{\partial D_{\mathrm{N}}}{\partial t}\, \mathrm{d}\tau_{\mathrm{N}-1} = \frac{\partial}{\partial t} \int D_{\mathrm{N}}\, \mathrm{d}\tau_{\mathrm{N}-1} = \frac{\partial f(\boldsymbol{r}, \boldsymbol{p})}{\partial t}\;. \tag{1.36}$$

Considering the second term in (1.35), we first split it into three parts ($i = 1$, $i = 2$ and

$i > 2$):

$$\int \sum_{i=1}^{\mathrm{N}} \left(\frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}_i} \cdot \frac{\partial h}{\partial \boldsymbol{r}_i} - \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{r}_i} \cdot \frac{\partial h}{\partial \boldsymbol{p}_i} \right) \mathrm{d}\tau_{\mathrm{N}-1} = \int \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}} \cdot \frac{\partial h}{\partial \boldsymbol{r}} \mathrm{d}\tau_{\mathrm{N}-1} - \int \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{r}} \cdot \frac{\partial h}{\partial \boldsymbol{p}} \mathrm{d}\tau_{\mathrm{N}-1} + \int \left(\frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}_2} \cdot \frac{\partial h}{\partial \boldsymbol{r}_2} - \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial h}{\partial \boldsymbol{p}_2} \right) \mathrm{d}\tau_{\mathrm{N}-1} + \int \sum_{i=3}^{\mathrm{N}} \left(\frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}_i} \cdot \frac{\partial h}{\partial \boldsymbol{r}_i} - \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{r}_i} \cdot \frac{\partial h}{\partial \boldsymbol{p}_i} \right) \mathrm{d}\tau_{\mathrm{N}-1} \,. \tag{1.37}$$

The specific and very important property of the first two terms on the right-hand side of eq. (1.37) is that (i) h is function of just $\boldsymbol{r}$ and $\boldsymbol{p}$ and (ii) the partial derivation is with respect to these quantities over which we do not integrate. Hence, we may put the terms $\partial h/\partial \boldsymbol{r}$ and $\partial h/\partial \boldsymbol{p}$ out of the integrals and, under the assumptions of derivative of integral with respect to a parameter, we may write:

$$\int \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}} \cdot \frac{\partial h}{\partial \boldsymbol{r}} \mathrm{d}\tau_{\mathrm{N}-1} - \int \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{r}} \cdot \frac{\partial h}{\partial \boldsymbol{p}} \mathrm{d}\tau_{\mathrm{N}-1} = \frac{\partial h}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} - \frac{\partial h}{\partial \boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} \,. \tag{1.38}$$

Finally, we will show that the second (and analogically any subsequent) term of the sum is zero. Let's reorganise the differentials in $\mathrm{d}\tau_{\mathrm{N}-1}$ and evaluate a sub-integral

$$I_1 \equiv \int \int \left(\frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}_2} \cdot \frac{\partial h}{\partial \boldsymbol{r}_2} - \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial h}{\partial \boldsymbol{p}_2} \right) \mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 \,. \tag{1.39}$$

Performing integration per partes with respect to variable $\boldsymbol{p}_2$ and $\boldsymbol{r}_2$ in the first and second term, respectively, we will get

$$I_1 = \int \left[D_{\mathrm{N}} \frac{\partial h}{\partial \boldsymbol{r}_2} \right]_{\boldsymbol{p}_2 \to \infty} \mathrm{d}^3\boldsymbol{r}_2 - \int \int D_{\mathrm{N}} \frac{\partial h}{\partial \boldsymbol{r}_2 \, \partial \boldsymbol{p}_2} \mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 - \int \left[D_{\mathrm{N}} \frac{\partial h}{\partial \boldsymbol{p}_2} \right]_{\boldsymbol{r}_2 \to \infty} \mathrm{d}^3\boldsymbol{p}_2 + \int \int D_{\mathrm{N}} \frac{\partial h}{\partial \boldsymbol{p}_2 \, \partial \boldsymbol{r}_2} \mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 \tag{1.40}$$

the notation of individual terms in (1.40) has to be taken somewhat fuzzy. More precisely, one should perform the integration over components of vectors $\boldsymbol{r}_2$ and $\boldsymbol{p}_2$. In the first and third term on the right-hand side of (1.40), the argument of squared brackets is to be evaluated in appropriate infinity. Hence, these terms are both zero as D_{N} goes to zero at infinity which is a necessary condition for the normalisation (1.18) to be fulfilled. The second and fourth term in (1.40) cancel out provided the function $h(\boldsymbol{r}, \boldsymbol{p})$ is smoothly differentiable with respect to both arguments (which is a condition for a possible exchange of the order of differentiation).

As mentioned above, each individual part of the sum in the third right-hand side term of eq. (1.37) gives zero after the manipulation analogical to that presented above. Equation (1.35) then reduces to

$$\frac{\partial f}{\partial t} + \frac{\partial h}{\partial \boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{\partial h}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \int \sum_{i,j} \frac{\partial w(\boldsymbol{r}_i, \boldsymbol{r}_j)}{\partial \boldsymbol{r}_i} \cdot \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}_i} \mathrm{d}\tau_{\mathrm{N}-1} \,. \tag{1.41}$$

The right-hand term of eq. (1.41) could be simplified by using a similar approach when dealing with (1.37). It can be easily shown that only $\mathrm{N}-1$ terms with $i = 1$ are non-zero and they are all identical due to symmetries of D_N. With an alternate notation $\boldsymbol{r}' \equiv \boldsymbol{r}_2$ and $\boldsymbol{p}' \equiv \boldsymbol{p}_2$ we write

$$\begin{aligned}
\frac{\partial f}{\partial t} + \frac{\partial h}{\partial \boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{\partial h}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} &= (\mathrm{N}-1) \int \frac{\partial w(\boldsymbol{r}, \boldsymbol{r}')}{\partial \boldsymbol{r}} \cdot \frac{\partial D_\mathrm{N}}{\partial \boldsymbol{p}} \, \frac{\mathrm{d}^3\boldsymbol{r}'\mathrm{d}^3\boldsymbol{p}'\mathrm{d}^3\boldsymbol{r}_3\mathrm{d}^3\boldsymbol{p}_3 \ldots \mathrm{d}^3\boldsymbol{r}_\mathrm{N}\mathrm{d}^3\boldsymbol{p}_\mathrm{N}}{(\mathrm{N}-1)!} \\
&= \int \mathrm{d}^3\boldsymbol{r}'\mathrm{d}^3\boldsymbol{p}' \, \frac{\partial w(\boldsymbol{r}, \boldsymbol{r}')}{\partial \boldsymbol{r}} \cdot \int \frac{\partial D_\mathrm{N}}{\partial \boldsymbol{p}} \, \mathrm{d}\tau_{\mathrm{N}-2} \\
&= \int \frac{\partial w(\boldsymbol{r}, \boldsymbol{r}')}{\partial \boldsymbol{r}} \cdot \frac{\partial f_2(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{r}', \boldsymbol{p}')}{\partial \boldsymbol{p}} \, \mathrm{d}^3\boldsymbol{r}'\mathrm{d}^3\boldsymbol{p}' \; . \qquad (1.42)
\end{aligned}$$

In the last step we, again, assumed the possibility to exchange the order of integration and differentiation with respect to $\boldsymbol{p}$. The right-hand side of eq. (1.42) covers mutual interaction of the particles and is usually called the collision term, being denoted as $(\partial f/\partial t)_\mathrm{coll}$. Using the Hamilton canonical equations, we may write one of several "standard" forms of *Boltzmann equation*:

$$\frac{\partial f}{\partial t} + \frac{\partial h}{\partial \boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{\partial h}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \left(\frac{\partial f}{\partial t}\right)_\mathrm{coll} \; . \qquad (1.43)$$

Keep in mind that this is just a notation and the collision term still involves integral of the two-body distribution function. This is a principial problem which makes it impossible to solve the equation. There are several ways how to overcome this obstacle.

1.4.1 BBGKY hierarchy

Deriving partial differential equation for two-body distribution function, f_2, in a similar way, i.e. integrating the Liouville equation over $6(\mathrm{N}-2)$ variables may seem to be a straightforward way how to make the system of equations closed. Unfortunately, the problem would reappear – in the equation for f_2 we would obtain an integral term including three-body distribution function. The procedure may be repeated again and again, leading to a closed set of $N-1$ equations. This set is called the BBGKY hierarchy (after scientists who independently studied this problem: Bogolyubov, Born, Green, Kirkwood and Yvon) but in its completeness, it is no more suitable for solving than the Liouville equation.

A general class of approximations with an intent to make a smaller set of equations closed lies in assumption that the higher-order distribution function can be written as a combination of lower-order distribution functions, e.g. $f_2(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{r}', \boldsymbol{p}') = f(\boldsymbol{r}, \boldsymbol{p}) \, f(\boldsymbol{r}', \boldsymbol{p}')$.

1.4.2 Vlasov approximation

The simplest, above presented approximation is, actually, often used and is the basis of the Vlasov equation. Let's start without the approximation and rewrite f_2 in a following way:

$$f_2(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{r}', \boldsymbol{p}') = f(\boldsymbol{r}, \boldsymbol{p}) \, f(\boldsymbol{r}', \boldsymbol{p}') + g_2(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{r}', \boldsymbol{p}') \; . \qquad (1.44)$$

The first term on the right-hand side of formula (1.44), a product of two single particle distribution functions, represents probablility density of two uncorrelated events (finding one particle at position $\boldsymbol{r}$ with momentum $\boldsymbol{p}$ and another one at position $\boldsymbol{r}'$ and momentum $\boldsymbol{p}'$. Function g_2 is then a correlation function which gives the probablity density of finding one particle at position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$ *due to* finding another particle at position $\boldsymbol{r}'$ and momentum $\boldsymbol{p}'$ and vice versa.

The Vlasov approximation lies in an assumption that the correlation function, g_2 may be neglected with respect to the uncorrelated probability density, i.e., $f_2(\boldsymbol{r},\boldsymbol{p},\boldsymbol{r}',\boldsymbol{p}') = f(\boldsymbol{r},\boldsymbol{p})\, f(\boldsymbol{r}',\boldsymbol{p}')$. Then, the collision term in the Boltzmann's equation (1.43) reads,

$$\begin{aligned}\left(\frac{\partial f}{\partial t}\right)_{\rm coll} &= \int \frac{\partial w(\boldsymbol{r},\boldsymbol{r}')}{\partial \boldsymbol{r}} \cdot \frac{\partial f(\boldsymbol{r},\boldsymbol{p})}{\partial \boldsymbol{p}}\, f(\boldsymbol{r}',\boldsymbol{p}')\, {\rm d}^3\boldsymbol{r}'{\rm d}^3\boldsymbol{p}' = \\ &= \frac{\partial f(\boldsymbol{r},\boldsymbol{p})}{\partial \boldsymbol{p}} \cdot \frac{\partial}{\partial \boldsymbol{r}} \int w(\boldsymbol{r},\boldsymbol{r}')\, f(\boldsymbol{r}',\boldsymbol{p}')\, {\rm d}^3\boldsymbol{r}'{\rm d}^3\boldsymbol{p}' \,. \end{aligned} \tag{1.45}$$

Here, we put out of the integral the term that does not depend on the integration variables and, similarly to several cases above, we assumed that integration and derivation with respect to the parameter $\boldsymbol{r}$ may be mutually exchanged in order. The integral in (1.45) is, by definition, mean value of quantity $w(\boldsymbol{r},\boldsymbol{r}')$. For the case of example (1.34), this represents mean value of gravitational potential energy of the system of N particles distributed according to the distribution function $f(\boldsymbol{r}',\boldsymbol{p}')$ at (arbitrary) position $\boldsymbol{r}$. Let us denote

$$\Phi(\boldsymbol{r}) \equiv \langle W\rangle(\boldsymbol{r}) = \int w(\boldsymbol{r},\boldsymbol{r}')\, f(\boldsymbol{r}',\boldsymbol{p}')\, {\rm d}^3\boldsymbol{r}'{\rm d}^3\boldsymbol{p}' = -\int \frac{Gm^2}{|\boldsymbol{r}-\boldsymbol{r}'|} f(\boldsymbol{r}',\boldsymbol{p}')\, {\rm d}^3\boldsymbol{r}'{\rm d}^3\boldsymbol{p}' \tag{1.46}$$

and, consequently, we may rewrite Boltzmann's equation (1.43) in Vlasov approximation:

$$\frac{\partial f}{\partial t} + \frac{\partial h}{\partial \boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \boldsymbol{\nabla}\left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0 \,, \tag{1.47}$$

which will be called Vlasov's equation hereafter.

For the case of the system described by Hamiltonian (1.31) with interaction term (1.34), we have $\boldsymbol{v} \equiv \dot{\boldsymbol{r}} = \boldsymbol{p}/m$ which allows us to rewrite the Vlasov's equation in the form

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \tfrac{1}{m}\boldsymbol{\nabla}\left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0 \,. \tag{1.48}$$

Above, we considered the distribution function f to be explicit function of $\boldsymbol{r}$ and $\boldsymbol{p}$ with $\boldsymbol{p}$ being the *canonical* impulse and with normalisation $\int f(\boldsymbol{r},\, \boldsymbol{p})\, {\rm d}^3\boldsymbol{r}\, {\rm d}^3\boldsymbol{p} = 1$.

Quite often, the Vlasov's equation is formulated for another distribution function, $f(\boldsymbol{r},\boldsymbol{v})$,

$$\frac{\partial f(\boldsymbol{r},\boldsymbol{v})}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f(\boldsymbol{r},\boldsymbol{v})}{\partial \boldsymbol{r}} + \frac{\boldsymbol{F}(\boldsymbol{r})}{m} \cdot \frac{\partial f(\boldsymbol{r},\boldsymbol{v})}{\partial \boldsymbol{v}} = 0 \tag{1.49}$$

with $\boldsymbol{F}(\boldsymbol{r})$ representing all external and mean internal forces and with normalisation $\int f(\boldsymbol{r},\, \boldsymbol{v})\, {\rm d}^3\boldsymbol{r}\, {\rm d}^3\boldsymbol{v} = 1$. Note that, while being trivial for $\boldsymbol{v} = \boldsymbol{p}/m$, the transition from eq. (1.48) to (1.49) needs to be carried out carefully when dealing with systems of charged particles in which case the linear relation between velocity and momentum does not hold. This is (currently) beyond the scope of this text and we will assume, that even more generalised version of Vlasov's equation (1.49) holds, in which $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}, \boldsymbol{v})$ may also cover the Lorentz force.

1.5 Fluid equations

Some variant of Boltzmann or Vlasov equation appears to be suitable for solving various physical systems. Nevertheless, in other cases, it is either too difficult, or unnecessary to consider all information held by the distribution function. Fluid equations, which subsequently allow various heuristic modifications are quite commonly used. Usually, they are derived from the Vlasov equation in the form (1.48).

The basic quantities which stand in the fluid equations are defined as various, linearly independent moments of the single-particle distribution function:

$$\begin{pmatrix} \rho(\boldsymbol{r};t) \\ \rho(\boldsymbol{r};t)\,\boldsymbol{u}(\boldsymbol{r};t) \\ \rho(\boldsymbol{r};t)\,\mathcal{E}(\boldsymbol{r};t) \end{pmatrix} \equiv \int \begin{pmatrix} m \\ m\,\boldsymbol{v} \\ \frac{1}{2}m\,|\boldsymbol{v}-\boldsymbol{u}|^2 \end{pmatrix} f(\boldsymbol{r};\boldsymbol{t},\boldsymbol{v})\,\mathrm{d}^3\boldsymbol{v}\ . \tag{1.50}$$

The quantities ρ, $\boldsymbol{u}$ and $\mathcal{E}$ are mean values of density, velocity and specific (per unit mass and unit volume) internal energy of the statistically described system. They are functions of position and time, but we have dropped the dependency on velocity $\boldsymbol{v}$. In the following, we will (again) drop the implicit dependence on time.

The fluid equations are obtained by multiplying of Boltzmann or Vlasov equation by m, $\boldsymbol{v}$ and $\frac{1}{2}mv^2$ and subsequent integration over the velocity space. They are sometimes called moments of the Boltzmann/Vlasov equation.

1.5.1 Continuity equation

The simplest to derive is the continuity equation which starts from

$$\int m\frac{\partial f}{\partial t}\,\mathrm{d}^3\boldsymbol{v} + \int m\boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{r}}\,\mathrm{d}^3\boldsymbol{v} - \int \boldsymbol{\nabla}\left[V(\boldsymbol{r})+\Phi(\boldsymbol{r})\right]\cdot\frac{\partial f}{\partial \boldsymbol{v}}\,\mathrm{d}^3\boldsymbol{v} = 0\ . \tag{1.51}$$

Under certain assumptions on the form of $f(\boldsymbol{r},\,\boldsymbol{v})$, we may exchange the order of integration and derivation with respect to t and $\boldsymbol{r}$ in the first and second term, respectively. In the last term, we may put all but the $\partial f/\partial\boldsymbol{v}$ term out of the integral:

$$\frac{\partial}{\partial t}\int m\,f\,\mathrm{d}^3\boldsymbol{v} + \frac{\partial}{\partial \boldsymbol{r}}\cdot\int m\,\boldsymbol{v}f\,\mathrm{d}^3\boldsymbol{v} - \boldsymbol{\nabla}\left[V(\boldsymbol{r})+\Phi(\boldsymbol{r})\right]\cdot\int\frac{\partial f}{\partial \boldsymbol{v}}\,\mathrm{d}^3\boldsymbol{v} = 0\ . \tag{1.52}$$

First two integrals are by definion ρ and $\rho\,\boldsymbol{u}$, the third integral evaluates to $f(\boldsymbol{r},\,\boldsymbol{v})$ evaluated in infinity which is zero due to the normalisation condition (see Sec. 1.4 for discussion of properties of the distribution function implied by the normalisation constraint). Hence, the zeroth moment of the Boltzmann/Vlasov equation reads:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{u}) = 0\ . \tag{1.53}$$

1.5.2 Euler equation

First moment of the Vlasov equation is sometimes called Euler equation or fluid equation of motion. It is a vector equation and for the sake of better clarity, we will write it in an index notation, using the Einstein summation convention:

$$\int m\, v_i \frac{\partial f}{\partial t}\, \mathrm{d}^3\boldsymbol{v} + \int m\, v_i\, v_j \frac{\partial f}{\partial r_j}\, \mathrm{d}^3\boldsymbol{v} - \int v_i \frac{\partial}{\partial r_j} \left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] \frac{\partial f}{\partial v_j}\, \mathrm{d}^3\boldsymbol{v} = 0 \;. \tag{1.54}$$

Let's assume that derivation with respect to t and r_j may be exchanged with integration over the velocity space. We will get

$$\frac{\partial \rho\, u_i}{\partial t} + \frac{\partial}{\partial r_j}(\rho \langle v_i\, v_j \rangle) - \frac{\partial}{\partial r_j} \left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] \int v_i \frac{\partial f}{\partial v_j}\, \mathrm{d}^3\boldsymbol{v} = 0 \;, \tag{1.55}$$

where for arbitrary function $\mathcal{X}$ we denote $\rho\langle \mathcal{X} \rangle \equiv \int m\, \mathcal{X}\, f\, \mathrm{d}^3\boldsymbol{v}$. It can be easily shown (e.g. by means of integration by parts) that

$$\int v_i \frac{\partial f}{\partial v_j}\, \mathrm{d}^3\boldsymbol{v} = -\frac{\rho}{m}\, \delta_{ij} \tag{1.56}$$

and

$$\langle v_i\, v_j \rangle = u_i\, u_j + \langle w_i\, w_j \rangle \;,^{2} \tag{1.57}$$

where $w_i \equiv v_i - u_i$ represents "random" deviations of particles' motion from the mean bulk velocity. The symmetric tensor $\rho\langle w_i\, w_j\rangle$ can be split into its trace, $P \equiv \frac{1}{3}\rho\langle |\boldsymbol{w}|^2\rangle$, and traceless tensor $\pi_{ij} \equiv \rho\langle \frac{1}{3}|\boldsymbol{w}|^2\delta_{ij} - w_i\, w_j\rangle$ taken with a negative sign. The newly introduced quantities P and π_{ij} are called pressure and stress tensor, respectively. Using them, we can rewrite Euler equation to the form

$$\frac{\partial}{\partial t}(\rho\, u_i) + \frac{\partial}{\partial r_j}(\rho\, u_i\, u_j + P\delta_{ij} - \pi_{ij}) = -\frac{\rho}{m}\frac{\partial}{\partial r_i} \left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] \tag{1.58}$$

and with further small alterations, we obtain

$$u_i \frac{\partial \rho}{\partial t} + u_i \frac{\partial}{\partial r_j}(\rho\, u_j) + \rho \frac{\partial u_i}{\partial t} + \rho\, u_j \frac{\partial u_i}{\partial r_j} = -\frac{\rho}{m}\frac{\partial}{\partial r_i} \left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] - \frac{\partial P}{\partial r_i} + \frac{\partial \pi_{ij}}{\partial r_j} \;. \tag{1.59}$$

The sum of the first two terms on the left-hand side of eq. (1.59) is zero as it is the left-hand side of the continuity equation (1.53) multiplied by u_i. Hence, the standard form of the Euler equation reads:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial r_j} = -\frac{1}{m}\frac{\partial}{\partial r_i}\left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] - \frac{1}{\rho}\frac{\partial P}{\partial r_i} + \frac{1}{\rho}\frac{\partial \pi_{ij}}{\partial r_j} \;. \tag{1.60}$$

[2] $\langle v_i v_j\rangle = \frac{m}{\rho}\int \mathrm{d}^3\boldsymbol{v}\, f v_i v_j = \frac{m}{\rho}\int \mathrm{d}^3\boldsymbol{v}\, f(u_i + w_i)(u_j + w_j) = \frac{m}{\rho}\int \mathrm{d}^3\boldsymbol{v}\, [f u_i u_j + f u_i w_j + f u_j w_i + w_i w_j]$. Considering that u_i may be taken out of the integrals and $\int w_i f \mathrm{d}^3\boldsymbol{v} = \int (v_i - u_i) f \mathrm{d}^3\boldsymbol{v} = \int v_i f \mathrm{d}^3\boldsymbol{v} - u_i \int f \mathrm{d}^3\boldsymbol{v} = \frac{\rho}{m} u_i - u_i \frac{\rho}{m} = 0$ and using the definition of the mean value, we have $\langle v_i\, v_j\rangle = u_i\, u_j + \langle w_i\, w_j\rangle$.

Let's put down also a common vector notation of the Euler equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{f} - \frac{1}{\rho}\boldsymbol{\nabla}P + \frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\pi}\ , \tag{1.61}$$

where we have replaced gradient of potential energy by the specific force density $\boldsymbol{f}$. Let's put stress on meaning of the term $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$, which can be inferred from comparison of eqs. (1.60) and (1.61). The so called "convective derivative" is an imporant term in expressing the full time derivative,

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}, \tag{1.62}$$

hence the convective derivative represents the change of the vector field due to the motion of the particle, where the partial derivative represents the temporal evolution of the field at the given position.

1.5.3 Energy equation

Energy balance equation will be obtained multiplying eq. (1.48) with $\frac{1}{2}m|\boldsymbol{v}|^2$ and integrating over the velocity space. We will be briefer than in the previous two cases as the steps done are qualitatively the same. Full derivation may be found in Appendix A.1. Hence, after some algebra, we will obtain

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho(u^2+\langle w^2\rangle)\right] + \frac{\partial}{\partial r_i}\left[\tfrac{1}{2}\rho\langle(u_i+w_i)|\boldsymbol{u}+\boldsymbol{w}|^2\rangle\right] = -\frac{\rho}{m}\,u_i\frac{\partial}{\partial r_i}(V+\Phi)\ , \tag{1.63}$$

where we denote $u^2 \equiv |\boldsymbol{u}|^2$ for the sake of brevity. Introducing the (conductive) heat flow, $\boldsymbol{F} \equiv \frac{1}{2}\rho\langle \boldsymbol{w}\,w^2\rangle$ and keeping in mind that $\rho\mathcal{E} \equiv \frac{1}{2}\rho\langle w^2\rangle \equiv \frac{3}{2}P$, we rewrite (1.63) to the form

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + \rho\mathcal{E}\right) + \frac{\partial}{\partial r_i}\left[\tfrac{1}{2}\rho u^2\,u_i + u_j(P\delta_{ij}-\pi_{ij}) + \rho\mathcal{E}\,u_i + F_i\right] = -\frac{\rho}{m}\,u_i\frac{\partial}{\partial r_i}(V+\Phi)\ , \tag{1.64}$$

which can be further simplified by subtracting the scalar product of the Euler equation (1.58) and u_i,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho\,u^2) + \frac{\partial}{\partial r_i}(\tfrac{1}{2}\rho\,u^2\,u_i) = -\frac{\rho}{m}\,u_i\frac{\partial}{\partial r_i}(V+\Phi) - u_i\frac{\partial P}{\partial r_i} + u_i\frac{\partial \pi_{ij}}{\partial r_j}\ , \tag{1.65}$$

yielding

$$\frac{\partial}{\partial t}(\rho\,\mathcal{E}) + \frac{\partial}{\partial r_i}(\rho\,\mathcal{E}\,u_i) = -P\frac{\partial u_i}{\partial r_i} - \frac{\partial F_i}{\partial r_i} + \Psi\ , \tag{1.66}$$

where $\Psi \equiv \pi_{ij}\,\partial u_i/\partial r_j$ is a local dissipation measure. Equation (1.66) can be rewritten with help of the continuity equation into vector form

$$\rho\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -P\,\boldsymbol{\nabla}\cdot\boldsymbol{u} - \boldsymbol{\nabla}\cdot\boldsymbol{F} + \Psi\ . \tag{1.67}$$

Sometimes, there are extra terms added rather heuristically to the right-hand side of the energy balance equation, e.g. divergence of radiative heat transfer, in order to match the considered physical system. Note, however, that the rigorous addition of such terms from the first principles is usually not trivial.

1.5.4 Closing the equations set

We have obtained a set of five linearly independent momentum equations,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{u}) \;=\; 0\;, \tag{1.68}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} \;=\; \boldsymbol{f} - \frac{1}{\rho}\boldsymbol{\nabla} P + \frac{1}{\rho}\boldsymbol{\nabla} \cdot \boldsymbol{\pi}\;, \tag{1.69}$$

$$\rho \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \;=\; -P\,\boldsymbol{\nabla} \cdot \boldsymbol{u} - \boldsymbol{\nabla} \cdot \boldsymbol{F} + \Psi \tag{1.70}$$

for 14 independent variables, ρ, $\boldsymbol{u}$ (3 components), P, $\boldsymbol{\pi}$ (5 independent components), $\boldsymbol{F}$ (3 components) and Φ (which is hidden in $\boldsymbol{f}$). Internal energy $\mathcal{E}$ is not independent, as it is proportional to pressure p, similarly the local dissipation measure Ψ is defined using the stress tensor $\boldsymbol{\pi}$ and $\boldsymbol{u}$. Such a description of a physical problem is clearly not solvable. Hence, various approximations are used to overcome this problem.

A large simplification, just sufficient to close the problem from the mathematical point of view, is achieved for isotropic systems with $f(\boldsymbol{r},\, \boldsymbol{v}) = f(\boldsymbol{r},\, |\boldsymbol{v}|)$, which implies that $\pi_{ij} = 0$ and $F_i = 0$ (see Appendix A.2).

Quite often, a so-called Navière-Stokes approximation is used, assuming the stress tensor to be proportional to the tensor of deformation,

$$\pi_{ij} = \mu D_{ij} = \mu \left[\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} - \frac{3}{2}\frac{\partial u_k}{\partial r_k}\delta_{ij} \right]\;, \tag{1.71}$$

which reduces the number of independent variables by 5. The constant of proportionality, μ, is called dynamical viscosity.

Another step towards a closed set of equations may lie in heuristic introduction of temperature, T, equation of ideal gas, $P = \rho/m\, K_{\mathrm{B}} T$ and an assumption

$$\boldsymbol{F} = -\kappa\, \boldsymbol{\nabla} T\;. \tag{1.72}$$

With κ being a constant determining the heat conduction.[3]

Finally, if an internal potential Φ is not neglected, some variant of Poisson equation, e.g.

$$\triangle \Phi = 4\pi\, G\, \rho\;, \tag{1.73}$$

completes the set. Note that a similar Poisson equation holds for the external potential

$$\triangle V = 4\pi\, G\, \rho_{\mathrm{ext}}\;, \tag{1.74}$$

where ρ_{ext} indicates the external density generating the external potential.

[3] Quite often the conductive heat flow, $\boldsymbol{F}$, is, actually, assumed to be zero or negligible but, on the other hand, radiative heat flow is considered. In such a case, equation of radiative transfer, formally similar to (1.72) is introduced with κ being an opacity.

1.6 Thermodynamical equilibria

So far, we attempted to downscale the problem of evolution of the statistically described system to a space (much) less-dimensional than the phase space. There is, however, a good reason to describe the system in its full complexity in a specific case – when it is in a thermodynamic equilibrium. We postulate that the distribution function corresponding to the equilibrium state is the one for which the statistical entropy,

$$S(D) \equiv -K_{\rm B} {\rm Tr}\left(D \ln D\right) , \tag{1.75}$$

reaches a maximum value. (Where to add a section on the statistical entropy? Here, or to Sec. 1.2?) For example, let's consider a system with a finite number, N, of possible states. The distribution function (probability in this case) for each state i is then $D(i) = 1/N$. Nevertheless, the definition of the equilibrium system is more general and can be applied also to statistical systems with some additional information. In particular, we usually consider a given mean value, $\langle A \rangle \equiv {\rm Tr}(AD)$, of some physical quantity which can be assigned to the state of the system (i.e. measured in the real world) to play the role of the constraint on the distribution function.

Let's have a set of n physical quantities A_i, mean values of which are given. We are looking for the form of the distribution function which maximises value of $S(D)$ and fulfills $n+1$ constraints $\langle A_i \rangle = {\rm Tr}(A_i D)$ and ${\rm Tr}D = 1$. A possible way to solve this task is via variation:

$$\eth \left\{ \frac{1}{K_{\rm B}} S(D) - \sum_{i=0}^{n} \lambda_i \left[{\rm Tr}(A_i D) - \langle A_i \rangle\right] \right\} = 0 \,, \tag{1.76}$$

where λ_i are Lagrange multipliers and $A_0 \equiv \mathbb{1}$. Using the explicit form of A_0 and $S(D)$ equation (1.76) leads to

$$-{\rm Tr}\left[(\ln D + \mathbb{1})\eth D\right] - \lambda_0 {\rm Tr}(\eth D) - \sum_{i=1}^{n} \lambda_i {\rm Tr}(A_i \eth D) = 0 \,, \tag{1.77}$$

which may be further rewritten as

$$-{\rm Tr}\left[\ln(D)\eth D\right] - \lambda_0' {\rm Tr}(\eth D) - \sum_{i=1}^{n} \lambda_i {\rm Tr}(A_i \eth D) = 0 \,, \tag{1.78}$$

where $\lambda_0' \equiv \lambda_0 + 1$. Finally, dropping prime from λ_0', we obtain:

$${\rm Tr}\left[\left(-\ln D - \sum_{i=0}^{n} \lambda_i A_i\right) \eth D\right] = 0 \,. \tag{1.79}$$

It can be shown (do it!) that, in order to fulfill relation (1.79) for arbitrary variation $\eth D$, the term in round brackets has to be zero, i.e.

$$D = \exp\left[-\sum_{i=0}^{n} \lambda_i A_i\right] . \tag{1.80}$$

Hence, we have an explicit form of the distribution function (functions/operators A_i are given) with $n+1$ parameters λ_i which are related to given values of $\langle A_i \rangle$. Let us express λ_0 for convenience. First, rewrite (1.80) to

$$D = \mathrm{e}^{-\lambda_0} \exp\left[-\sum_{i=1}^{n} \lambda_i A_i\right] . \tag{1.81}$$

Then, consider that $\langle A_0 \rangle = \mathrm{Tr} D = 1$ and apply the operator Tr to equation (1.81) which gives

$$\mathrm{e}^{-\lambda_0} \mathrm{Tr} \left\{ \exp\left[-\sum_{i=1}^{n} \lambda_i A_i\right] \right\} = 1 . \tag{1.82}$$

Instead of λ_0 itself, let us introduce so-called partition function (also known as partition sum):

$$Z \equiv \mathrm{e}^{\lambda_0} = \mathrm{Tr} \left\{ \exp\left[-\sum_{i=1}^{n} \lambda_i A_i\right] \right\} . \tag{1.83}$$

The equilibrium distribution function 1.80 can be then rewritten as

$$D = \frac{1}{Z} \exp\left[-\sum_{i=1}^{n} \lambda_i A_i\right] . \tag{1.84}$$

Partition function Z keeps a lot of information about the system. It can be easily shown that, e.g.

$$\frac{\partial \ln Z}{\partial \lambda_i} = -\langle A_i \rangle \quad \text{and} \quad \frac{\partial^2 \ln Z}{\partial \lambda_i \partial \lambda_j} = \langle A_i A_j \rangle - \langle A_i \rangle \langle A_j \rangle . \tag{1.85}$$

Similarly, we may write relation for entropy S and the Lagrange multipliers:

$$\frac{\partial S}{\partial \langle A_i \rangle} = K_{\mathrm{B}} \lambda_i . \tag{1.86}$$

(Can the equilibrium state (of isolated system) be reached starting from non-equilibrium one?)

1.6.1 Statistical ensembles

The statistical systems appear often static from a human perspective (despite microscopic motions, which are not directly observable) and this allows us to describe these systems by a set of macroscopic variables. Such systems may be described by *statistical ensembles*, which are assumed to be in a statistical equilibrium, and given various macroscopic constraints, these ensembles depend on a few observable parameters. Three important thermodynamic ensembles were defined for a fixed volume Ω.

Microcanonical ensemble is a statistical ensemble, where the total energy of the system and the number of particles in the system are known (they each have particular fixed values). The system is totally isolated, no exchange of the energy or particles with the environment is allowed. Each member of the ensemble has the same total energy and the same total number of particles. Following (1.84) the distribution function holds the shape of

$$D_{\mathrm{N}} = \frac{1}{Z_{\mathrm{M}}}. \tag{1.87}$$

Canonical ensemble is a statistical ensemble, where the energy is not known exactly, but the number of particles is fixed. The canonical ensemble is suitable for the description of a system, which has heat exchange with a surrounding environment, however, there is not particle exchange with the surroundings. Each member of the ensemble has the same number of particles, their energies are not defined, however, there is a constraint on mean energy (usually expressed by a temperature), which is given by the temperature of the surrounding bath. Following (1.84) the distribution function holds the shape of

$$D_{\mathrm{N}} = \frac{1}{Z_{\mathrm{C}}} \exp\left[-\beta H\right], \tag{1.88}$$

where $\beta \equiv \lambda_H$ is the Lagrange multiplier for the Hamiltonian, which can be expressed by using the temperature T as $\beta = 1/K_{\mathrm{B}}T$, and Z_{C} is the partition function for the canonical ensemble. Note that the distribution function bears the index N because the number of particles is known.

Grand-canonical ensemble is a statistical ensemble, where neither the energy nor the particle number is fixed. Similarly to the previous case, instead of the total energy the temperature is defined, and in place of the total number of particles, the chemical potential is defined. The grand-canonical ensemble is appropriate for the description of open systems, which has a thermal contact with the surroundings and exchanges particles with a reservoir. For members of the grand-canonical ensemble the temperature and the chemical potential of the reservoir places constraints to the mean energy of each member and mean number of particles. Following (1.84) the distribution function takes form

$$D = \frac{1}{Z_{\mathrm{G}}} \exp\left[-\beta H + \alpha N\right], \tag{1.89}$$

where $\alpha \equiv -\lambda_N \equiv \beta\mu$ represents the Lagrange multiplier for number of particles and may be expressed also by using the chemical potential μ.

In the following, we will use the properties of both the canonical and grand-canonical ensembles in order to derive the equilibrium particle distribution function.

1.6.2 Maxwell-Boltzmann distribution

We will derive the particle distribution function for an open system of particles. We will use the calculation for a grandcanonical ensemble. We remind for clarity that the usage of such a statistical ensemble means that we do not know the exact total energy of the particles in the volume nor the total number of particles. However, the constraints are given to mean values of energy and mean number of particles, these quantities are further denoted by a temperature T and a chemical potential μ. Elaborating Eq. (1.84) for this case the distribution function is given by

$$D = \frac{1}{Z_\mathrm{G}} \exp\left[-\beta H + \alpha N\right], \tag{1.90}$$

where Z_G is a partition function of the grandcanonical ensemble and the two operators connected with the two known constraints are H, Hamiltonian as an operator of energy and N is a formal operator of number of particles. These two operators use Lagrange multipliers β, which represents the temperature, and α representing the chemical potential. Note that the argument of the exponential is a hybrid function, because the hamiltonian H is continuous, whereas the operator of number of particles N is discrete.

The grandcanonical partition function may be written using its canonical counterpart Z_C

$$Z_\mathrm{G} = \sum_{N=0}^{\infty} e^{\alpha N} Z_\mathrm{C}(\beta, N), \tag{1.91}$$

which basically says that the partition function (also termed as *sum over states*) simply is a sum of partition functions of the canonical ensembles with a fixed number of particles N with a given temperature hidden in β.

For a hamiltonian H_N of N noninteracting particles (point masses) in the external potential $V(\boldsymbol{r})$ given by

$$H_N = \sum_i H_1(\boldsymbol{r}_i, \boldsymbol{p}_i), \tag{1.92}$$

where

$$H_1(\boldsymbol{r}, \boldsymbol{p}) = p^2/(2m) + V(\boldsymbol{r}) \tag{1.93}$$

is a hamiltonian of one particle in the external potential, we may write the canonical partition function as

$$Z_\mathrm{C} = \int_{-\infty}^{+\infty} \exp\left(-\beta H_N\right) \mathrm{d}\tau_N, \tag{1.94}$$

where for $\mathrm{d}\tau_N$ we use a slightly modified definition

$$\mathrm{d}\tau_N \equiv \frac{\mathrm{d}^3\boldsymbol{r}_1 \mathrm{d}^3\boldsymbol{p}_1 \ldots \mathrm{d}^3\boldsymbol{r}_N \mathrm{d}^3\boldsymbol{p}_N}{\hbar^{3N} N!}. \tag{1.95}$$

The advantage of utilising the factor $\hbar^{3N}$ in the denominator is that $\mathrm{d}\tau_N$ is then dimension-less, as is a corresponding distribution function D.

Turning the sum over N particles into a multiplication (it is possible because the functions H_1 are the same) we obtain

$$Z_{\mathrm{C}}(\beta,N)=\int\exp\left[-\beta H_N\right]\mathrm{d}\tau_N=\int\limits_{-\infty}^{+\infty}\prod_{i=1}^{N}\left(e^{-\beta p_i^2/(2m)}e^{-\beta V(\boldsymbol{r}_i)}\mathrm{d}^3\boldsymbol{r}_i\mathrm{d}^3\boldsymbol{p}_i\right)\frac{1}{\hbar^{3N}N!}=$$

$$=\frac{1}{\hbar^{3N}N!}\left[\int\limits_{-\infty}^{+\infty}e^{-\beta p^2/(2m)}e^{-\beta V(\boldsymbol{r})}\mathrm{d}^3\boldsymbol{r}\mathrm{d}^3\boldsymbol{p}\right]^N. \tag{1.96}$$

Thanks to the unification of the description of the energy we may perform integrations both over the positions and momenta. The integration over the momenta is simpler, as it is in fact an Gauss integral $\int_{-\infty}^{+\infty}\exp\left(-x^2\right)\mathrm{d}x=\sqrt{\pi}$. The integration in the spatial domain requires a bit of the justification.

In order to be able to perform the integration in the spatial domain, we must define the external potential V. To keep things simple, we would require an integrable potential, which is zero outside the region of interest (outside our volume Ω, where we study the problem), and which has some value inside Ω. It is easy to show that the potential well will have exactly those properties. Then the integration of $\boldsymbol{r}$ leads to a scaled volume of our region of interest.

$$\Omega\Xi\equiv\int\limits_{-\infty}^{+\infty}e^{-\beta V(\boldsymbol{r})}\mathrm{d}^3\boldsymbol{r}. \tag{1.97}$$

Hence, we have

$$Z_{\mathrm{C}}(\beta,N)=\frac{1}{\hbar^{3N}N!}\left(2\pi mK_{\mathrm{B}}T\right)^{3N/2}\Omega^N\Xi^N\equiv\frac{Z_1^N}{N!}, \tag{1.98}$$

where

$$Z_1\equiv\frac{\left(2\pi mK_{\mathrm{B}}T\right)^{3/2}}{\hbar^{3N}}\Omega\Xi. \tag{1.99}$$

By choosing the potential V in a form of an infinite potential well (i.e. $V(\boldsymbol{r})=0$ for $\boldsymbol{r}\in\Omega$ and $V(\boldsymbol{r})=+\infty$ for $\boldsymbol{r}\notin\Omega$), we set $\Xi=1$ and the integration of the spatial part in (1.96) exactly leads to the volume of the region of interest. Beware of a conflicting notation, when Ω means both the regions of interest and the volume of this region. Further note that we defined a particle partition function $Z_1(\beta)\equiv Z_{\mathrm{C}}(\beta,N=1)$.

In the following, we will apply the above discussed approximation of the infinite potential well. By combining (1.91) and (1.98) we have

$$Z_{\mathrm{G}}=\sum_{N=0}^{\infty}\frac{1}{N!}\left(e^{\alpha}Z_1\right)^N=\exp\left[e^{\alpha}Z_1\right], \tag{1.100}$$

where we used the Taylor expansion of the exponential function.

Now we search for a probability of finding a particle in spatial point $\boldsymbol{r}$ with a momentum of $\boldsymbol{p}$. The procedure is somewhat complicated because we don't know the exact number of particles in our open system. The sought probability f is[4], however, given by

$$f(\boldsymbol{r},\boldsymbol{p}) \equiv \sum_{N=0}^{\infty} \left[\mathcal{P}(N) \sum_{i=1}^{N} \mathcal{P}_N(i,\boldsymbol{r},\boldsymbol{p}) \right], \tag{1.101}$$

where $\mathcal{P}(N)$ evaluates the probability that there is exactly N particles in the ensemble and $\mathcal{P}_N(i,\boldsymbol{r},\boldsymbol{p})$ indicates the probability that i-th particle (from fixed number of N particles) can be found around coordinates $(\boldsymbol{r},\boldsymbol{p})$ in the phase space. $\mathcal{P}(N)$ may be intuitively written as a trace of the full-system distribution function D[5] from (1.90):

$$D \sim \begin{bmatrix} \frac{1}{Z_\mathrm{G}} & & & \\ & \frac{1}{Z_\mathrm{G}} e^{\alpha} \exp(-\beta H_1) & & \\ & & \frac{1}{Z_\mathrm{G}} e^{2\alpha} \exp(-\beta [H_1(\boldsymbol{r}_1,\boldsymbol{p}_1) + H_1(\boldsymbol{r}_2,\boldsymbol{p}_2)]) & \\ & & & \ddots \end{bmatrix} \tag{1.102}$$

with diagonal terms $D_{N,N}$ defined by

$$D_{N,N} = \frac{1}{Z_\mathrm{G}} e^{\alpha N} \exp\left[-\beta \sum_{i=1}^{N} H_1(\boldsymbol{r}_i,\boldsymbol{p}_i) \right]. \tag{1.103}$$

Hence (using eq. **??**)

$$\mathcal{P}(N) = \int_{-\infty}^{+\infty} D_{N,N} \mathrm{d}\tau_N = \frac{e^{\alpha N} Z_\mathrm{C}(\beta,N)}{Z_\mathrm{G}(\alpha,\beta)}. \tag{1.104}$$

Further, $\mathcal{P}_N$ is simply defined for 1-particle probability (probability of finding particle carrying flag 1 on phase coordinates $(\boldsymbol{r},\boldsymbol{p})$) by

$$\mathcal{P}_N(1,\boldsymbol{r},\boldsymbol{p}) = \int_{-\infty}^{+\infty} D_N \frac{\mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 \ldots \mathrm{d}^3\boldsymbol{r}_N \mathrm{d}^3\boldsymbol{p}_N}{\hbar^{3N} N!} = \frac{1}{N\hbar^3} \int_{-\infty}^{+\infty} D_N \mathrm{d}\tau_{N-1} \tag{1.105}$$

where $D_N = e^{-\beta H_N}/Z_\mathrm{C}(\beta,N)$ is the distribution function for the canonical ensemble.

[4]Let us take a short excursion to the past chapters. The notation $f(\boldsymbol{r},\boldsymbol{p})$ was already used when we discussed the reduced distribution functions of N particles – see eq. (1.24). The question is, what is the relation of (1.24) to (1.101). We need to keep in mind that in the past chapters our ensemble consisted of a fixed number of particles. Therefore strictly speaking, for the previous case $\mathcal{P}(N) = 1$ and the first sum disappears. Then the two fs have the same meaning.

[5]Note that D does not have a simple dimension. It belongs to within the Fock space, which is formed as a direct sum of tensor products of Hilbert spaces with varying dimensions. A proper treatment is beyond the scope of this text, the interested reader is pointed to elsewhere.

We further use (1.98) to have

$$\mathcal{P}_N(1,\boldsymbol{r},\boldsymbol{p}) = \frac{1}{N\hbar^3}\int\limits_{-\infty}^{+\infty}\frac{e^{-\beta H_N}\mathrm{d}\tau_{N-1}}{\frac{1}{N!}Z_1^N(\beta)} = \frac{1}{N\hbar^3}\int\limits_{-\infty}^{+\infty}\frac{N(N-1)!e^{-\beta H_N}\mathrm{d}\tau_{N-1}}{Z_1^N(\beta)} =$$
$$= \frac{(N-1)!}{\hbar^3}e^{-\beta H_1(\boldsymbol{r},\boldsymbol{p})}\frac{1}{Z_1^N}\int e^{-\beta H_{N-1}}\mathrm{d}\tau_{N-1} = \frac{1}{\hbar^3}\frac{e^{-\beta H_1(\boldsymbol{r},\boldsymbol{p})}}{Z_1(\beta)}. \tag{1.106}$$

Note that the factor of $\exp\left(-\beta H_1(\boldsymbol{r},\boldsymbol{p})\right)$ could be taken out of the integral, because we do not integrate over $\boldsymbol{r}$ and $\boldsymbol{p}$ coordinates. Furthermore, in the last step we used

$$\int e^{-\beta H_{N-1}}\mathrm{d}\tau_{N-1} = \frac{1}{(N-1)!}Z_1^{(N-1)}, \tag{1.107}$$

which follows from (1.94) and (1.98). The 1-particle hamiltonian H_1 is defined by (1.93).

Then (1.101) takes a form of

$$f(\boldsymbol{r},\boldsymbol{p}) = \sum_{N=0}^{\infty}\frac{e^{\alpha N}Z_{\mathrm{C}}(\beta,N)}{Z_{\mathrm{G}}(\beta,\alpha)}N\frac{1}{\hbar^3}\frac{e^{-\beta H_1}}{Z_1}, \tag{1.108}$$

where the second sum over i in (1.101) was turned into multiplication by N, as we have N undistinguishable particles. We further elaborate (1.108) by using (1.98) and by moving terms not depending on N to the front of the sum to obtain:

$$f(\boldsymbol{r},\boldsymbol{p}) = \frac{1}{\hbar^3 Z_{\mathrm{G}}}\sum_{N=0}^{\infty}\frac{e^{\alpha N}Z_1^N}{N!}N\frac{e^{-\beta H_1}}{Z_1} = \frac{e^{\alpha}e^{-\beta H_1}}{\hbar^3}\frac{\sum\limits_{N=1}^{\infty}\frac{e^{\alpha(N-1)}Z_1^{N-1}}{(N-1)!}}{\exp\left[e^{\alpha}Z_1\right]}, \tag{1.109}$$

where used (1.100) and realised that the summation index N must start from 1 instead of the original 0, because the expression $(N-1)!$ is not defined for $N=0$[6]. Now we realise that the sum in the numerator is the Taylor expansion of the denominator and hence finally

$$f(\boldsymbol{r},\boldsymbol{p}) = \frac{e^{\alpha-\beta H_1(\boldsymbol{r},\boldsymbol{p})}}{\hbar^3}. \tag{1.110}$$

Now we use the fact that the mean values of physical quantities may be expressed from the partition function (1.85):

$$\langle N\rangle = \frac{\partial \ln Z_{\mathrm{G}}}{\partial\alpha} = \frac{\partial}{\partial\alpha}e^{\alpha}Z_1 = e^{\alpha}Z_1 \tag{1.111}$$

and our previous finding that $\int_{-\infty}^{+\infty}\exp[-\beta V(\boldsymbol{r})] = \Omega$ to finally have

$$f'(\boldsymbol{p}) = \int\limits_{-\infty}^{+\infty} f(\boldsymbol{r},\boldsymbol{p})\mathrm{d}^3\boldsymbol{r} = \int\limits_{-\infty}^{+\infty}\mathrm{d}^3\boldsymbol{r}\frac{1}{\hbar^3}e^{\alpha}\exp\left\{-\beta\left[p^2/(2m)+V(\boldsymbol{r})\right]\right\} =$$
$$= \frac{\Omega}{\hbar^3}\exp\left[\alpha-\beta\frac{p^2}{2m}\right] = \langle N\rangle g(\boldsymbol{p}). \tag{1.112}$$

[6]Or alternatively we may say that for $N=0$ the fraction $N/N! = 0$ and therefore the first term in the sum vanished.

Note that the expression $\langle N\rangle/\Omega \equiv n$ has the meaning of the particle density n.

Function $g(\boldsymbol{p})$ is what is our final solution to the problem and is usually termed the *Maxwell-Boltzmann distribution function.* Explicitly written by using (1.111) and (1.98) we have

$$g(\boldsymbol{p}) = \frac{\Omega}{\hbar^3} e^{-\frac{\beta p^2}{2m}} \frac{1}{Z_1} = \frac{e^{-\beta p^2/2m}}{(2\pi K_\mathrm{B} m T)^{3/2}} = \frac{e^{-\frac{p^2}{2K_\mathrm{B}Tm}}}{(2\pi K_\mathrm{B} m T)^{3/2}}. \tag{1.113}$$

1.6.3 Chemical equilibrium, Saha's equation

Let's consider a system of three different kinds of particles – free electrons and protons and neutral hydrogen atoms. They play a role in a chemical equation $\mathrm{e} + \mathrm{p} \rightleftarrows \mathrm{H}$, where the double arrow indicates that the system is in the equilibrium, therefore the speed of the reactions to the left is the same as the speed of the reverse reaction. Let's define the system by a simple hamiltonian

$$H = \sum_{i=1}^{\mathrm{N^e}} H_1^\mathrm{e}(\boldsymbol{r}_i\, \boldsymbol{p}_i) + \sum_{i=1}^{\mathrm{N^p}} H_1^\mathrm{p}(\boldsymbol{r}_i\, \boldsymbol{p}_i) + \sum_{i=1}^{\mathrm{N^H}} H_1^\mathrm{H}(\boldsymbol{r}_i\, \boldsymbol{p}_i) \tag{1.114}$$

with

$$H_1^a(\boldsymbol{r},\, \boldsymbol{p}) = \frac{p^2}{2m^a} + V(\boldsymbol{r}) + h_1^a\ , \quad a = \{\mathrm{e},\, \mathrm{p},\, \mathrm{H}\}\ . \tag{1.115}$$

Hamiltonians h_1^a correspond to internal degrees of freedom and we will assume them to have a discrete spectrum. Partition function of this grand-canonical ensemble can be written in the form

$$Z_\mathrm{G}(\beta,\, \mu^\mathrm{e},\, \mu^\mathrm{p},\, \mu^\mathrm{H}) = \prod_a \{\mathrm{Tr}\,[\exp(\beta\mu^a N^a)]\ \mathrm{Tr}\,[\exp(-\beta H^a)]\}\ . \tag{1.116}$$

The partition function of a canonical ensemble of N^a particles of the species a reads:

$$\begin{aligned} Z_\mathrm{C}^a(\beta,\, \mathrm{N}^a) &= \frac{1}{\mathrm{N}^a!}\left(\int \exp\left[-\beta\left(\frac{p^2}{2m^a} + V(\boldsymbol{r})\right)\right] \frac{\mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{p}}{\hbar^3}\right)^{\mathrm{N}^a} \left(\mathrm{Tr}\big\{\exp\left(-\beta h_1^a\right)\big\}\right)^{\mathrm{N}^a} \\ &= \frac{1}{\mathrm{N}^a!}\left[\frac{1}{\hbar^3}\left(2\pi m^a K_\mathrm{B} T\right)^{3/2} \Omega\, \zeta^a(T)\right]^{\mathrm{N}^a} = \frac{1}{\mathrm{N}^a!}\left[Z_1^a(T)\,\zeta^a(T)\right]^{\mathrm{N}^a}\ , \end{aligned} \tag{1.117}$$

(Dat stabni kulturu vnorenym zavorkam) here again we used an integration over the spatial domain which defines the fixed volume of the configuration space accessible to the particles (here we assume $V(\boldsymbol{r})$ to represent an infinitely deep potential well),

$$\Omega \equiv \int \exp\left[-\beta\, V(\boldsymbol{r})\right] \mathrm{d}^3\boldsymbol{r}\ , \tag{1.118}$$

Furthermore,

$$\zeta^a(T) \equiv \mathrm{Tr}\big\{\exp\left(-\beta h_1^a\right)\big\} = \sum d_i^a \exp\left[-\beta \mathcal{E}_i^a\right]\ , \tag{1.119}$$

where d_i^a is degeneracy of the energy level $\mathcal{E}_i^a$. Finally, let us assume for simplicity that each type of particles has only one internal energy level,

$$\begin{aligned} \zeta^{\mathrm{e}} &= d^{\mathrm{e}} \exp\left[-\beta \mathcal{E}_0^{\mathrm{e}}\right] \\ \zeta^{\mathrm{p}} &= d^{\mathrm{p}} \exp\left[-\beta \mathcal{E}_0^{\mathrm{p}}\right] \\ \zeta^{\mathrm{H}} &= d_0 \exp\left[-\beta(\mathcal{E}_0^{\mathrm{e}} + \mathcal{E}_0^{\mathrm{p}} - U_{\mathrm{i}})\right] , \end{aligned} \tag{1.120}$$

where $d^{\mathrm{e}} = d^{\mathrm{p}} = 2$ due to spin degeneracy of free electrons and protons and d_0 denotes the degeneracy of the ground state of the hydrogen atom with (positive) ionisation energy U_{i}. Inserting (1.117) into (1.116) and replacing trace in the space of N^a by sum, we obtain

$$Z_{\mathrm{G}}(\beta,\, \mu^{\mathrm{e}},\, \mu^{\mathrm{p}},\, \mu^{\mathrm{H}}) = \prod_a \sum_{\mathrm{N}^a} \frac{1}{\mathrm{N}^a!} \left[\mathrm{e}^{\beta\mu^a} Z_1^a\, \zeta^a(T)\right]^{\mathrm{N}^a} = \prod_a \exp\left[\mathrm{e}^{\beta\mu^a} Z_1^a\, \zeta^a(T)\right] . \tag{1.121}$$

This allows us to evaluate the mean number of particles of species a,

$$\langle \mathrm{N}^a \rangle = \frac{1}{\beta} \frac{\partial \ln Z_{\mathrm{G}}}{\partial \mu^a} = \frac{(2\pi m^a K_{\mathrm{B}} T)^{3/2}}{\hbar^3} \Omega\, \zeta^a(T) \exp\left(\frac{\mu^a}{K_{\mathrm{B}} T}\right) . \tag{1.122}$$

Chemical equilibrium means that the rate of the reaction $\mathrm{e} + \mathrm{p} \rightleftarrows \mathrm{H}$ is equal in both directions, which can in the thermodynamical limit be expressed in terms of chemical potentials,

$$\mu^{\mathrm{e}} + \mu^{\mathrm{p}} = \mu^{\mathrm{H}} . \tag{1.123}$$

Equality (1.123) can be rewritten in more complicated way:

$$\prod_a \left[\exp\left(\frac{\mu^a}{K_{\mathrm{B}} T}\right)\right]^{\nu^a} = 1 \,, \text{ with } \quad \nu^{\mathrm{e}} = \nu^{\mathrm{p}} = 1,\ \nu^{\mathrm{H}} = -1 . \tag{1.124}$$

Inserting (1.122) into (1.124) and denoting $n_a \equiv \langle \mathrm{N}^a \rangle / \Omega$, we obtain:

$$\prod_a \left[n^a \frac{\hbar^3}{(2\pi m^a K_{\mathrm{B}} T)^{3/2}} \frac{1}{\zeta^a(T)} \right]^{\nu^a} = 1 \tag{1.125}$$

which may be finally rewritten with use of (1.121), $m_{\mathrm{H}} \approx m_{\mathrm{p}}$ and $n_{\mathrm{e}} \approx n_{\mathrm{p}}$ to the so-called Saha's equation for equilibrium of ionised and neutral atoms,

$$\frac{n_{\mathrm{e}}}{n_{\mathrm{H}}} = \frac{4}{d_0} \frac{1}{n_{\mathrm{e}}} \frac{(2\pi m^{\mathrm{e}} K_{\mathrm{B}} T)^{3/2}}{\hbar^3} \exp\left(-\frac{U_{\mathrm{i}}}{K_{\mathrm{B}} T}\right) . \tag{1.126}$$

Chapter 2

Plasma

Plasma is being defined as a *quasi-neutral gas of mixture of charged and neutral particles showing a collective behaviour*. It is characterised by its density n, temperature T (which is usually bound to the width of the velocity distribution, the Maxwell velocity distribution (1.113) for the equilibrium states), and the ionisation degree (usually well described by the Saha equation (1.126)). The *collective behaviour* indicates that the electromagnetic interaction is long-range one, hence the behaviour of each particle of the ensemble influence the behaviour of all other particles[1]. The *quasineutrality* indicates that on the large scales, the possible charge concentrations are efficiently shielded and thus not observable from the outside. The shielding length scale is termed a *Debye length* λ_{D}.

In the system of charged particles, each particle induces an electric potential, which interacts with the electric potential of the surrounding particles. The particles are free to move. Assuming that the system is in the thermodynamical equilibrium or close to it, any disbalance of the charge is almost immediately shielded in order to preserve the whole system electrically quasi neutral. An illustrative conception of the shielding process may be descibed in the following model (see Fig. 2.1). In the system of charged

[1]The Coulomb electric interaction decreases as $1/r^2$, where r is the distance, whereas for the given solid angle (where $\Delta r/r = \mathrm{const}$) the volume of the plasma increases as r^3. Hence altogether the electric force does not vanish to large distances within the plasma.

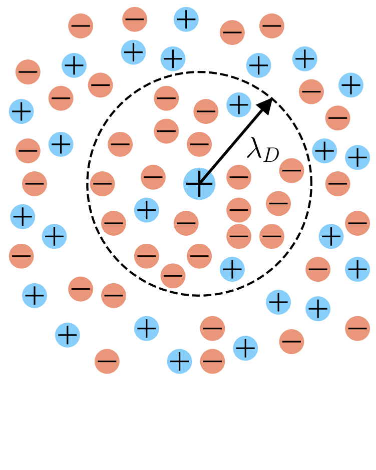

Figure 2.1: The concept of Debye shielding.

particles, suddenly a concentration of one charge appears. There are random mechanisms that can drive such clumping. The particles of the opposite charge then cluster around this clump to remove the charge disbalance. We need to mention explicitly that the cluster of the opposite charge is not stationary. Thermal motions of particles are very large, so particles propagate through shielding region almost freely, only the electric potential of the initial charge concentration slows them down. The propagating particles spend longer time in the shielding region creating an apparent opposite charge clump. In the system, there exists a characteristic length scale, on which the charge concentration is shielded.

In order to derive this length scale, we will start with the Poisson equation, which binds the electric intensity $\boldsymbol{E}$ and the charge density ρ_{e}:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{\rho_{\mathrm{e}}(\boldsymbol{r})}{\varepsilon_0}, \tag{2.1}$$

where ε_0 is a permitivity of the vacuum. The electric-field intensity may be expressed by using the potential of the electric field ϕ as

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r}), \tag{2.2}$$

hence the Poisson equation gains the form

$$\triangle\phi(\boldsymbol{r}) = -\frac{\rho_{\mathrm{e}}(\boldsymbol{r})}{\varepsilon_0}. \tag{2.3}$$

Next we search for the ϕ which solves the Poisson equation. Assuming that the plasma is near equilibrium, we may obtain the expression for ρ_{e} from the Maxwell-Boltzmann distribution function. For particles of type α in the equilibrium when the motion of particles is influenced by an external potential $q_\alpha\phi$, the velocity distribution function gains form

$$f_\alpha(\boldsymbol{r}, \boldsymbol{v}) = A_\alpha \exp\left[-\frac{\frac{1}{2}m_\alpha v^2 + q_\alpha\phi(\boldsymbol{r})}{K_{\mathrm{B}}T_\alpha}\right], \tag{2.4}$$

where A_α is a normalisation constant. The integration of (2.4) over the velocity space yields the particle density n_α. In this case, the density may be a function of the electric potential ϕ. We remind the reader that in principle, ϕ may be a function of position:

$$n_\alpha(\phi) = \int_0^\infty f_\alpha(\boldsymbol{v})\mathrm{d}^3\boldsymbol{v} = \exp\left[-\frac{q_\alpha\phi(\boldsymbol{r})}{K_{\mathrm{B}}T_\alpha}\right]\int_0^\infty A_\alpha \exp\left[-\frac{\frac{1}{2}m_\alpha v^2}{K_{\mathrm{B}}T_\alpha}\right]\mathrm{d}^3\boldsymbol{v}. \tag{2.5}$$

If there was not electric potential, i.e., when $\phi = 0$, then the integral in the above equation would directly yield the particle density. In the case $\phi \neq 0$ it can be interpreted as a *background* or unperturbed density. The functional form for the particle density then yields

$$n_\alpha(\phi) = n_{\alpha,0} \exp\left[-\frac{q_\alpha\phi(\boldsymbol{r})}{K_{\mathrm{B}}T_\alpha}\right]. \tag{2.6}$$

Using the particle density we may now express the charge density as a sum of the respective particle densities multiplied by the particle charges:

$$\rho_{\mathrm{e}}(\phi) = \sum_{\alpha} q_\alpha n_\alpha(\phi(\boldsymbol{r})). \tag{2.7}$$

Consequently, (2.3) may be written as

$$\triangle\phi(\boldsymbol{r}) = -\frac{1}{\varepsilon_0} \sum_{\alpha} q_\alpha n_{\alpha,0} \exp\left[-\frac{q_\alpha \phi(\boldsymbol{r})}{K_{\mathrm{B}} T_\alpha}\right]. \tag{2.8}$$

We will further assume that the electric potential will only perturb the background density, hence we may expand the exponential in a Taylor expansion:

$$\triangle\phi(\boldsymbol{r}) = -\frac{1}{\varepsilon_0} \left[\sum_{\alpha} q_\alpha n_{\alpha,0} - \sum_{\alpha} \frac{q_\alpha^2 n_{\alpha,0} \phi(\boldsymbol{r})}{K_{\mathrm{B}} T_\alpha} + \ldots\right]. \tag{2.9}$$

The constant term in the Taylor expansion must vanish, otherwise the quasineutrality of the plasma cannot be kept. Finally we arrive at the suitable form of the Poisson equation

$$\triangle\phi(\boldsymbol{r}) = \left[\sum_{\alpha} \frac{q_\alpha^2 n_{\alpha,0}}{\varepsilon_0 K_{\mathrm{B}} T_\alpha}\right] \phi(\boldsymbol{r}). \tag{2.10}$$

In the following, let us assume that we will pick a particular testing particle having the charge Q. We will further assume the spherical geometry of the electric potential around this selected particle. In that case only the radial part of the Poisson equation is relevant:

$$\frac{1}{r} \frac{\mathrm{d}^2}{\mathrm{d}r^2} \left[r\phi(r)\right] = a\phi(r), \tag{2.11}$$

where we defined

$$a = \sum_{\alpha} \frac{q_\alpha^2 n_{\alpha,0}}{\varepsilon_0 K_{\mathrm{B}} T_\alpha} \tag{2.12}$$

for simplicity. Then by substituting $\psi(r) \equiv r\phi(r)$ equation (2.11) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} \psi(r) = a\psi(r), \tag{2.13}$$

which has a solution in a form

$$\psi(r) = c_1 e^{\sqrt{a}r} + c_2 e^{-\sqrt{a}r}. \tag{2.14}$$

The constant c_1 must equal identically to zero, otherwise the solution diverges for large r, which is unphysical. The solution for the electric potencial then has a form

$$\phi(r) = \frac{c_2}{r} e^{-\sqrt{a}r}. \tag{2.15}$$

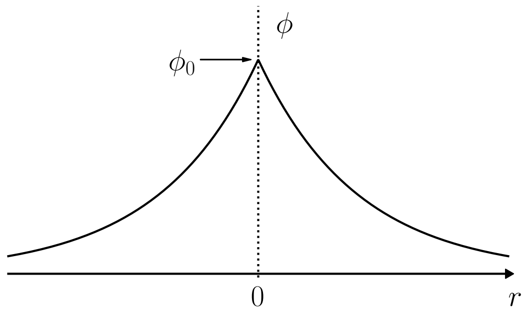

Figure 2.2: The modelled profile of the attenuation of the electric potential around the charge concentration.

The form of c_2 may be expressed in a limit of $r \to 0$, when the potential should converge to the coulombic solution of an isolated charge. Obviously

$$\frac{c_2}{r} \to \frac{Q}{4\pi\varepsilon_0 r}. \tag{2.16}$$

A final solution then takes a form of

$$\phi(r) = \frac{Q}{4\pi\varepsilon_0 r} \exp\left[-\frac{r}{\lambda_\mathrm{D}}\right], \tag{2.17}$$

where

$$\lambda_\mathrm{D} \equiv \left[\sum_\alpha \frac{q_\alpha^2 n_{\alpha,0}}{\varepsilon_0 K_\mathrm{B} T_\alpha}\right]^{-1/2} \tag{2.18}$$

is a definition of the Debye shielding length. This gives a characteristic scale on which the charge of the selected testing particle is damped in plasmas.

By going back to our simplistic model with a positive charge, when assuming that only the electrons will be responsible for the shielding one gets a simplified definition of the Debye length scale

$$\lambda_\mathrm{D} \equiv \sqrt{\frac{\varepsilon_0 K_\mathrm{B} T_\mathrm{e}}{n_0 e^2}}, \tag{2.19}$$

which is usually used in applications. Thus in such a system with an artificially inserted ion charge the electrons will concentrate so that the inserted charge is exponentially attenuated (see Fig. 2.2).

Note that Debye length increases with temperature, when the thermal motions of electrons are larger and the shielding is thus less effective (the electrons are running out of the shielding region) and decreases with increasing density (the more electrons, the more is the shielding effective).

A few additional notes.

- In physics, the temperature is a measure of the width of the particle velocity distribution and usually is being expressed in Kelvins. In plasma physics, a quite often used unit expressed by the kinetic energy (in *electronvolts* eV) of the particle having the given temperature. From

$$E_{\rm k} = \frac{1}{2} m \bar{v}^2 = K_{\rm B} T \tag{2.20}$$

where $\bar{v}$ is the root-mean-square velocity, we find a useful working relation that 1eV corresponds to 11 600K.

- Plasma may have many temperatures at once. Electrons and ions may have different velocity distributions and thus different temperatures. This is due to the fact that the frequencies of ion-ion and electron-electron collisions are much higher than the frequency of ion-electron collisions. The velocity distribution of a single particle species does not have to be isotropic, e.g., when a magnetic field is present. Then the temperature of the plasma is usually higher along the field than across the field.

- Plasma densities vary over many orders in the universe. The density of 10^6 m^{-3} is the typical plasma density in the interplanetary space around the Earth, it might be many orders less in the interstellar or intergalactic space. It is around 10^8 m^{-3} in the giant molecular clouds, from which the stars are born. The plasma density in the solar photosphere is in the order of 10^{22} m^{-3}, and in the solar core 10^{31} m^{-3}. In the laboratory plasmas, the densities between 10^{20}–10^{28} m^{-3} are considered for the experiments with the thermonuclear fusion.

- Various approaches can be practically used to describe plasmas and their dynamics. The most detailed is the kinetic description using the Boltzmann equation. The dynamics of plasmas is also being described using the fluid and magnetohydrodynamic approaches. The motion of individual or testing particles is investigated using the drift description. Using some of these approaches, we may state the conditions for the plasma equilibrium, to study instabilities and conditions for the propagation, excitation and damping of waves and other oscillatory motions. We will study all these aspects in the following chapters.

- The ionisation degree may be estimated using the Saha equation (1.126). It strongly depends on the temperature. For example, for the air at the room temperature (assumed that the air is composed from nitrogen only), one gets the estimate (assuming $n_0 \sim 3 \times 10^{25}$ m^{-3}, $T \sim 300$ K, and $U_{\rm i} \sim 14.5$ eV)) of the ionisation degree to be $n_i/n_0 \sim 10^{-122}$. The air in the room certainly is not a plasma.

- The ionosphere holds the name resembling that it contains ions. Assessing the situation of hydrogen: the temperature in the ionosphere is about $K_{\rm B}T_{\rm e} \sim 0.1$ eV, the ionisation potential is $U^{\rm H} = 13.6$ eV, the particle density is about $n \sim 10^{12}$ m^{-3}. Using the Saha equation one gets the expected ion density of $n_{\rm i}^2 \sim 10^{-52}$ m^{-3}. The ionisation degree is tiny. However, the ionosphere is not composed of hydrogen. There are various chemical reactions with a smaller ionisation potential, which indeed fill the ionosphere with ions and electrons. E.g.,

$$O_2^+ + e^- \rightarrow O + O + 6.6\text{eV} \tag{2.21}$$

or

$$O_2^+ + N_2 \rightarrow NO^+ + NO + 0.87\text{eV}. \tag{2.22}$$

Then the matter in the ionosphere may be considered to be a plasma.

- In practice, there are three criteria, which help to decide whether the studied system may be considered plasma:

    1. The system is much larger than the Debye length, hence the quasineutrality holds. $L \gg \lambda_D$, where L is the characteristic size of the system.
    2. Shielding is effective. The number of particles in the Debye sphere is much larger than one, $N_D \gg 1$. In case it is not, the studied system is probably a bound system.
    3. The particle interactions are dominated by electromagnetic forces and not by collisions. $\omega\tau > 1$, where ω is a characteristic oscillation frequency of plasma and τ is a characterictic time between particle collisions. For example an exhaust of the jet engine can't be considered plasma despite the huge temperature and high ionisation, because the particle interactions are purely collisional and electromagnetic forces play negligible role.

Chapter 3

Charged particle motion in electromagnetic fields

In plasma, we sometimes have to consider motions of each single particle to capture some of the effects. The densities of plasmas are not large enough to be allowed to completely ignore the motions of individual particles. The plasmas usually are not dominated by collisions. Hence motions of individual particles play an important role. In this approach we study the motion of test particle in the *prescribed* electric and magnetic fields. In general, the trajectory of such motion may be unambiguously found by solving the equation of motion

$$m\frac{\mathrm{d}\boldsymbol{v}(\boldsymbol{r},t)}{\mathrm{d}t} = q\left[\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{v}(\boldsymbol{r},t)\times\boldsymbol{B}(\boldsymbol{r},t)\right], \tag{3.1}$$

In the following sections we will discuss some examples by giving particular analytical forms to background fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$. We will derive the relations for some *drifts*, thus systematic motion of charged particles which are not captured when we use other approaches to study plasmas. It is important to study drifts of particles because these drifts often drive instabilities, which is critical e.g. for laboratory plasma in plasma confinement and thermonuclear fusion.

3.1 Homogeneous magnetic field

Let us define the system with constant homogeneous magnetic field and no electric field, where we define Cartesian coordinate system so that the z axis is parallel to the direction of magnetic induction. Then we have the vector equation of motion in a form of

$$m\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = q\left[\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}\right], \tag{3.2}$$

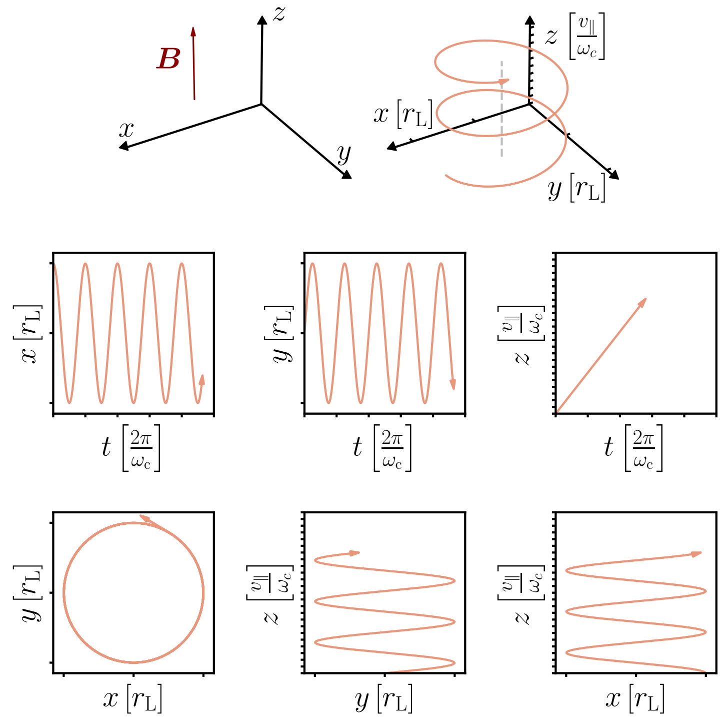

Figure 3.1: Motion of an electron in homogeneous magnetic field. The 3-D view of the motion, trajectories in the coordinate–time and coordinate–coordinate spaces.

where m is the mass of the particle charged with electrical charge q. This equation becomes a system of scalar differential equations:

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = qBv_y, \tag{3.3}$$

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = -qBv_x, \tag{3.4}$$

$$m\frac{\mathrm{d}v_z}{\mathrm{d}t} = 0. \tag{3.5}$$

We immediately see that the parallel component of velocity has a trivial solution, which is the constant motion along the magnetic field. By differentiating the perpendicular components with respect to time we have

$$\frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} = \frac{q}{m}B\frac{\mathrm{d}v_y}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2v_y}{\mathrm{d}t^2} = -\frac{q}{m}B\frac{\mathrm{d}v_x}{\mathrm{d}t}. \tag{3.6}$$

Their combination then gives

$$\frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} = -\left(\frac{q}{m}B\right)^2 v_x \quad \text{and} \quad \frac{\mathrm{d}^2v_y}{\mathrm{d}t^2} = -\left(\frac{q}{m}B\right)^2 v_y, \tag{3.7}$$

which has trivial solution in the form of harmonic functions. We define

$$\omega_\mathrm{c} \equiv \frac{qB\,\mathrm{sgn}\,q}{m} = \frac{|q|B}{m} \tag{3.8}$$

to be a *cyclotron* frequency. Let's take the solution of the second order ordinary differential equations as

$$v_{x,y} = v_\perp \exp\left(\mathrm{i}\omega_\mathrm{c} t + \mathrm{i}\delta_{x,y}\right), \tag{3.9}$$

where $v_\perp$ is the amplitude of the velocity and it is obviously[1] identical for both perpendicular components and is usually initially driven by thermal motion. Both perpendicular components are allowed to have different phase shifts $\delta_{x,y}$. The trajectory in the phase space obviously is a circle. Let's set the spacetime coordinate system so that $\delta_x = 0$. Then

$$v_x = v_\perp \exp \mathrm{i}\omega_\mathrm{c} t, \tag{3.10}$$

and we consider only the real part to have a physical meaning. Then

$$v_y = \frac{m}{qB}\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{m}{qB} v_\perp \mathrm{i}\omega_\mathrm{c} \exp \mathrm{i}\omega_\mathrm{c} t = \frac{m}{qB} v_\perp \mathrm{i}\frac{qB \operatorname{sgn} q}{m} \exp \mathrm{i}\omega_\mathrm{c} t = v_\perp \mathrm{i} \operatorname{sgn} q \exp \mathrm{i}\omega_\mathrm{c} t. \tag{3.11}$$

We see that the sense of rotation in the phase space depends on the charge of the particle, thus being opposite for electrons and ions.

To obtain the trajectory, one needs to integrate (3.10) and (3.11):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Re[v_x] = \Re[v_\perp \exp \mathrm{i}\omega_\mathrm{c} t] \rightarrow x - x_0 = \Re\left[-\mathrm{i}\frac{v_\perp}{\omega_\mathrm{c}} \exp \mathrm{i}\omega_\mathrm{c} t\right] = \frac{v_\perp}{\omega_\mathrm{c}} \sin \omega_\mathrm{c} t \tag{3.12}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \Re[v_y] = \Re[v_\perp \mathrm{i} \exp \mathrm{i}\omega_\mathrm{c} t] \rightarrow y - y_0 = \Re\left[\frac{v_\perp}{\omega_\mathrm{c}} \operatorname{sgn} q \exp \mathrm{i}\omega_\mathrm{c} t\right] = \frac{v_\perp}{\omega_\mathrm{c}} \operatorname{sgn} q \cos \omega_\mathrm{c} t \tag{3.13}$$

Of course, we should not forget about the parallel component

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \Re[v_z] = 0 \rightarrow z - z_0 = v_\parallel t. \tag{3.14}$$

Then the total trajectory is a helix. The ratio $v_\perp/\omega_\mathrm{c} = r_\mathrm{L}$ is the *Larmor radius*, the typical radius of gyration. Note that

$$r_\mathrm{L} = \frac{v_\perp m}{qB \operatorname{sgn} q} = \frac{v_\perp m}{|q|B} \tag{3.15}$$

depends on the perpendicular velocity and the mass of the particle. More massive particle has larger Larmor radius and thus smaller cyclotron frequency. Also, stronger magnetic field causes the particles to gyrate using tighter orbits.

Electrons and ions circulate the magnetic field lines in the opposite directions (see Fig. 3.1), but they both follow the field lines (we will generalise this case further). The charge travelling along the helical trajectory induces it's own magnetic field (according to the Maxwell's equations), which has the sign opposite to the background magnetic field. Thus the plasma weakens the background magnetic field, it is *diamagnetic*.

[1] It might be shown that if we multiply (3.3) by v_x, (3.4) by v_y and sum the two, we obtain $\frac{\mathrm{d}v_x^2}{\mathrm{d}t} + \frac{\mathrm{d}v_y^2}{\mathrm{d}t} = \frac{\mathrm{d}v^2}{\mathrm{d}t} = 0$, hence the amplitude of the velocity is constant in time.

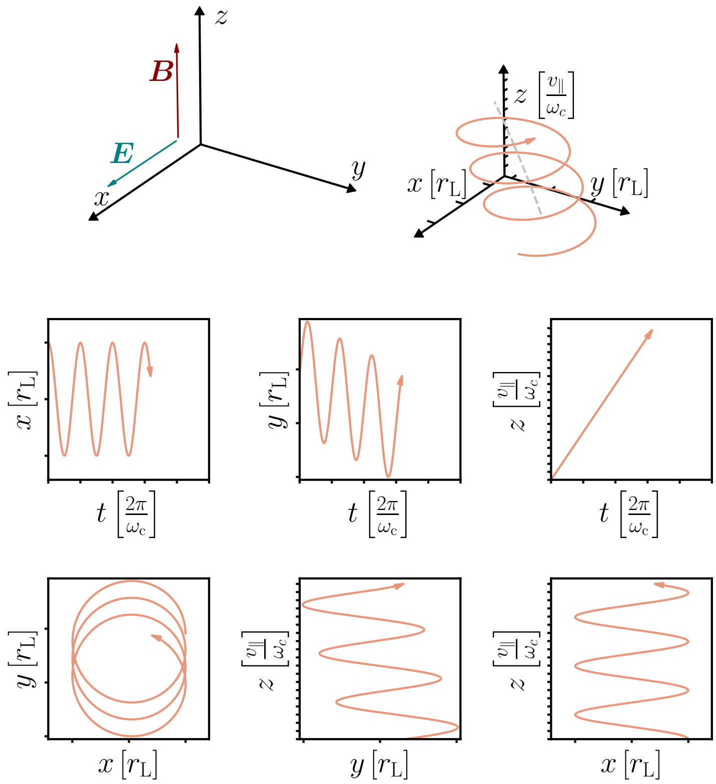

Figure 3.2: Motion of an electron in the crossed electric and magnetic fields.

3.2 Homogeneous electric field

Now we add a homogeneous constant electric field to the problem, and thus we choose the coordinate system so that the magnetic field is still along the z axis, and the electric field has non-trivial components only in x and z directions (see Fig. 3.2. Then the equation of motion is

$$m\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = q\left[\begin{pmatrix} E_x \\ 0 \\ E_z \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}\right]. \tag{3.16}$$

The parallel component can be solved immediately:

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = \frac{q}{m}E_z \rightarrow z - z_0 = \frac{q}{2m}E_z t^2 + v_{z,0}t. \tag{3.17}$$

There is a uniform acceleration of the particle along the magnetic field caused by the parallel component of the electric field. By using the definition of the cyclotron frequency, the remaining two components of the equation of motion are

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{q}{m}E_x + \omega_\mathrm{c}\,\mathrm{sgn}\,q v_y, \tag{3.18}$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\omega_\mathrm{c}\,\mathrm{sgn}\,q v_x. \tag{3.19}$$

Similarly to the previous case, we take the time derivative of (3.18) and (3.19) and combine the two. We have

$$\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} = \omega_\mathrm{c}\,\mathrm{sgn}\,q\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\,\mathrm{sgn}^2 q\omega_\mathrm{c}^2 v_x = -\omega_\mathrm{c}^2 v_x, \tag{3.20}$$

where we assumed that $q \neq 0$, and

$$\frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\,\mathrm{sgn}\,q\omega_\mathrm{c}\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\,\mathrm{sgn}\,q\omega_\mathrm{c}\left(\frac{q}{m}E_x + \omega_\mathrm{c}\,\mathrm{sgn}\,q v_y\right) = -\omega_\mathrm{c}^2\left(\frac{E_x}{B} + v_y\right). \tag{3.21}$$

We solved (3.20) already in the previous case and obtained (3.10):

$$v_x = v_\perp \exp \mathrm{i}\omega_\mathrm{c} t. \tag{3.22}$$

For (3.21) we note that E_x/B does not depend on time, thus we may simply add it to v_y on the left-hand side of the equation which then becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(v_y + \frac{E_x}{B}\right) = -\omega_\mathrm{c}^2\left(v_y + \frac{E_x}{B}\right), \tag{3.23}$$

having a trivial solution in a form of

$$v_y = v_\perp \mathrm{i}\,\mathrm{sgn}\,q \exp \mathrm{i}\omega_\mathrm{c} t - \frac{E_x}{B}. \tag{3.24}$$

We see that in the solution and additional term, *drift* appeared. Let's find a general solution in a vector form. The equation of motion is

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{3.25}$$

If we substitute $\boldsymbol{v}' = \boldsymbol{v} - \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}$ (we transform to a different coordinate system, which moves with a speed of $-\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}$ with respect to the original one). As we will find out later, in this moving coordinate system the parallel components of the electric field disappear. Thus (3.25) becomes:

$$\begin{aligned} m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t} &= q\left(\boldsymbol{E} + \boldsymbol{v}' \times \boldsymbol{B} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} \times \boldsymbol{B}\right) = \\ &= q\left[\cancel{\boldsymbol{E}} + \boldsymbol{v}' \times \boldsymbol{B} - \frac{1}{\cancel{B^2}}\cancel{\boldsymbol{E}B^2} + \frac{\boldsymbol{B}(\boldsymbol{E}\cdot\boldsymbol{B})}{B^2}\right] = \\ &= q\left[\boldsymbol{v}' \times \boldsymbol{B} + \boldsymbol{E}_\parallel\right], \end{aligned} \tag{3.26}$$

where we defined the parallel component of the electric field

$$\boldsymbol{E}_{\parallel} \equiv \frac{\boldsymbol{B}(\boldsymbol{E}\cdot\boldsymbol{B})}{B^2}. \tag{3.27}$$

We transformed the new problem to the previous case, as in this moving coordinate system the effects of the electric field (causing the acceleration along the magnetic field lines) and the magnetic field (causing the gyration around the magnetic field lines) are separable. Thus we solve the problem in the moving coordinate system and transform back to the original coordinate system

$$\boldsymbol{v} = \boldsymbol{v}' + \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}. \tag{3.28}$$

We see that a new component of the particle velocity appeared,

$$\boldsymbol{v}_E = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}, \tag{3.29}$$

which is the expression for the *drift of the gyration centre* (illustrated in Fig. 3.2), the *E-B drift*. Note that the sign of this drift does neither depend on the particle mass nor on its electric charge. This drift does not cause a separation of the charge.

Note that the E-B drift naturally shows up as a consequence of the Lorentz transform of the electromagnetic field tensor. For details, see section B.1 in the Appendix.

3.2.1 Drift in the field of general force

Let's make a generalisation, which we will use further. In the case of E-B drift, we have an additional force (the force of the electric field) acting on the motion of the particle. This force (the Lorentz force) is simply given by

$$\boldsymbol{F}_E = q\boldsymbol{E}. \tag{3.30}$$

After substitution into (3.29) we have

$$\boldsymbol{v}_E = \frac{1}{q}\frac{\boldsymbol{F}_E\times\boldsymbol{B}}{B^2}. \tag{3.31}$$

Could this expression be generalised for *any* force $\boldsymbol{F}$ other than Lorentz? The answer is yes, if and only if the force in question acts homogeneously, in particular, when the vector of the force does not depend on speed and position. Then we have a drift of gyration centre $\boldsymbol{v}_{\mathrm{gs}}$

$$\boldsymbol{v}_{\mathrm{gs}} = \frac{1}{q}\frac{\boldsymbol{F}\times\boldsymbol{B}}{B^2}. \tag{3.32}$$

An example? The action of gravity field may be considered homogeneous in laboratory or at least when the small spatial scales (compared to the curvature of the gravity field lines) are in question. Then we have the *gravity drift* $\boldsymbol{v}_g$

$$\boldsymbol{v}_g = \frac{m}{q}\frac{\boldsymbol{g}\times\boldsymbol{B}}{B^2}, \tag{3.33}$$

where $\boldsymbol{g}$ is the vector of gravity acceleration. How about the centrifugal force $\boldsymbol{F}_{\mathrm{c}} = \frac{mv_{\parallel}^2}{R^2}\boldsymbol{R}$ acting in the curved magnetic field? Is the expression for the curvature drift $\boldsymbol{v}_{\mathrm{c}}$

$$\boldsymbol{v}_{\mathrm{c}} = \frac{mv_{\parallel}^2}{qR^2B^2}\boldsymbol{R}\times\boldsymbol{B} \tag{3.34}$$

correct? It is not, because we violated one of the assumptions written above: the vector of the acting force *does depend on position* in this case. Magnetic field is not homogeneous anymore and we must find another correction. (A figure justifying the notation $v_{\parallel}$ in the above equations would be suitable here. It may contain both macroscopic view in which the force as well as B would not be homogenous and a zoomed view with both of them being approximately homogenous, i.e. justifying use of (3.32) above.)

3.3 Inhomogeneous magnetic field

Let us study the effects of the non-homogeneous magnetic field. We will keep our coordinate system, thus having magnetic field in the z-direction, however we will allow for a smooth change in the y-direction. What does the word *smooth* mean? Let's remind that there is a characteristic scale of the particle motion even in the case of homogeneous field: the Larmor radius $r_{\mathrm{L}} = \frac{mv_{\perp}}{|q|B}$. By *smooth* we further mean that the characteristic spatial change of the magnetic field occurs on much larger scales than the Larmor gyration. The characteristic scale of the spatial change of the magnetic field L may be estimated by comparing the gradient of the field to its magnitude, thus

$$L \sim \frac{|\boldsymbol{B}|}{|\nabla B|}. \tag{3.35}$$

We will study the case when $L \gg r_{\mathrm{L}}$. The ratio of the two is a small number, thus can be used as a small parameter to expand the physical functions to the Taylor series and neglect the higher order of this small parameter. Hence in our case of the inhomogeneous magnetic field we have to the first order

$$\boldsymbol{B} = \boldsymbol{B}_0 + (\boldsymbol{r}\cdot\nabla)\boldsymbol{B} \tag{3.36}$$

and in our coordinate system we have

$$B_z = B_0 + y\frac{\partial B_z}{\partial y}. \tag{3.37}$$

What effect do we expect? By waving our hands we can say that when we study the gyration of the testing particle, we see that the particles still gyrate along the magnetic field lines. However, in that part of the orbit, where the magnetic field is stronger, the Larmor radius is smaller and vice versa. We should see a drift in the direction perpendicular to both $\boldsymbol{B}$ and $\nabla|\boldsymbol{B}|$. We would expect the effect to increase with the gradient of the field, when the difference between the effective radii in the "upper" and "lower" part of the orbit is larger.

In our case, only the y component of the Lorentz force is perturbed

$$F_y = -qv_xB_z = -qv_x\left(B_0 + y\frac{\partial B_z}{\partial y}\right). \tag{3.38}$$

The first term on the right-hand side is responsible for the normal Larmor rotation, the second term is a correction caused by the inhomogeneous magnetic field and thus causes the expected drift. The approach we will take is that we will separate the effects of the B_0 causing the Larmor rotation and the effect of the small correction from $\nabla|\boldsymbol{B}|$. To investigate the correction, we will average the particle motion over one Larmor rotation. Let's remind again that this is possible if and only if the Larmor rotation is the dominant motion of the studied particle and thus when $|\partial B_z/\partial y| \ll |B_0|$. Then we use the solution for v_x from (3.10) and for y from (3.13). We have

$$F_y = -qv_\perp \cos\omega_c t\left(B_0 + r_L\ \operatorname{sgn} q \cos\omega_c t\frac{\partial B_z}{\partial y}\right). \tag{3.39}$$

The averaged (over Larmor orbit) Lorentz force is thus

$$\begin{aligned}\langle F_y\rangle &= \frac{1}{2\pi}\int_0^{2\pi} F_y \mathrm{d}(\omega_c t) = -\frac{1}{2\pi}\int_0^{2\pi} qv_\perp \cos\omega_c t\left(B_0 + r_L\ \operatorname{sgn} q\cos\omega_c t\frac{\partial B_z}{\partial y}\right)\mathrm{d}(\omega_c t) = \\ &= -qv_\perp B_0\frac{1}{2\pi}\int_0^{2\pi}\cos\omega_c t\mathrm{d}(\omega_c t) - qv_\perp r_L\ \operatorname{sgn} q\frac{\partial B_z}{\partial y}\frac{1}{2\pi}\int_0^{2\pi}\cos^2\omega_c t\,\mathrm{d}(\omega_c t) = \\ &= 0 - \frac{1}{2}v_\perp r_L\, q\operatorname{sgn} q\frac{\partial B_z}{\partial y}.\end{aligned} \tag{3.40}$$

We leave it to the reader to prove that the integral over one period of the sine or cosine function vanishes and that the integral over one period of the sine squared or cosine squared equals to π.

We found a correction term. Since the averages of the remaining components of the Lorentz force vanish, we may generalise

$$\langle\boldsymbol{F}\rangle = -\frac{1}{2}v_\perp r_L\, q\operatorname{sgn} q\nabla|\boldsymbol{B}|, \tag{3.41}$$

and use it in the expression for the drift in the field of general force (3.32). Note that under approximations we took the additional force defined by (3.41) is homogeneous and does not depend on velocity. Hence we have a drift of the gyration centre, the *grad-B drift* $\boldsymbol{v}_{\nabla B}$:

$$\boldsymbol{v}_{\nabla B} = \frac{1}{2}\operatorname{sgn} q v_\perp r_L\,\frac{\boldsymbol{B}\times\nabla|\boldsymbol{B}|}{B^2}. \tag{3.42}$$

Note that the sign does depend on the sign of the particle charge and thus the grad-B drift may cause the separation of charges and to drive additional electric field. See Fig. 3.3.

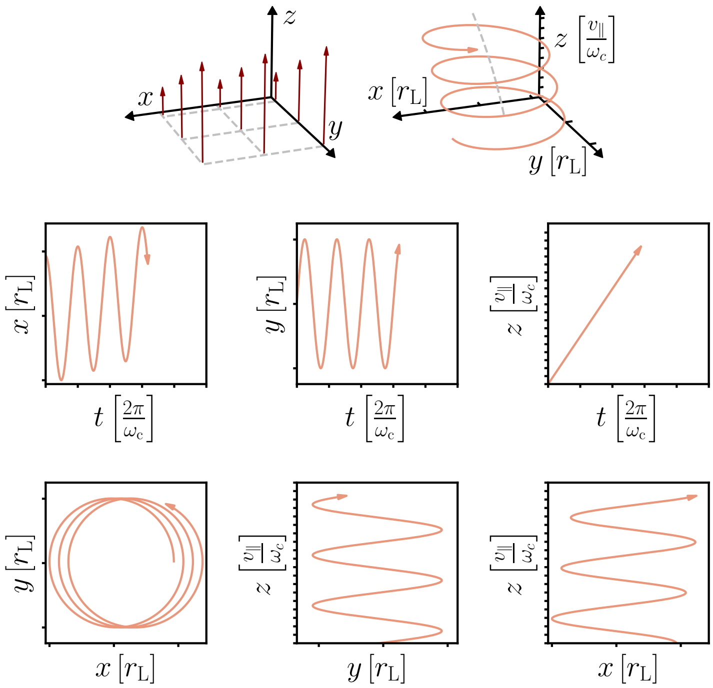

Figure 3.3: Drift of an electron in an inhomogeneous magnetic field.

At the beginning we used the assumption of small spatial changes of the magnetic field to derive this relation and we should check that the assumption holds and thus the solution is consistent. We simply compute how much does the particle move due to the drift after one Larmor rotation, hence after one period $\tau = 2\pi/\omega_{\rm c}$. We will further use (3.35) and $v_\perp/\omega_{\rm c} = r_{\rm L}$ to evaluate the displacement, Δ, of the gyration centre over one rotation:

$$\Delta = \tau|\boldsymbol{v}_{\nabla B}| = \frac{2\pi}{\omega_{\rm c}}\frac{1}{2}v_\perp r_{\rm L}\frac{|\nabla B|}{|B|} = \pi r_{\rm L}\frac{r_{\rm L}}{L} \ll r_{\rm L} \tag{3.43}$$

under the assumption $L \gg r_{\rm L}$. The solution is consistent with the assumptions, the dominant motion still is the Larmor rotation.

Now we may continue to investigate the motion of particles in the curved magnetic field. Obviously, additionally to the curvature drift (3.34) we need to account for inherited gradient of the field. Let's make a qualitative estimate. The Cartesian coordinates are no longer convenient to investigate this case, cylindrical coordinates (R, ϑ, z) are more appropriate (they naturally incorporate the "curvature" of the field). In any case, we assumed the changes of the magnetic field to be smooth, thus we may always introduce *local* cylindrical coordinates to approximate the real configuration of the field lines. In vacuum we have

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} = 0, \tag{3.44}$$

hence when we assume that the dominant direction of the magnetic field lines is along the azimuthal ϑ direction, $\nabla|\boldsymbol{B}|$ must have only the component in R direction. Hence

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{R}\frac{\partial}{\partial R}(RB_\vartheta) = 0 \tag{3.45}$$

and therefore

$$B_\vartheta = \frac{\mathcal{C}}{R}, \tag{3.46}$$

where $\mathcal{C}$ is an integration constant. We further obtain

$$\nabla B = \frac{\partial}{\partial R}\frac{\mathcal{C}}{R} = -\mathcal{C}\frac{1}{R^2}\frac{\boldsymbol{R}}{R} \tag{3.47}$$

and when expressing $\mathcal{C}$ from (3.46) we finaly have

$$\nabla B = -\frac{|B|}{R^2}\boldsymbol{R}. \tag{3.48}$$

Now we can ingest this expression for grad-B into (3.42) and (when also explicitly writing the expression for r_L) have

$$\boldsymbol{v}_{\nabla B} = -\frac{1}{2}\,\mathrm{sgn}\,q\frac{v_\perp r_\mathrm{L}}{B^2}\boldsymbol{B} \times |\boldsymbol{B}|\frac{\boldsymbol{R}}{R^2} = \frac{1}{2}\frac{mv_\perp^2}{qB^2R^2}\boldsymbol{R} \times \boldsymbol{B}. \tag{3.49}$$

After adding the curvature drift (3.34), we finally have an expression for the drift in the curved magnetic field:

$$\boldsymbol{v}_\mathrm{gc} = \boldsymbol{v}_\mathrm{c} + \boldsymbol{v}_{\nabla B} = \frac{m}{q}\frac{\boldsymbol{R} \times \boldsymbol{B}}{R^2B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right). \tag{3.50}$$

This has important consequences, especially for laboratory plasmas. It shows that the plasma cannot be confined in the simply curved magnetic field. Both important drifts, the curvature drift and the grad-B drift, sum up and thus act to displace the plasma. Both these drifts are drivers for *kink instability.* Let's imagine that the plasma flows along the cylindrical tube of the magnetic field (the flux tube) with a prevalently poloidal configuration of the magnetic field, which is generated by the current. When this flux tube is displaced (e.g. by some random motions), the tube gets curved with the gradient of the magnetic field pointing against the displacement. The drifts in the curved field cause the charge separation in the direction perpendicular to the plane of the kink. The separated changes introduce a small electric field and the E-B drift cause the instability to grow.

When the configuration of the magnetic field in the flux tube is rather helical than cylindrical, E-B drift may cancel the drift in the curved magnetic field and the plasma may remain confined within this tube. This is the famous tokamak configuration.

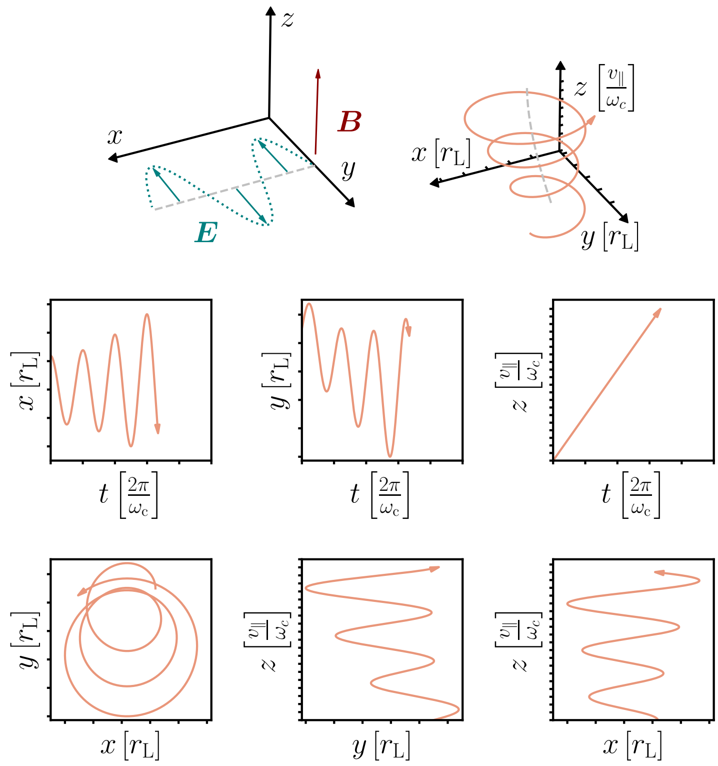

Figure 3.4: Drift of electrons in the inhomogeneous electric field.

3.4 Inhomogeneous electric field

Also the inhomogenous electric field requires the correction to the normal E-B drift of particles. Let us investigate one example only, when the electric field has a harmonic character. Such configuration is not uncommon, later on we will study the plasma oscillations and waves propagating through plasmas, where we will see that there are way too many harmonic motions in plasmas, that may be approximated as a superposition of simple harmonic functions. Let us assume that we have a periodic electric field in the x direction (see Fig. 3.4),

$$\boldsymbol{E} = E_0 \cos ky \boldsymbol{e}_x, \tag{3.51}$$

where k is the wave number, which is related to the wavelength ($\lambda = 2\pi/k$), which is the characteristic spatial scale of the perturbation. Otherwise we still keep the same configuration of the coordinate system as we did in the previous cases. Hence the perpendicular components of equation of motion are

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{qB}{m}v_y + \frac{q}{m}E_x(y) \quad \text{and} \quad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -\frac{qB}{m}v_x. \tag{3.52}$$

The standard way to solve this system is to take the time derivative of these equations and combine the two, hence

$$\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} = -\omega_\mathrm{c}^2 v_x + \omega_\mathrm{c} \frac{\operatorname{sgn} q}{B}\frac{\mathrm{d}E_x}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\omega_\mathrm{c}^2 v_y - \omega_\mathrm{c}^2 \frac{E_x(y)}{B}. \tag{3.53}$$

This is a system of implicit ordinary differential equations, which cannot be trivially solved. We would need to proceed iteratively, since in order to find the solution, we need to know $\boldsymbol{E}$, which is the function of the solution. Let's focus to the case of small electric fields and apply the averaging formalism again in order to find the correction. In this approach we simply use the unperturbed solution and average over one Larmor rotation.

By substituting (3.13) into (3.53) we have

$$\frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\omega_\mathrm{c}^2 v_y - \omega_\mathrm{c}^2 \frac{E_0}{B} \cos\left[k(y_0 + r_\mathrm{L} \operatorname{sgn} q \cos \omega_\mathrm{c} t)\right], \tag{3.54}$$

which we average over one Larmor orbit to have

$$\left\langle \frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} \right\rangle = -\omega_\mathrm{c}^2 \langle v_y \rangle - \omega_\mathrm{c}^2 \frac{E_0}{B} \langle \cos\left[k(y_0 + r_\mathrm{L} \operatorname{sgn} q \cos \omega_\mathrm{c} t)\right] \rangle. \tag{3.55}$$

The action of the additional force oscilates (we do not expect a secular trend in time; this assumption should be validated in the solution), therefore when averaged over one Larmor orbit, it vanishes, thus

$$\left\langle \frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} \right\rangle = 0. \tag{3.56}$$

We approximate the cosine term by expansion into Taylor series in the parameter kr_L, which we assume to be small[2]. Essentially again the electric field is perturbed on scales much larger than the Larmor radius

$$\begin{aligned}\cos\left[k(y_0\right. &+ \left. r_\mathrm{L} \operatorname{sgn} q \cos \omega_\mathrm{c} t)\right] = \\ &= \cos k y_0 \cos\left(k r_\mathrm{L} \operatorname{sgn} q \cos \omega_\mathrm{c} t\right) - \sin k y_0 \sin\left(k r_\mathrm{L} \operatorname{sgn} q \cos \omega_\mathrm{c} t\right) = \\ &= \cos k y_0 \cos\left(k r_\mathrm{L} \cos \omega_\mathrm{c} t\right) - \operatorname{sgn} q \sin k y_0 \sin\left(k r_\mathrm{L} \cos \omega_\mathrm{c} t\right) \\ &= \cos k y_0 \left[1 - \frac{1}{2} k^2 {r_\mathrm{L}}^2 \cos^2 \omega_\mathrm{c} t\right] - \operatorname{sgn} q \sin k y_0 (k r_\mathrm{L} \cos \omega_\mathrm{c} t)\end{aligned} \tag{3.57}$$

Hence after averaging the term with the sine vanishes, while the term with the cosine squared becomes 1/2. Finally

$$\langle v_y \rangle = -\frac{E_0}{B} \cos k y_0 \left(1 - \frac{1}{4} k^2 {r_\mathrm{L}}^2\right) = -\frac{E_x(y)}{B}\left(1 - \frac{1}{4} k^2 {r_\mathrm{L}}^2\right). \tag{3.58}$$

We may generalise this expression to

$$\boldsymbol{v}_E = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}\left(1 - \frac{1}{4} k^2 {r_\mathrm{L}}^2\right) \tag{3.59}$$

[2] We remind that $\cos\varepsilon \approx 1 - \frac{1}{2}\varepsilon^2$ and $\sin\varepsilon \approx \varepsilon$ for $\varepsilon \to 0$.

and we keep it to the reader to prove that in our case it is the correct generalisation. In the section dealing with waves in plasmas we will often use the solution as a superposition of various modes, when for each mode we may conviniently solve the equations in the Fourier space. The Fourier representation is convenient because differential operators become multiplications. We ask the reader to see (5.13) for a short moment and apply it to the equation we just obtained. We may generalise this expression to

$$\boldsymbol{v}_E = -\frac{\boldsymbol{B}\times}{B^2}\left(1 + \frac{1}{4}{r_{\mathrm{L}}}^2 \triangle\right)\boldsymbol{E}. \tag{3.60}$$

The correction term captures the effect of the finite Larmor radius. Since r_{L} is larger for ions, ions "feel" the changes in the background electric field stronger than electrons. When an electric field appears in plasma, the *drift instability* causes this electric field to grow. Note that the effect of the inhomogeneous electric field is proportional to $(kr_{\mathrm{L}})^2$, whereas the effect of the inhomogeneous magnetic field is proportional to kr_{L} (when applying the Fourier transform formalism to (3.42)). Hence, the effects of the non-uniform E-field are relatively more important at large k, therefore at small spatial scales.

3.5 Drift currents

Some of the drifts discussed above and summarised graphically in Fig. 3.6 depend on the mass or charge of the particles, thus lead to the separation of charges and give rise to the *drift currents*. In plasmas we have to consider k different particle species, each having particle density n_k, mass m_k and q_k. From the general equation for the drift velocity (3.32), we derive the relation for the drift current

$$\boldsymbol{j}_{\mathrm{drift}} = \sum_k n_k q_k \boldsymbol{v}_{\mathrm{drift},k} = \frac{\sum_k n_k \boldsymbol{F}_k}{B^2} \times \boldsymbol{B}. \tag{3.61}$$

For instance, the drift current due to the grad-B drift is

$$\boldsymbol{j}_{\nabla B} = \frac{1}{2}\sum_k \frac{m_k n_k v_{\perp,k}^2}{B^2}(\boldsymbol{B}\times\nabla B). \tag{3.62}$$

This current leads to the charge separation. An example of the natural drift current is the ring current in the Earth's magnetosphere (see Fig. 3.5).

3.6 Guiding centre motion

Let us show that it is possible to derive all the above discussed drifts and much more in a very elegant and general way[3]. We will correctly solve the equations of motion

[3]Following de Blank 2012: *Guiding center motion*, Fusion Science and Technology 61(2T), 61–68

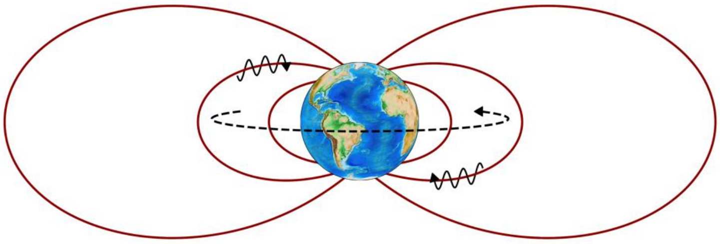

Figure 3.5: A combination of drifts in the curved magnetic field in the Earth's magnetosphere results in the formation of the *ring current* in the equatorial plane (the dashed line indicates the drift of the electrons).

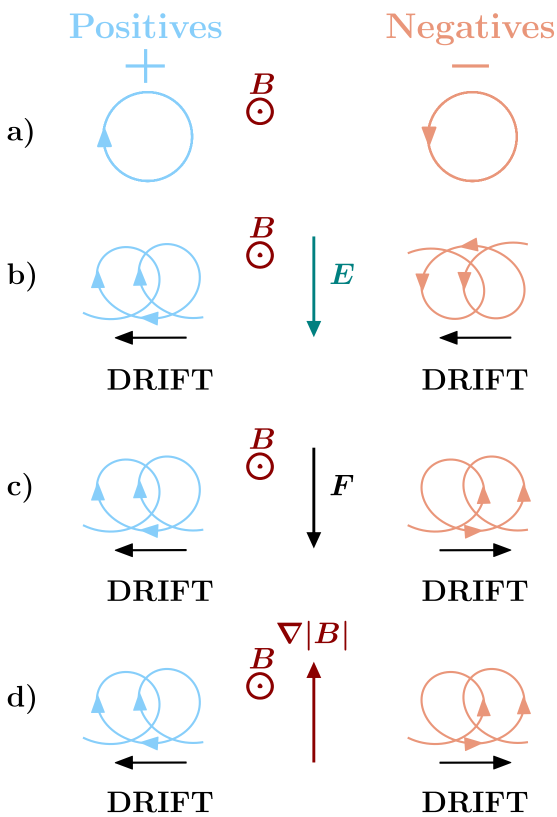

Figure 3.6: Charged particle drifts in a homogeneous magnetic field. (A) No disturbing force, (B) With an electric field, E (C) With an independent force, F (e.g. gravity) (D) In an inhomogeneous magnetic field, grad B. Based on Hannes Alfvén's, Cosmical Electrodynamics (1950); (cc) Stannered, redesigned by Ian Tresman.

and will find the corrections to the normal gyration under the approximation of slowly varying fields. That means that we limit ourselves to the case when

$$\frac{r_\mathrm{L}}{L} = \frac{v_\perp}{\omega_\mathrm{c} L} \ll 1 \quad \text{and} \quad \frac{1}{\omega_\mathrm{c}\tau} \ll 1, \tag{3.63}$$

where L and τ are the characteristic spatial and temporal changes in the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively. We see that $\omega_\mathrm{c}^{-1} \propto m/q$, which is small[4]. Thus we define $\epsilon \equiv m/q$ to be a small parameter of the system and we will expand the equations to the Taylor series using this parameter. Then $r_\mathrm{L} \propto \epsilon L$ and $\tau^{-1} \propto \epsilon\omega_\mathrm{c}$. We solve a system of ordinary differential equations

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v} \tag{3.64}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{q}{m}\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right). \tag{3.65}$$

We will expand these equations in ϵ. But first, we introduce the coordinate transformation $(\boldsymbol{x}, \boldsymbol{v}) \to (\boldsymbol{R}, \boldsymbol{u})$. Coordinates $(\boldsymbol{R}, \boldsymbol{u})$ are functions of time.

$$\boldsymbol{R}(t) = \boldsymbol{x}(t) - \boldsymbol{\rho}(t), \tag{3.66}$$

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_\mathrm{E}. \tag{3.67}$$

Here $\boldsymbol{R}$ is the instantaneous position of the gyration center, $\boldsymbol{x}$ is the instantaneous position of the given particle (with respect to the fixed reference coordinate system), $\boldsymbol{v}_\mathrm{E} \equiv \boldsymbol{E} \times \boldsymbol{B}/B^2$ is the velocity of the E-B drift [see (3.29)], and $\boldsymbol{\rho}$ is the instantaneous position of the particle with respect to the gyration centre. From the previous section [see e.g. (3.15)] we may see that

$$\boldsymbol{\rho} \equiv \frac{\epsilon}{B}\boldsymbol{b} \times \boldsymbol{u}, \tag{3.68}$$

where $\boldsymbol{b}(\boldsymbol{x}, t) = \boldsymbol{B}(\boldsymbol{x}, t)/|\boldsymbol{B}(\boldsymbol{x}, t)|$ is the unity vector along the field lines. Vectors $\boldsymbol{b}$ and $\boldsymbol{v}_\mathrm{E}$ are functions of both space $\boldsymbol{x}$ and time t and their values are measured with respect to the particle. Note that

$$\omega_\mathrm{c}(t) = \frac{B(\boldsymbol{x}, t)}{\epsilon} \tag{3.69}$$

is an instantaneous cyclotron frequency.

We further introduce the local cylindrical coordinates in the plane perpendicular to the magnetic field with two direction vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2)$, so that

$$\boldsymbol{u} = v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp, \quad \text{where} \quad \boldsymbol{e}_\perp \equiv \boldsymbol{e}_1 \cos\phi + \boldsymbol{e}_2 \sin\phi, \quad \text{and} \quad \phi = \phi_0 - \omega_\mathrm{c} t. \tag{3.70}$$

Note again that the base vectors of this local coordinate system $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{b})$ vary in both the space and time. Unit vectors $\boldsymbol{e}_\perp$ and

$$\boldsymbol{e}_\rho \equiv \boldsymbol{e}_2 \cos\phi - \boldsymbol{e}_1 \sin\phi \tag{3.71}$$

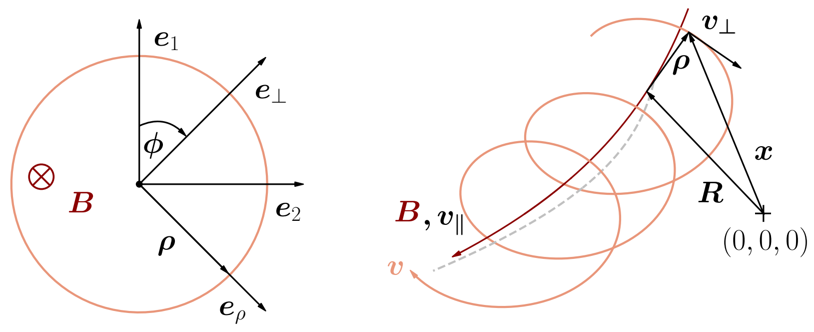

Figure 3.7: Sketch of the coordinate system of the guiding centre variables. The dashed line indicates the position of the gyration centre which deviates from the magnetic field line in general.

are projections of the particle's velocity $\boldsymbol{u}$ and relative position $\boldsymbol{\rho}$, respectively, to the plane perpendicular to $\boldsymbol{b}$.

Hence we derived useful relations defining the guiding centre variables (see also Fig. 3.7).

$$\boldsymbol{x} = \boldsymbol{R} + \boldsymbol{\rho}, \tag{3.72}$$

$$\boldsymbol{\rho} = -\frac{m}{qB^2}\boldsymbol{u} \times \boldsymbol{B} = \rho \operatorname{sgn} q\, \boldsymbol{e}_\rho, \tag{3.73}$$

$$\boldsymbol{u}_\perp = u_\perp \boldsymbol{e}_\perp \quad \text{and} \tag{3.74}$$

$$\boldsymbol{v} = v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E}. \tag{3.75}$$

Now we transform (3.64) and (3.65) to the newly introduced coordinates. From (3.66) we have:

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} &= \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} - \frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t} = \boldsymbol{v} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\epsilon}{B}\boldsymbol{b} \times \boldsymbol{u}\right) = \\
&= v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E} + \epsilon \boldsymbol{u} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{b}}{B}\right) - \frac{\epsilon}{B}\boldsymbol{b} \times \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \\
&= v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E} + \epsilon \boldsymbol{u} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{b}}{B}\right) - \frac{\epsilon}{B}\boldsymbol{b} \times \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} - \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right) = \\
&= v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \epsilon \boldsymbol{u} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{b}}{B}\right) + \frac{\epsilon}{B}\boldsymbol{b} \times \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t},
\end{aligned} \tag{3.76}$$

[4] As m/q is not a dimensionless quantity, its value depend on the system of units. It's smallness needs to be understood in terms of equation (3.63) in which it parametrises ω_c with respect to the characteristic time scale(s) of changes of the fields.

where we used the equation of motion (3.65) to evaluate the term $-\frac{\epsilon}{B}\boldsymbol{b}\times\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$:

$$\begin{aligned}
-\frac{\epsilon}{B}\boldsymbol{b}\times\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= -\frac{\epsilon}{B}\boldsymbol{b}\times\left[\frac{1}{\epsilon}\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)\right] = \\
&= -\frac{1}{B}\boldsymbol{b}\times\boldsymbol{E} - \frac{1}{B}(\boldsymbol{v}\,\boldsymbol{b}\cdot\boldsymbol{B} - \boldsymbol{B}\,\boldsymbol{b}\cdot\boldsymbol{v}) = -\frac{1}{B}\boldsymbol{b}\times\boldsymbol{E} - \boldsymbol{v}\frac{\boldsymbol{B}\cdot\boldsymbol{B}}{B^2} + v_\parallel\frac{\boldsymbol{B}}{B} = \\
&= \boldsymbol{v}_\mathrm{E} - \boldsymbol{v} + v_\parallel\boldsymbol{b} = -u_\perp\boldsymbol{e}_\perp. \qquad (3.77)
\end{aligned}$$

For the time derivative of velocity, we expand $\boldsymbol{v}$ using (3.75) to have

$$\begin{aligned}
\frac{\mathrm{d}(v_\parallel\boldsymbol{b})}{\mathrm{d}t} + \frac{\mathrm{d}(u_\perp\boldsymbol{e}_\perp)}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t} &= \frac{1}{\epsilon}\left[\boldsymbol{E} + (v_\parallel\boldsymbol{b} + u_\perp\boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E})\times\boldsymbol{B}\right] = \\
&= \frac{1}{\epsilon}\left[\boldsymbol{E} + v_\parallel\boldsymbol{b}\times\boldsymbol{B} + u_\perp\boldsymbol{e}_\perp\times\boldsymbol{B} + \boldsymbol{v}_\mathrm{E}\times\boldsymbol{B}\right] = \\
&= \frac{1}{\epsilon}\left[\cancel{\boldsymbol{E}} + u_\perp\boldsymbol{e}_\perp\times\boldsymbol{B} - \cancel{\frac{\boldsymbol{E}}{B^2}\boldsymbol{B}\cdot\boldsymbol{B}} + \frac{\boldsymbol{E}\cdot\boldsymbol{B}}{B^2}\boldsymbol{B}\right] = \\
&= \frac{1}{\epsilon}\left[E_\parallel\boldsymbol{b} + u_\perp\boldsymbol{e}_\perp\times\boldsymbol{B}\right], \qquad (3.78)
\end{aligned}$$

where we defined $E_\parallel \equiv \frac{\boldsymbol{E}\cdot\boldsymbol{B}}{|\boldsymbol{B}|}$. Next, we project this equation to the directions $\boldsymbol{e}_\perp$, $\boldsymbol{b}$, and $\boldsymbol{e}_\rho$. For the first one, we have

$$\frac{1}{\epsilon}\left[E_\parallel\boldsymbol{b}\cdot\boldsymbol{e}_\perp + u_\perp(\boldsymbol{e}_\perp\times\boldsymbol{B})\cdot\boldsymbol{e}_\perp\right] = \left[\frac{\mathrm{d}(v_\parallel\boldsymbol{b})}{\mathrm{d}t} + \frac{\mathrm{d}(u_\perp\boldsymbol{e}_\perp)}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right]\cdot\boldsymbol{e}_\perp. \qquad (3.79)$$

The left-hand side is zero since $\boldsymbol{b}\perp\boldsymbol{e}_\perp$ and $(\boldsymbol{e}_\perp\times\boldsymbol{B})\cdot\boldsymbol{e}_\perp = (\boldsymbol{e}_\perp\times\boldsymbol{e}_\perp)\cdot\boldsymbol{B} = 0$. Thus we have

$$0 = \boldsymbol{e}_\perp\cdot v_\parallel\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \boldsymbol{e}_\perp\cdot\boldsymbol{b}\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} + \boldsymbol{e}_\perp\cdot\boldsymbol{e}_\perp\frac{\mathrm{d}u_\perp}{\mathrm{d}t} + u_\perp\boldsymbol{e}_\perp\cdot\frac{\mathrm{d}\boldsymbol{e}_\perp}{\mathrm{d}t} + \boldsymbol{e}_\perp\cdot\frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}. \qquad (3.80)$$

The second term vanishes because $\boldsymbol{e}_\perp\perp\boldsymbol{b}$ and the fourth term is

$$\boldsymbol{e}_\perp\cdot\frac{\mathrm{d}\boldsymbol{e}_\perp}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{e}_\perp\cdot\boldsymbol{e}_\perp) = 0, \qquad (3.81)$$

because $\boldsymbol{e}_\perp\cdot\boldsymbol{e}_\perp = 1$. Finally, we have

$$\frac{\mathrm{d}u_\perp}{\mathrm{d}t} = -\boldsymbol{e}_\perp\cdot\left(v_\parallel\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right). \qquad (3.82)$$

For the projection into the $\boldsymbol{b}$ direction we have:

$$\frac{1}{\epsilon}\left[E_\parallel\boldsymbol{b}\cdot\boldsymbol{b} + u_\perp(\boldsymbol{e}_\perp\times\boldsymbol{B})\cdot\boldsymbol{b}\right] = \left[\frac{\mathrm{d}(v_\parallel\boldsymbol{b})}{\mathrm{d}t} + \frac{\mathrm{d}(u_\perp\boldsymbol{e}_\perp)}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right]\cdot\boldsymbol{b}. \qquad (3.83)$$

The second term on the left-hand side of (3.83) equals to zero because $(\boldsymbol{e}_\perp\times\boldsymbol{B})\cdot\boldsymbol{b} = -(\boldsymbol{b}\times\boldsymbol{B})\cdot\boldsymbol{e}_\perp = 0$. Writing out the terms on the right-hand side explicitly, we obtain

$$\frac{E_\parallel}{\epsilon} = \boldsymbol{b}\cdot\boldsymbol{b}\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} + v_\parallel\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{e}_\perp\frac{\mathrm{d}u_\perp}{\mathrm{d}t} + u_\perp\boldsymbol{b}\cdot\frac{\mathrm{d}\boldsymbol{e}_\perp}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}\cdot\boldsymbol{b}}{\mathrm{d}t} - \boldsymbol{v}_\mathrm{E}\cdot\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}. \qquad (3.84)$$

The second term on the right-hand side of (3.84) is zero from the same arguments as in (3.81). The third term vanishes as $\boldsymbol{b} \perp \boldsymbol{e}_\perp$ and the fifth term also vanishes, because $\boldsymbol{v}_\mathrm{E} \perp \boldsymbol{b}$. The fourth term may be rewritten as

$$u_\perp \boldsymbol{b} \cdot \frac{\mathrm{d}\boldsymbol{e}_\perp}{\mathrm{d}t} = u_\perp \frac{\mathrm{d}\boldsymbol{e}_\perp \cdot \boldsymbol{b}}{\mathrm{d}t} - u_\perp \boldsymbol{e}_\perp \cdot \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} = -u_\perp \boldsymbol{e}_\perp \cdot \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}. \tag{3.85}$$

Finally, we have

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{E_\parallel}{\epsilon} + (u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E}) \cdot \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}. \tag{3.86}$$

The most intense algebra concerns the projection into the $\boldsymbol{e}_\rho$ direction, where we will evaluate the individual terms one by one.

$$\frac{1}{\epsilon}\left[E_\parallel \overset{\textcircled{1}}{\boldsymbol{b}} \cdot \boldsymbol{e}_\rho + u_\perp(\boldsymbol{e}_\perp \overset{\textcircled{2}}{\times} \boldsymbol{B}) \cdot \boldsymbol{e}_\rho\right] = \boldsymbol{e}_\rho \cdot \overset{\textcircled{3}}{\frac{\mathrm{d}(v_\parallel \boldsymbol{b})}{\mathrm{d}t}} + \boldsymbol{e}_\rho \cdot \overset{\textcircled{4}}{\frac{\mathrm{d}(u_\perp \boldsymbol{e}_\perp)}{\mathrm{d}t}} + \boldsymbol{e}_\rho \cdot \overset{\textcircled{5}}{\frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}}. \tag{3.87}$$

① :

$$\boldsymbol{b} \perp \boldsymbol{e}_\rho \Rightarrow \boldsymbol{b} \cdot \boldsymbol{e}_\rho = 0 \tag{3.88}$$

② :

$$u_\perp(\boldsymbol{e}_\perp \times \boldsymbol{B}) \cdot \boldsymbol{e}_\rho = u_\perp(\boldsymbol{e}_\rho \times \boldsymbol{e}_\perp) \cdot \boldsymbol{B} = u_\perp \left[\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}\right] \cdot \boldsymbol{B} =$$

$$= u_\perp \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} = -u_\perp B \tag{3.89}$$

③ :

$$\boldsymbol{e}_\rho \cdot \frac{\mathrm{d}(v_\parallel \boldsymbol{b})}{\mathrm{d}t} = \boldsymbol{e}_\rho \cdot v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \cancel{\boldsymbol{e}_\rho \cdot \boldsymbol{b}\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}} \tag{3.90}$$

④ :

$$\begin{aligned}
\boldsymbol{e}_\rho \cdot \frac{\mathrm{d}(u_\perp \boldsymbol{e}_\perp)}{\mathrm{d}t} &= u_\perp(\boldsymbol{e}_2\cos\phi - \boldsymbol{e}_1\sin\phi) \cdot \frac{\mathrm{d}(\boldsymbol{e}_1\cos\phi + \boldsymbol{e}_2\sin\phi)}{\mathrm{d}t} + \cancel{\boldsymbol{e}_\rho \cdot \boldsymbol{e}_\perp \frac{\mathrm{d}u_\perp}{\mathrm{d}t}} = \\
&= u_\perp \left[(\boldsymbol{e}_2\cos\phi - \boldsymbol{e}_1\sin\phi) \cdot (\cos\phi \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \boldsymbol{e}_1 \sin\phi \frac{\mathrm{d}\phi}{\mathrm{d}t} + \sin\phi \frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t} + \boldsymbol{e}_2\cos\phi \frac{\mathrm{d}\phi}{\mathrm{d}t})\right] = \\
&= u_\perp \left[\cos^2\phi\, \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \cancel{\sin\phi\cos\phi\, \boldsymbol{e}_1 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}} + \cancel{\sin\phi\cos\phi \frac{\mathrm{d}\phi}{\mathrm{d}t} \boldsymbol{e}_1 \cdot \boldsymbol{e}_2} + \sin^2\phi \frac{\mathrm{d}\phi}{\mathrm{d}t} \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 + \right. \\
&\left. + \cancel{\sin\phi\cos\phi\, \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t}} - \sin^2\phi\, \boldsymbol{e}_1 \cdot \frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t} + \cos^2 \frac{\mathrm{d}\phi}{\mathrm{d}t} \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 - \cancel{\sin\phi\cos\phi \frac{\mathrm{d}\phi}{\mathrm{d}t} \boldsymbol{e}_1 \cdot \boldsymbol{e}_2}\right] = \\
&= u_\perp \left[(\sin^2\phi + \cos^2\phi)\left(\frac{\mathrm{d}\phi}{\mathrm{d}t} + \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}\right) - \sin^2\phi \left(\boldsymbol{e}_1 \cdot \frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t} + \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}\right)\right] = \\
&= u_\perp \left[\frac{\mathrm{d}\phi}{\mathrm{d}t} + \boldsymbol{e}_2 \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \sin^2\phi \left(\cancel{\frac{\mathrm{d}(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)}{\mathrm{d}t}} - \cancel{\boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}} + \cancel{\boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}}\right)\right],
\end{aligned} \tag{3.91}$$

where in the last-but-one step we added and subtracted the term $\sin^2\phi\, \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}$.

⑤ remains unchanged

Hence, together we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{B}{\epsilon} - \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \frac{1}{u_\perp}\boldsymbol{e}_\rho \cdot \left(v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t} \right). \tag{3.92}$$

Let us repeat that now we have again six differential equations for modified variables the position of the gyration center $\boldsymbol{R}$ and for variables in the plane perpendicular to the field lines. Thus we have to solve the following set of equations:

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \epsilon \boldsymbol{u} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{b}}{B}\right) + \frac{\epsilon}{B}\boldsymbol{b} \times \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}, \tag{3.93}$$

$$\frac{\mathrm{d}u_\perp}{\mathrm{d}t} = -\boldsymbol{e}_\perp \cdot \left(v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t} \right), \tag{3.94}$$

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{E_\parallel}{\epsilon} + (u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E}) \cdot \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}, \tag{3.95}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{B}{\epsilon} - \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \frac{1}{u_\perp}\boldsymbol{e}_\rho \cdot \left(v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t} \right). \tag{3.96}$$

To proceed further, we need to replace the total time derivative by the derivative along the trajectory

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp \right) \cdot \boldsymbol{\nabla} \tag{3.97}$$

Let us do the proper derivation for (3.93). Let's first decompose $\boldsymbol{u}$ according to formula (3.70) and use relation (3.97) to express $\mathrm{d}/\mathrm{d}t$ in (3.93):

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = {} & v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \frac{\epsilon}{B}\left(v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp \right) \times \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp \right) \cdot \boldsymbol{\nabla} \right] \boldsymbol{b} + \\ & \epsilon \left(v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp \right) \times \boldsymbol{b} \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp \right) \cdot \boldsymbol{\nabla} \right] \left(\frac{1}{B} \right) + \\ & \frac{\epsilon}{B}\boldsymbol{b} \times \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} \right) \cdot \boldsymbol{\nabla} \right] \boldsymbol{v}_\mathrm{E} + \frac{\epsilon}{B}\boldsymbol{b} \times \left(u_\perp \boldsymbol{e}_\perp \cdot \boldsymbol{\nabla} \right) \boldsymbol{v}_\mathrm{E} \, . \end{aligned} \tag{3.98}$$

Next, we add a zero term

$$\frac{\epsilon}{B}\boldsymbol{b} \times \boldsymbol{b} \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} \right) \cdot \boldsymbol{\nabla} \right] v_\parallel = 0 \tag{3.99}$$

to the right hand side of (3.98). Then, we drop several terms containing $\boldsymbol{b} \times \boldsymbol{b}$ and rearrange the equation to obtain

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = {} & v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \frac{\epsilon}{B}\boldsymbol{b} \times D_t \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} \right) + \frac{\epsilon}{B} v_\parallel \boldsymbol{b} \times \left[u_\perp \boldsymbol{e}_\perp \cdot \boldsymbol{\nabla} \right] \boldsymbol{b} + \\ & \frac{\epsilon u_\perp}{B}\boldsymbol{e}_\perp \times \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp \right) \cdot \boldsymbol{\nabla} \right] \boldsymbol{b} - \\ & \frac{\epsilon u_\perp}{B^2}\boldsymbol{e}_\perp \times \boldsymbol{b} \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp \right) \cdot \boldsymbol{\nabla} \right] B + \frac{\epsilon}{B}\boldsymbol{b} \times \left(u_\perp \boldsymbol{e}_\perp \cdot \boldsymbol{\nabla} \right) \boldsymbol{v}_\mathrm{E} \, , \end{aligned} \tag{3.100}$$

where we denote

$$D_t \equiv \frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E}\right) \cdot \boldsymbol{\nabla} \ . \tag{3.101}$$

Equation (3.100) contains a lot of terms varying on different time scale. Obviously, the terms containing factors $\boldsymbol{e}_\perp$ or $\boldsymbol{e}_\rho$ will change with the gyration of the particle. Therefore we may call these terms as *fast* terms. Recalling back the "classical" derivation of the ∇B drift, we treated a similar problem by "averaging" over one Larmor rotation. In the case of (3.100) we will proceed by analogy. We will average (3.100) over ϕ.

The averaging has a mathematically correct basis. The set of equations (3.93) to (3.96) can be symbolically written in a form

$$\frac{\mathrm{d}z}{\mathrm{d}t} = f_z(z) = \langle f_z \rangle + \tilde{f}_z, \tag{3.102}$$

where $z \in \{\boldsymbol{R}, v_\parallel, u_\perp, \phi\}$. The angled brackets denote the average over the gyroangle ϕ, hence $\langle \bullet \rangle = (2\pi)^{-1} \int \mathrm{d}\phi(\bullet)$. The function $\tilde{f}_z$ is periodic in the fast-oscillating gyroangle ϕ. Such separation may always be done. We will show that the oscillating component can be removed to the leading order by means of the small parameter, ϵ, by the redefinition of the variables z to

$$\bar{z} = z + \frac{\epsilon}{B} \int_0^\phi \tilde{f}_z(\phi')\mathrm{d}\phi'. \tag{3.103}$$

From (3.96) we have:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{B}{\epsilon} - \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \frac{1}{u_\perp}\boldsymbol{e}_\rho \cdot \left(v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right) = -\frac{B}{\epsilon} + \mathcal{O}(\epsilon^0). \tag{3.104}$$

The total time derivative (3.97) in terms of the guiding centre variables reads

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} \cdot \frac{\partial}{\partial \boldsymbol{R}} + \frac{\mathrm{d}v_\parallel}{\mathrm{d}t}\frac{\partial}{\partial v_\parallel} + \frac{\mathrm{d}u_\perp}{\mathrm{d}t}\frac{\partial}{\partial u_\perp} + \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\partial}{\partial \phi}. \tag{3.105}$$

The last term is the leading order term, where for $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ we use (3.104). Note that actually the third term on the right-hand side is a leading order term ($\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} \sim \frac{E_\parallel}{\epsilon} \sim \mathcal{O}(\epsilon^{-1})$), however we silently assume here that $E_\parallel \sim \mathcal{O}(\epsilon)$, otherwise the assumption of small changes gets violated. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} = -\frac{B}{\epsilon}\frac{\partial}{\partial \phi} + \mathcal{O}(\epsilon^0). \tag{3.106}$$

Note that such an approximation is far from being general and is valid only for the system we investigate.

By applying the leading order time derivative (3.106) to the transformed variables (3.103) we have

$$\begin{aligned} \frac{\mathrm{d}\bar{z}}{\mathrm{d}t} &= \frac{\mathrm{d}z}{\mathrm{d}t} - \frac{B}{\epsilon}\frac{\partial}{\partial\phi}\frac{\epsilon}{B}\int_\phi \tilde{f}_z(\phi')\mathrm{d}\phi' = \\ &= \frac{\mathrm{d}z}{\mathrm{d}t} - \tilde{f}_z + \mathcal{O}(\epsilon) = \langle f_z(z)\rangle + \tilde{f}_z - \tilde{f}_z + \mathcal{O}(\epsilon) = \langle f_z(z)\rangle + \mathcal{O}(\epsilon). \end{aligned} \tag{3.107}$$

Note that the final expression contains z, e.g., non-averaged variables as arguments. Therefore we have a mixture of the $\bar{z}$ averaged (guiding-centre based) arguments on the left-hand side and non-averaged (particle-based) arguments on the right-hand side. Fortunately, the right-hand side can approximately be written in the guiding centre variables $(\boldsymbol{R}, v_\parallel, u_\perp, \phi)$. First, the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ (and consequently vectors $\boldsymbol{b}$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{v}_\mathrm{E}$) at the particle position are expanded in a Taylor series around the position of the guiding center. For instance:

$$\boldsymbol{b}(\boldsymbol{x}, t) = \boldsymbol{b}(\boldsymbol{R}, t) + \boldsymbol{\rho} \cdot \nabla \boldsymbol{b}(\boldsymbol{R}, t) + \dots = \boldsymbol{b}(\boldsymbol{R}, t) + \epsilon \frac{u_\perp}{B} \boldsymbol{e}_\rho \cdot \nabla \boldsymbol{b}(\boldsymbol{R}, t) + \dots \tag{3.108}$$

For the averaged variables even the first order term vanishes, because $\langle \boldsymbol{e}_\rho \rangle = 0$ and thus for (3.108) we have

$$\langle \boldsymbol{b}(\boldsymbol{x}, t) \rangle = \boldsymbol{b}(\boldsymbol{R}, t) + \mathcal{O}(\epsilon). \tag{3.109}$$

Then, finally:

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} = \langle f_z(\bar{z}) \rangle + \mathcal{O}(\epsilon), \quad \text{Q.E.D.} \tag{3.110}$$

Let us go back to averaging (3.100) over ϕ. For the terms independent of $\boldsymbol{e}_\perp$, this step does not lead to any change. On the other hand, terms linear in $\boldsymbol{e}_\perp$ are zero as $\langle \sin \psi \rangle = \langle \cos \psi \rangle = 0$. The terms quadratic in $\boldsymbol{e}_\perp$ are non-zero in general and have to be treated carefully.

$$\left\langle \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} \right\rangle = \left\langle v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \frac{\epsilon}{B} \boldsymbol{b} \times D_t \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} \right) + \frac{\epsilon u_\perp^2}{B} \boldsymbol{e}_\perp \times \left(\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla} \right) \boldsymbol{b} - \frac{\epsilon u_\perp^2}{B^2} \boldsymbol{e}_\perp \times \boldsymbol{b} \left(\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla} \right) B \right\rangle . \tag{3.111}$$

Now, let us consider the last term in (3.111). Using a notation $\nabla_1 \equiv \boldsymbol{e}_1 \cdot \boldsymbol{\nabla}$ and $\nabla_2 \equiv \boldsymbol{e}_2 \cdot \boldsymbol{\nabla}$ we may write:

$$\left(\boldsymbol{e}_\perp \times \boldsymbol{b} \right) \left(\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla} \right) B = \left(\boldsymbol{e}_1 \sin \phi - \boldsymbol{e}_2 \cos \phi \right) \left(\cos \phi \nabla_1 + \sin \phi \nabla_2 \right) B \tag{3.112}$$

Considering $\langle \sin \phi \cos \phi \rangle = 0$ and $\langle \sin^2 \phi \rangle = \langle \cos^2 \phi \rangle = \frac{1}{2}$, averaging of (3.112) over ϕ gives:

$$\left\langle \left(\boldsymbol{e}_\perp \times \boldsymbol{b} \right) \left(\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla} \right) B \right\rangle = \tfrac{1}{2} \left(\boldsymbol{e}_1 \nabla_2 B - \boldsymbol{e}_2 \nabla_1 B \right) = -\tfrac{1}{2} \left(\boldsymbol{b} \times \boldsymbol{\nabla} B \right) . \tag{3.113}$$

Note that an alternative derivation of this relation can be found in Appendix B.2.

A similar approach will be applied to the fourth term of the right-hand side of eq. (3.111). In this case, we have to keep in mind that the vector field $\boldsymbol{b}$ has components (0, 0, 1) at the point $\boldsymbol{R}$ though, but derivatives of all of its components are non-zero in general. Hence,

$$\begin{aligned}
\boldsymbol{e}_\perp \times \left(\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla} \right) \boldsymbol{b} &= \left(\boldsymbol{e}_1 \cos \phi + \boldsymbol{e}_2 \sin \phi \right) \times \left(\cos \phi \nabla_1 + \sin \phi \nabla_2 \right) \boldsymbol{b} \\
&= \left(\boldsymbol{e}_1 \cos \phi + \boldsymbol{e}_2 \sin \phi \right) \times \left[\boldsymbol{e}_1 \left(\cos \phi \nabla_1 + \sin \phi \nabla_2 \right) b_1 + \right. \\
&\quad \left. \boldsymbol{e}_2 \left(\cos \phi \nabla_1 + \sin \phi \nabla_2 \right) b_2 + \boldsymbol{e}_3 \left(\cos \phi \nabla_1 + \sin \phi \nabla_2 \right) b_3 \right] \\
&= \boldsymbol{e}_1 \left[\sin \phi \left(\cos \phi \nabla_1 + \sin \phi \nabla_2 \right) b_3 \right] - \boldsymbol{e}_2 \left[\cos \phi \left(\cos \phi \nabla_1 + \sin \phi \nabla_2 \right) b_3 \right] + \\
&\quad \boldsymbol{e}_3 \left[\cos \phi \left(\cos \phi \nabla_1 + \sin \phi \nabla_2 \right) b_2 - \sin \phi \left(\cos \phi \nabla_1 + \sin \phi \nabla_2 \right) \right] b_1
\end{aligned} \tag{3.114}$$

Averaging over ϕ then leads to

$$\langle \boldsymbol{e}_\perp \times (\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla})\, \boldsymbol{b} \rangle = \tfrac{1}{2}\left(\boldsymbol{e}_1 \nabla_2 - \boldsymbol{e}_2 \nabla_1\right) b_3 + \tfrac{1}{2}\boldsymbol{e}_3 \left(\nabla_1 b_2 - \nabla_2 b_1\right) \ . \tag{3.115}$$

First term on the right hand side of eq. (3.115) may be neglected as $\nabla_i b_3 = \mathcal{O}((\nabla_i b_{[1,2]})^2)$.[5]

The second term on the right-hand side of eq. (3.115) can be rewritten in the vector notation:

$$\langle \boldsymbol{e}_\perp \times (\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla})\, \boldsymbol{b} \rangle = \tfrac{1}{2}\boldsymbol{e}_3 \left(\boldsymbol{e}_3 \cdot \nabla \times \boldsymbol{b}\right) = \tfrac{1}{2}\boldsymbol{b} \left(\boldsymbol{b} \cdot \nabla \times \boldsymbol{b}\right) \ , \tag{3.118}$$

where we used the identity $\boldsymbol{b} \equiv \boldsymbol{e}_3$. Note that an alternative derivation of this relation can be found in Appendix B.2. Inserting (3.113) and (3.118) gives

$$\left\langle \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} \right\rangle = \overset{\textcircled{1}}{v_\parallel \boldsymbol{b}} + \overset{\textcircled{2}}{\boldsymbol{v}_\mathrm{E}} + \overset{\textcircled{3}}{\frac{\epsilon}{B}\boldsymbol{b} \times D_t \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E}\right)} + \overset{\textcircled{4}}{\frac{\epsilon u_\perp^2}{2B}\left(\boldsymbol{b} \cdot \boldsymbol{\nabla} \times \boldsymbol{b}\right) \boldsymbol{b}} + \overset{\textcircled{5}}{\frac{\epsilon u_\perp^2}{2B^2}\boldsymbol{b} \times \boldsymbol{\nabla} B} + \mathcal{O}(\epsilon^2) \ . \tag{3.119}$$

This is the equation for the guiding centre motion. It contains all drift we discussed in the previous chapters and much more.

① This term represents the movement along the field lines. Local vector $\boldsymbol{b}$ is curved in $(\boldsymbol{x}, t)$ space, nevertheless, the motion of the particle is always dominantly parallel to the local vector of the magnetic induction.

② This is the E-B drift.

③ This term contains the generalised curvature drift (term $\sim \frac{\epsilon}{B}\boldsymbol{b} \times (v_\parallel \boldsymbol{b} \cdot \boldsymbol{\nabla})(v_\parallel \boldsymbol{b})^6$), the polarisation drift[7] and the effects of the non-homogeneous electric field.

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[5] The fact that $\nabla_i b_3 = \mathcal{O}((\nabla_i b_{[1,2]})^2)$ is a consequence of normalisation $|\boldsymbol{b}| = 1$. Vector $\boldsymbol{b}$ at a small displacement $\boldsymbol{\delta R}$ from $\boldsymbol{R}$ can be expressed by means of Taylor series:

$$\boldsymbol{b}(\boldsymbol{R} + \boldsymbol{\delta R}) = \boldsymbol{b}(\boldsymbol{R}) + \boldsymbol{\delta R} \cdot \boldsymbol{\nabla} \boldsymbol{b}(\boldsymbol{R}) + \mathcal{O}(\delta R^2) \ . \tag{3.116}$$

Denoting $\boldsymbol{b}' \equiv \boldsymbol{b}(\boldsymbol{R} + \boldsymbol{\delta R})$ and $\delta b_i \equiv \boldsymbol{\delta R} \cdot \boldsymbol{\nabla} b_i(\boldsymbol{R})$, we write $\boldsymbol{b}' = \boldsymbol{b} + \delta b_1 \boldsymbol{e}_1 + \delta b_2 \boldsymbol{e}_2 + \delta b_3 \boldsymbol{e}_3$. We further consider $\boldsymbol{b} = (0,\, 0,\, 1)$ and both $\boldsymbol{b}$ and $\boldsymbol{b}'$ be normalised. Hence $|\boldsymbol{b}'|^2 = 1$ which implies

$$|\boldsymbol{b}'|^2 = 1 = \left| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \delta b_1 \\ \delta b_2 \\ \delta b_3 \end{pmatrix} \right|^2 = \delta b_1^2 + \delta b_2^2 + (1 + \delta b_3)^2 \sim \delta b_1^2 + \delta b_2^2 + \delta b_3^2 + 2\delta b_3 + 1. \tag{3.117}$$

and, therefore, $2\delta b_3 \approx -\delta b_1^2 - \delta b_2^2 - \delta b_3^2 \sim \mathcal{O}(\delta b_{[1,2]}^2)$, Q.E.D.

[6] Deriving it more explicitly, let's assume the parallel velocity to be constant (or weakly variable), then

$$\begin{aligned} \boldsymbol{v}_\mathrm{c} &= \frac{m v_\parallel^2}{qB}\left[\boldsymbol{b} \times (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{b}\right] = \frac{m v_\parallel^2}{qB^2}\boldsymbol{B} \times \left[\boldsymbol{b} \cdot (\boldsymbol{e}_1 \nabla b_1 + \boldsymbol{e}_2 \nabla b_2 + \boldsymbol{b} \nabla b_3)\right] = \\ &= \frac{m v_\parallel^2}{qB^2}\boldsymbol{B} \times \nabla b_3 = -\frac{m v_\parallel^2}{qB^2 R^2}\boldsymbol{B} \times \boldsymbol{R}, \end{aligned} \tag{3.120}$$

where we noted that in the local cylindrical coordinates, b_3 is along the constant radius R and in order to fulfill $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ in a vaccuum the field must decrease as $1/R$ and hence the gradient of the field must decrease as $-1/R^2 \boldsymbol{R}$. This relation is exactly the same as (3.34).

[7] Let us handle the third term of 3.119 a little more in detail, in each step always by focusing to one

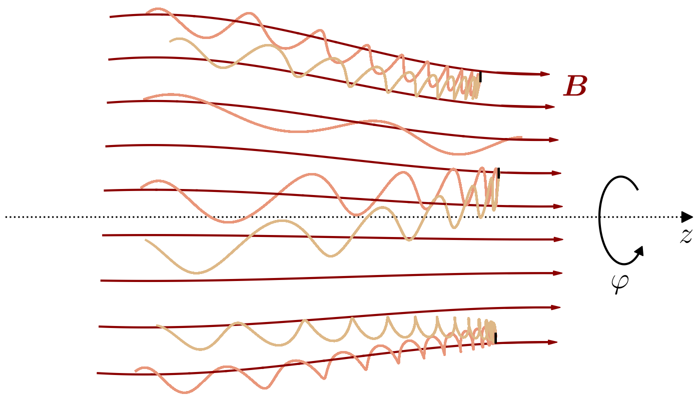

Figure 3.8: An illustration of the gyration of charged particles in the magnetic field with the configuration of the magnetic mirror. The particles gyrate along the field line, however they also undergo a drift between field-lines thanks to the drift in the curved magnetic field. As the particle approaches the strong-field region, it's motion in the parallel direction slows down. When it reaches zero, the particle reflects back to the region of the weak field. The reflection positions are indicated by vertical lines. Note that there is one particle trajectory indicated, which has a large initial parallel velocity, which does not reflect and in principle passes through the magnetic mirror to the other side.

(4) This is the correction to the parallel current: $\boldsymbol{b} \cdot \boldsymbol{\nabla} \times \boldsymbol{b} = \mu_0 j_\parallel / B$.

(5) This is the grad-B drift.

Similar equations might be derived for mean values of time derivatives of velocity components (see Appendix B.3), namely:

$$\left\langle \frac{\mathrm{d}v_\parallel}{\mathrm{d}t} \right\rangle = \frac{E_\parallel}{\epsilon} - \frac{u_\perp^2}{2B} \nabla_\parallel B + \boldsymbol{v}_\mathrm{E} \cdot D_t \boldsymbol{b} + \mathcal{O}(\epsilon) \tag{3.122}$$

$$\left\langle \frac{\mathrm{d}u_\perp}{\mathrm{d}t} \right\rangle = \frac{v_\parallel u_\perp}{2B} \nabla_\parallel B - \frac{u_\perp}{2} (\boldsymbol{\nabla} \cdot \boldsymbol{v}_\mathrm{E} - \boldsymbol{b} \cdot \nabla_\parallel \boldsymbol{v}_\mathrm{E}) + \mathcal{O}(\epsilon) \tag{3.123}$$

$$\left\langle \frac{\mathrm{d}\phi}{\mathrm{d}t} \right\rangle = -\frac{B}{\epsilon} - \boldsymbol{e}_2 \cdot D_t \boldsymbol{e}_1 - \frac{v_\parallel}{2} \boldsymbol{b} \cdot \boldsymbol{\nabla} \times (v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E}) + \mathcal{O}(\epsilon) \tag{3.124}$$

3.7 Magnetic mirrors

We deliberately did not study the motion of the testing particle in the case of an inhomogeneous magnetic field with $\nabla|\boldsymbol{B}| \parallel \boldsymbol{B}$. We will derive some useful properties of this configuration of the magnetic field now. Let us consider the axisymmetric configuration in cylindrical coordinates (such as that indicated in Fig. 3.8), where $\boldsymbol{B} = (B_r, B_\varphi, B_z)$ and $B_\varphi = 0$ and $\partial/\partial\varphi = 0$. From the Gauss's law for magnetism

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{1}{r}\frac{\partial}{\partial r} r B_r + \frac{1}{r}\frac{\partial B_\varphi}{\partial \varphi} + \frac{\partial B_z}{\partial z} = 0 \tag{3.125}$$

we immediately see that

$$r B_r = -\int_0^r r' \frac{\partial B_z}{\partial z} \mathrm{d}r'. \tag{3.126}$$

Assuming that we know $\frac{\partial B_z}{\partial z}$ on the axis and consider only particle motions close enough to the axis, we may neglect the radial dependence of this term and obtain

$$r B_r \sim -\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \int_0^r r' \mathrm{d}r' = -\frac{1}{2} r^2 \left[\frac{\partial B_z}{\partial z}\right]_{r=0}, \tag{3.127}$$

and hence

$$B_r \sim -\frac{1}{2} r \left[\frac{\partial B_z}{\partial z}\right]_{r=0}. \tag{3.128}$$

Thus we have an additional component to the Lorentz force, hence we may expect a drift. It will be a grad-B drift, but not the radial one, since $\partial B/\partial\varphi = 0$. The Lorentz force is

$$F = q\boldsymbol{v} \times \boldsymbol{B} = q \begin{pmatrix} v_r \\ v_\varphi \\ v_z \end{pmatrix} \times \begin{pmatrix} B_r \\ B_\varphi \\ B_z \end{pmatrix} = q \begin{pmatrix} v_\varphi B_z - v_z B_\varphi \\ v_z B_r - v_r B_z \\ v_r B_\varphi - v_\varphi B_r \end{pmatrix} =$$

$$= q \begin{pmatrix} v_\varphi B_z^{\textcircled{1}} \\ v_z B_r^{\textcircled{2}} - v_r B_z^{\textcircled{3}} \\ -v_\varphi B_r^{\textcircled{4}} \end{pmatrix}. \tag{3.129}$$

Terms ① and ③ are not interesting, as they are responsible for the Larmor rotation. Term ② vanishes around the axis, however even when we had relaxed the assumptions, this term would only cause the normal grad-B drift. Term ④ is something new.

particular term, while leaving the remaining terms unevaluated.

$$\frac{\epsilon}{B}\boldsymbol{b} \times D_t\left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E}\right) = \frac{\epsilon}{B}\boldsymbol{b} \times \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t} + \mathcal{A} = \frac{m}{qB^2}\boldsymbol{B} \times \frac{\mathrm{d}}{\mathrm{d}t}\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} + \mathcal{A} = \frac{m}{qB^4}\boldsymbol{B} \times \left(\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B}\right) + \mathcal{B} =$$

$$= \frac{m}{qB^4}\left(\boldsymbol{B}\cdot\boldsymbol{B}\frac{\partial \boldsymbol{E}}{\partial t} - \boldsymbol{B}\cdot\frac{\partial \boldsymbol{E}}{\partial t}\boldsymbol{B}\right) + \mathcal{B} = \frac{m}{qB^2}\frac{\partial \boldsymbol{E}}{\partial t} + \mathcal{C} = \frac{\operatorname{sgn} q}{\omega_\mathrm{c} B}\frac{\partial \boldsymbol{E}}{\partial t} + \mathcal{C}, \tag{3.121}$$

where the first term is the polarisation drift.

Hence we have an additional force

$$F_z = -qv_\varphi B_r = \frac{1}{2} qv_\varphi r \frac{\partial B_z}{\partial z}, \tag{3.130}$$

which we (in agreement with the procedure already used before) average over one Larmor rotation. By taking into account that $v_\varphi = -\operatorname{sgn} q v_\perp$, $r = r_\mathrm{L} = v_\perp/\omega_\mathrm{c}$ and $\omega_\mathrm{c} = q \operatorname{sgn} qB/m$ we have

$$F_z = -\frac{1}{2} qv_\perp \operatorname{sgn} q \frac{v_\perp m}{q \operatorname{sgn} qB} \frac{\partial B_z}{\partial z} = -\frac{1}{2} \frac{mv_\perp^2}{B} \frac{\partial B_z}{\partial z} = -\mu \frac{\partial B_z}{\partial z}, \tag{3.131}$$

where $\mu = \frac{1}{2} mv_\perp^2 / B$ is the *magnetic moment*. This equation may be generalised, when we identify z direction with the *parallel direction*, hence

$$F_\parallel = -\mu \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{s}} = -\mu \nabla_\parallel \boldsymbol{B}, \tag{3.132}$$

where $\mathrm{d}\boldsymbol{s}$ is the trajectory element.

We will further show that during the particle's motion, its r_L changes, but its magnetic moment remains invariant. First, we handle the parallel component of the equation of motion

$$m \frac{\mathrm{d}\boldsymbol{v}_\parallel}{\mathrm{d}t} = -\mu \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{s}} \tag{3.133}$$

by multiplying it by $\boldsymbol{v}_\parallel$:

$$m\boldsymbol{v}_\parallel \cdot \frac{\mathrm{d}\boldsymbol{v}_\parallel}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} mv_\parallel^2 \right) = -\mu \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{s}} \cdot \boldsymbol{v}_\parallel = -\mu \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{s}} \cdot \frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = -\mu \frac{\mathrm{d}B}{\mathrm{d}t}. \tag{3.134}$$

The energy conservation law gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} mv_\parallel^2 + \frac{1}{2} mv_\perp^2 \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} mv_\parallel^2 + \mu B \right) = 0, \tag{3.135}$$

where we used $\frac{1}{2} mv_\perp^2 = \mu B$. Finally we subtract (3.134) and (3.135) to have

$$0 = -\mu \frac{\mathrm{d}B}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t} \mu B = -\mu \frac{\mathrm{d}B}{\mathrm{d}t} + \mu \frac{\mathrm{d}B}{\mathrm{d}t} + B \frac{\mathrm{d}\mu}{\mathrm{d}t} \tag{3.136}$$

and hence

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0. \tag{3.137}$$

The magnetic moment remains invariant. Unfortunately, this conclusion cannot be considered general, as we took very restrictive assumptions at the beginning. It is interesting to study to what extent is the magnetic moment conservation valid. Here we will go back to the guiding centre equations we derived in the previous section. Let's work with (3.123) in a form

$$\frac{\mathrm{d}u_\perp}{\mathrm{d}t} = \frac{v_\parallel u_\perp}{2B} \boldsymbol{b} \cdot \nabla_\parallel B - \frac{u_\perp}{2} (\overset{\text{Ⓐ}}{\boldsymbol{\nabla} \cdot \boldsymbol{v}_\mathrm{E}} - \boldsymbol{b} \cdot \overset{\text{Ⓑ}}{\nabla_\parallel \boldsymbol{v}_\mathrm{E}}) + \mathcal{O}(\epsilon), \tag{3.138}$$

where $\nabla_\parallel \bullet \equiv \boldsymbol{b} \cdot \boldsymbol{\nabla} \bullet$. Let's trim the terms one-by-one:

(A)

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{v}_{\mathrm{E}} &= \boldsymbol{\nabla} \cdot \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} = \frac{1}{B^2}\left[\boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B})\right] + (\boldsymbol{E} \times \boldsymbol{B}) \cdot \nabla \frac{1}{B^2} = \\
&= -\frac{1}{B^2}\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \frac{\boldsymbol{E}_{\parallel} + \boldsymbol{E}_{\perp}}{B^2} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) - \frac{2(\boldsymbol{E} \times \boldsymbol{B})}{B^2} \cdot \frac{\nabla B}{B} = \\
&= -\frac{B}{B^2}\boldsymbol{b} \cdot \left[B\frac{\partial \boldsymbol{b}}{\partial t} + \boldsymbol{b}\frac{\partial B}{\partial t}\right] - \frac{\boldsymbol{E}_{\parallel}}{B^2} \cdot \left[(\nabla B) \times \boldsymbol{b} + B(\boldsymbol{\nabla} \times \boldsymbol{b})\right] - \\
&- \frac{\boldsymbol{E}_{\perp}}{B^2} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) - 2\boldsymbol{v}_{\mathrm{E}} \cdot \frac{\nabla B}{B} = -\boldsymbol{b} \cdot \frac{\partial \boldsymbol{b}}{\partial t} - \frac{1}{B}\frac{\partial B}{\partial t} - \\
&- \frac{E_{\parallel}}{B^2}\boldsymbol{b} \cdot \left[(\nabla B) \times \boldsymbol{b}\right] - \frac{E_{\parallel}}{B}\boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}),
\end{aligned}
\tag{3.139}
$$

where we used that $\boldsymbol{b} \cdot \frac{\partial \boldsymbol{b}}{\partial t} = 0$, defined the parallel amplitude of the electric intensity by $E_{\parallel} = \boldsymbol{E}_{\parallel} \cdot \boldsymbol{b}$, and also assumed $(\boldsymbol{\nabla} \times \boldsymbol{B}) \parallel \boldsymbol{B}$ (force-free approximation). We also applied Farraday's law $\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$.

(B)

$$
\begin{aligned}
\boldsymbol{b} \cdot \nabla_{\parallel} \boldsymbol{v}_{\mathrm{E}} &= \nabla_{\parallel}(\boldsymbol{b} \cdot \boldsymbol{v}_{\mathrm{E}}) - \boldsymbol{v}_{\mathrm{E}} \cdot (\nabla_{\parallel} \boldsymbol{b}) = -\boldsymbol{v}_{\mathrm{E}} \cdot \left[(\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{b}\right] = \\
&- \frac{\boldsymbol{v}_{\mathrm{E}}}{B} \cdot \left[(\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{B}\right] - (\boldsymbol{v}_{\mathrm{E}} \cdot \boldsymbol{B})\,(\boldsymbol{b} \cdot \boldsymbol{\nabla})\frac{1}{B} = -\frac{\boldsymbol{v}_{\mathrm{E}}}{B} \cdot \left[(\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{B}\right],
\end{aligned}
\tag{3.140}
$$

where we used $\boldsymbol{v}_{\mathrm{E}} \perp \boldsymbol{b}$. When adjusting the term $(\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{B}$ let us evaluate the term ∇B^2:

$$
\nabla B^2 = \nabla(\boldsymbol{B} \cdot \boldsymbol{B}) = 2\boldsymbol{B} \cdot \nabla \boldsymbol{B} = 2B\boldsymbol{b} \cdot \nabla \boldsymbol{B} \quad \text{and} \quad \nabla B^2 = 2B\nabla B. \tag{3.141}
$$

Hence $\boldsymbol{b} \cdot \boldsymbol{\nabla} B = \nabla B$.

We will further apply the assumption that $E_{\parallel} \sim \mathcal{O}(\epsilon)$, where we remind that $\epsilon = m/q$ is a small parameter. Hence all together (3.138) reads

$$
\begin{aligned}
\frac{\mathrm{d}u_{\perp}}{\mathrm{d}t} &= \frac{v_{\parallel} u_{\perp}}{2B}\boldsymbol{b} \cdot \nabla_{\parallel} B + \frac{u_{\perp}}{2}\frac{1}{B}\frac{\partial B}{\partial t} + \frac{u_{\perp}}{2}\boldsymbol{v}_{\mathrm{E}} \cdot \frac{\nabla B}{B} + \mathcal{O}(\epsilon) = \\
&= \frac{u_{\perp}}{2}\left[\frac{\partial}{\partial t} + (v_{\parallel}\boldsymbol{b} + \boldsymbol{v}_{\mathrm{E}}) \cdot \boldsymbol{\nabla}\right] B + \mathcal{O}(\epsilon) = \frac{u_{\perp}}{2B} D_t B + \mathcal{O}(\epsilon),
\end{aligned}
\tag{3.142}
$$

where again D_t is the full time derivative along the guiding centre trajectory.

Hence by taking the time derivative of the magnetic moment $\mu = u_{\perp}^2/2B$ and by using (3.142) we have

$$
\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{u_{\perp}\frac{\mathrm{d}}{\mathrm{d}t}u_{\perp}}{B} + \frac{u_{\perp}^2}{2B}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{B} = \frac{u_{\perp}^2}{2B^2} D_t B + \mathcal{O}(\epsilon) - \frac{u_{\perp}^2}{2B^2}\frac{\mathrm{d}B}{\mathrm{d}t} = \mathcal{O}(\epsilon). \tag{3.143}
$$

Hence the magnetic moment is invariant when the changes of the fields are small (the assumption of $\boldsymbol{E} \sim \mathcal{O}(\epsilon)$.

A measurable consequence of the magnetic moment conservation is the effect of *magnetic mirrors*. Let's imagine that a charged particle moves (due to the thermal motion for example) from the weak into the stronger magnetic field. It "sees" the stronger field, hence its Larmor radius decreases and perpendicular velocity increases, so that μ remains constant. But also the total kinetic energy must be conserved, hence the parallel velocity decreases as the particle enters the stronger field. For B large enough the parallel velocity drops to zero and the particle suffers from the reflection back to the weaker field (see Fig. 3.8).

The reflection is not perfect. For example particles with originally $v_\perp = 0$ have zero magnetic moments and thus penetrate the "magnetic trap". They "do not feel" the Lorentz force along the magnetic field lines. Similar behaviour may be expected for particles with small ratio of $v_\perp/v_\parallel$. Is there a limit to this ratio?

Let's assume that in the region of the weak field B_0, the particle has the following values of velocity components: $v_\perp = v_{\perp,0}$ and $v_\parallel = v_{\parallel,0}$. In the place of reflection we have $v_\perp = v'_\perp$ and $v_\parallel = 0$ and the field has the induction B'. Since

$$\mu = \text{const} = \frac{1}{2} m v_{\perp,0}^2 / B_0 = \frac{1}{2} m v'_\perp / B' \quad \text{and} \quad v'^2_\perp = v_{\perp,0}^2 + v_{\parallel,0}^2 \equiv v_0^2, \tag{3.144}$$

we have

$$\frac{B_0}{B'} = \frac{v_{\perp,0}^2}{v'^2_\perp} = \frac{v_{\perp,0}^2}{v_0^2} = \sin^2\vartheta, \tag{3.145}$$

where ϑ is a *pitch angle* in the region of the weak field. The smaller ϑ is, the stronger magnetic field needs to be reached at the place of reflection. Considering that there is some maximum finite value of magnetic field, $B_{\max}$, in the real setup, particles with sufficiently small value of ϑ may not get reflected. From the equation above it becomes that the maximal pitch angle for the particle to reflect is

$$\sin^2\vartheta_{\max} = \frac{B_0}{B_{\max}}. \tag{3.146}$$

If we plot all three components of velocity for the particles (Fig. 3.9), we find that $\vartheta_{\max}$ creates a cone in the phase space. Particles located inside this cone will escape the reflection. Hence the name *loss cone*. Note that the shape of the loss cone depends neither on a charge of the particle nor on its mass.

3.8 Adiabatic invariants

Let's have a hamiltonian of the system $\mathcal{H} = \mathcal{H}(\boldsymbol{p}, \boldsymbol{q})$ and a system of equations of motion

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial\boldsymbol{p}} \quad \text{and} \quad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial\boldsymbol{q}}. \tag{3.147}$$

Should any of the coordinates q_i by *cyclic*, thus

$$q_i(t) = q_i(t+T), \tag{3.148}$$

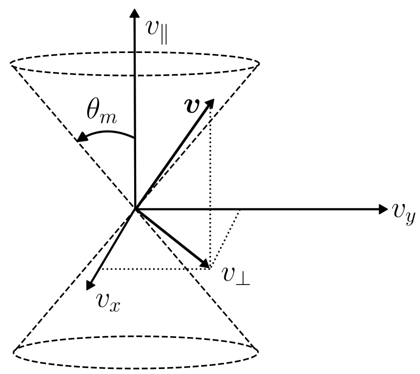

Figure 3.9: The illustration of the loss cone.

where T is the period, then

$$\oint p_i \mathrm{d}q_i = \text{const} \tag{3.149}$$

is the *integral of motion*. For example, in case of the mathematical pendulum, the deviation $\varphi(t) = \varphi_0 \cos \omega t$ is cyclic with period of $T = 2\pi/\omega$. Let's define the canonical pair $q = \varphi$ and $p = ml^2\dot{\varphi}$, where m and l are the mass and the length of the pendulum, respectively. Then we have

$$\int\limits_0^T p\mathrm{d}q = \int\limits_0^T (-ml^2\varphi_0\omega \sin \omega t)(-\varphi_0\omega \sin \omega t)\mathrm{d}(\omega t) =$$

$$= ml^2\varphi_0^2\omega \int\limits_0^T \sin^2 \omega t \mathrm{d}(\omega t) = \pi ml^2\varphi_0^2\omega = \text{const.} \tag{3.150}$$

When the integral of motion contains a parameter, which is slowly variable (slowly means that the time scale of the change of this parameter is much larger than period T of the cyclic coordinate), it turns into the *adiabatic invariant*. In the case of the pendulum, the invariant is conserved when we slowly change the length of the pendulum. Clearly, when l increases, ω must decrease in order for the invariant to remain constant. That is true only if the time scale, with which the length of the pendulum changes, is much longer than the period T of the cyclic coordinate. Should the time scale of the length changes be too short, the system behaves in a completely different regime.

3.8.1 First adiabatic invariant

Charged particles moving in the background electric and magnetic field have coordinate ϕ, which is obviously cyclic and connected to Larmor rotation, see (3.70). The canonically conjugated momentum is an *angular momentum* $p = mv_\perp r_\mathrm{L}$. Hence

$$J_1 = \oint p_i \mathrm{d}q_i = \int\limits_0^{2\pi} mv_\perp r_\mathrm{L} \,\mathrm{d}\phi = 2\pi v_\perp \frac{m^2 v_\perp}{q \operatorname{sgn} qB} = \frac{4\pi m}{q \operatorname{sgn} q}\mu = \text{const.} \tag{3.151}$$

It basically proves what we have derived in the previous chapters: for small changes the magnetic moment μ is an adiabatic invariant.

There are some important situations in which the magnetic moment is not invariant:

- *Magnetic pumping*: If the collision frequency is larger than the pump frequency, μ is no longer conserved. In particular, collisions allow net heating by transferring some of the perpendicular energy to parallel energy.

- *Cyclotron heating*: If $\boldsymbol{B}$ is oscillating at the cyclotron frequency, the condition for adiabatic invariance is violated and heating is possible. In particular, the induced electric field rotates in phase with some of the particles and continuously accelerates them.

- *Magnetic cusps*: The magnetic field at the center of a cusp vanishes, so the cyclotron frequency is automatically smaller than the rate of *any* changes. Thus the magnetic moment is not conserved and particles are scattered relatively easily into the loss cone.

3.8.2 Second adiabatic invariant

Let us consider a system with two magnetic mirrors and the particles out of the loss cone, thus cycling from one mirror to another. This motion is cyclic, thus this motion would in the normal system have a motion integral $I = \oint_a^b mv_\parallel \mathrm{d}s$. In plasmas the motions of particles are subjects of drifts, hence the absolute periodicity is violated. We would assume that the motion integral becomes an adiabatic invariant when the effects of the drifts are much smaller than the dominant motion of the system (reflection between two mirrors).

We will derive an *longitudinal invariant* J, which is defined only between two consecutive reflections, hence

$$J_2 = \int_a^b v_\parallel \mathrm{d}s \tag{3.152}$$

and we will show that J is invariant in the stationary magnetic field. Further, we will generalise this case when $\frac{\partial B}{\partial t}$ is small. When the distance between the two mirrors shortens, the reflecting particles gain kinetic energy, which is the basics of the *Fermi acceleration*.

Let's study the system of an asymmetrical magnetosphere, when due to the drifts the particle occupies different geometries ("a different field line"). The particle drifts from one field line to another, which has a different curvature R_k in time Δt (see Fig. 3.10). Then for small angles we have

$$\frac{\delta s}{R_k} = \frac{\delta s'}{R'_k}. \tag{3.153}$$

The fractional change of the trajectory computed by subtracting 1 from both sides of

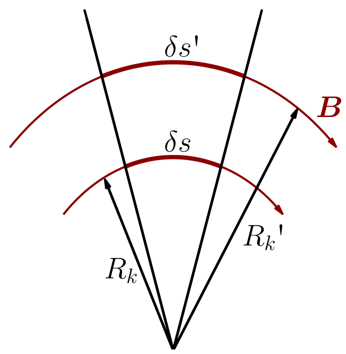

Figure 3.10: The scheme of the coordinate system used in the derivation of the second adiabatic invariant.

(3.153) and dividing them by Δt is

$$\frac{\delta s' - \delta s}{\Delta t \delta s} = \frac{R_k' - R_k}{\Delta t R_k} = \boldsymbol{v}_{\rm gc} \cdot \frac{\boldsymbol{R}_k}{R_k^2} = \frac{1}{\delta s}\frac{\Delta \delta s}{\Delta t} = \frac{1}{\delta s}\frac{\mathrm{d}\delta s}{\mathrm{d}t}, \tag{3.154}$$

where we used the radial component of the curvature drift to estimate the speed of jump between the field lines

$$\left|\frac{R_k' - R_k}{\Delta t}\right| = \boldsymbol{v}_{\rm gc} \cdot \frac{\boldsymbol{R}_k}{R_k}. \tag{3.155}$$

We studied the drift speed in the curved magnetic field earlier, thus we only apply results from (3.50) here:

$$\boldsymbol{v}_{\rm gc} = \frac{1}{2}\,\mathrm{sgn}\, q v_\perp r_{\rm L} \frac{\boldsymbol{B} \times \nabla B}{B^2} + \frac{m v_\parallel^2}{q}\frac{\boldsymbol{R_k} \times \boldsymbol{B}}{R_k^2 B^2}. \tag{3.156}$$

The second term does not have any component along $\boldsymbol{R}_k$, thus we do not have to consider it further. Hence

$$\frac{1}{\delta s}\frac{\mathrm{d}\delta s}{\mathrm{d}t} = \frac{1}{2}\frac{m}{q}\frac{v_\perp^2}{B^3}(\boldsymbol{B} \times \nabla B) \cdot \frac{\boldsymbol{R}_k}{R_k^2}, \tag{3.157}$$

which represents the fractional change of δs from the particle's point of view. Further, we will investigate the change of $v_\parallel$. Let's use the total energy of the particle W,

$$W = \frac{1}{2}m v_\parallel^2 + \frac{1}{2}m v_\perp^2 = \frac{1}{2}m v_\parallel^2 + \mu B = W_\parallel + W_\perp. \tag{3.158}$$

The parallel velocity is then $v_\parallel = \sqrt{\frac{2}{m}(W - \mu B)}$ and its time derivative

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{1}{2}\frac{1}{\sqrt{\frac{2}{m}(W - \mu B)}}\left(-\mu\frac{\mathrm{d}B}{\mathrm{d}t}\right)\frac{2}{m} \tag{3.159}$$

and the fractional change of $v_\parallel$ is

$$\frac{\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}}{v_\parallel} = -\frac{1}{2}\frac{\mu\frac{\mathrm{d}B}{\mathrm{d}t}}{W - \mu B} = -\frac{\mu\frac{\mathrm{d}B}{\mathrm{d}t}}{m v_\parallel^2}, \tag{3.160}$$

where we used (3.158) to express $W - \mu B$. The field $\boldsymbol{B}$ is stationary, but its derivative (seen by the moving particle) is generally not vanishing due to the drift :

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\mathrm{d}B}{\mathrm{d}R}\frac{\mathrm{d}R}{\mathrm{d}t} = \boldsymbol{v}_{\mathrm{gc}} \cdot \nabla B = \frac{mv_\parallel^2}{q}\frac{\boldsymbol{R}_k \times \boldsymbol{B}}{R_k^2 B^2} \cdot \nabla B. \tag{3.161}$$

We insert this useful relation into (3.160) and have

$$\frac{\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}}{v_\parallel} = -\frac{\mu}{q}\frac{(\boldsymbol{R}_k \times \boldsymbol{B}) \cdot \nabla B}{R_k^2 B^2} = -\frac{1}{2}\frac{mv_\perp^2}{Bq}\frac{(\boldsymbol{B} \times \nabla B) \cdot \boldsymbol{R}_k}{R_k^2 B^2}. \tag{3.162}$$

The goal is to evaluate the fractional change of $v_\parallel \delta s$ during the exchange of trajectory:

$$\frac{1}{v_\parallel \delta s}\frac{\mathrm{d}(v_\parallel \delta s)}{\mathrm{d}t} = \frac{1}{\delta s}\frac{\mathrm{d}\delta s}{\mathrm{d}t} + \frac{1}{v_\parallel}\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}. \tag{3.163}$$

The first term we already evaluated in (3.157) and the second in (3.162). They both equal except for the sign, hence

$$\frac{1}{v_\parallel \delta s}\frac{\mathrm{d}(v_\parallel \delta s)}{\mathrm{d}t} = 0 \tag{3.164}$$

and $v_\parallel \delta s$ remains constant when the particle exchanges the field line. This is not the claim we wanted to prove, which was that $J = \int_a^b v_\parallel \delta s$ is constant (as indicated in Fig. 3.11). However, we may split this integral into three terms,

$$J_2 = \int_a^{a'} v_\parallel \delta s + \int_{a'}^{b'} v_\parallel \delta s + \int_{b'}^{b} v_\parallel \delta s, \tag{3.165}$$

representing the different reflection points for the different field lines. When a and a' are close and b and b' are also close, then the contributions of the first and third terms are negligible and J remains approximately invariant. The smallness of these boundary contributions is emphasised by the fact that $v_\parallel$ is small near the reflection points (where the total velocity amplitude is dominated by the perpendicular component).

By replacing the local longitudinal velocity by its average $\langle v_\parallel \rangle$ over the distance between the two reflection points L, we have

$$J_2 = \langle v_\parallel \rangle L, \tag{3.166}$$

which is invariant. When L decreases, $\langle v_\parallel \rangle$ increases, which is the basics of the Fermi acceleration. High-energetic particles of the cosmic radiation of the galactic origin were mostly accelerated by this mechanism, e.g. by repeated reflections between the magnetic field embedded in the interstellar gaseous clouds, which moved towards each other.

In an axially nonsymmetric magnetosphere – such as the magnetosphere of the Earth – the gravitation drift drives the ring current, which drives the "jumping of the particles between different field lines". The second adiabatic invariant is conserved on both sides of the magnetosphere.

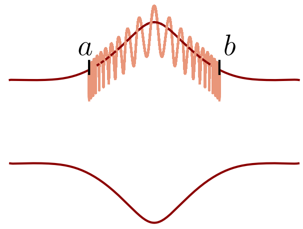

Figure 3.11: The scheme of gyrating particles trapped between two reflecting magnetic mirrors.

3.8.3 Third adiabatic invariant

This adiabatic invariant is defined for an axisymmetrical magnetic field when the curvature drift is also cyclical. Then the total magnetic flux enclosed by a drift surface

$$J_3 = \Phi = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \tag{3.167}$$

is invariant. It is associated with the periodic motion of mirror-trapped particles drifting around the axis of the system. The particle moves on the surface which adjusts itself to variations in the magnetic field so that the flux enclosed by this surface remains constant.

When the waves in the Earth's ionosphere (usually having long periods) are in resonance with the drifting particles, the energy of the drift converts to the energy of these waves. The drift is relatively slow, thus in practice, Φ is not conserved in real systems.

3.9 van Allen radiation belts

We will apply what we have just learned to a realistic problem, which is the particle trapping in the axially symmetric magnetosphere. This is the reason why there are radiation (known as van Allen) belts in the Earth's magnetosphere and also in the magnetospheres of other planets. We do not aim to solve the full problem, we aim more to show the basic principles using a rather simplistic approach. We will search for allowed trajectories of the particles.

Let us assume the axially symmetric magnetic field with components

$$\boldsymbol{B} = \left(B_R(R,z), 0, B_z(R,z)\right), \tag{3.168}$$

where B_R and B_z are the radial and vertical components of the magnetic field induction vector. The Gauss law must be fulfilled also in this case, hence

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 = \frac{1}{R}\frac{\partial}{\partial R} R B_R + \frac{\partial B_z}{\partial z}. \tag{3.169}$$

Following the recipe exploited further in Section 4.4.2, such condition is fulfilled when the radial and vertical components of the magnetic field are given (4.69) as

$$B_z = \frac{1}{R}\frac{\partial F}{\partial R} \quad \text{and} \quad B_R = -\frac{1}{R}\frac{\partial F}{\partial z}. \tag{3.170}$$

The function F is called the flux function and it can be explicitly computed for the given configuration of the magnetic field. E.g., for a dipole (see Appendix C.1)

$$F = \mathcal{M}\frac{R^2}{(R^2+z^2)^{3/2}}, \tag{3.171}$$

where $\mathcal{M}$ is the magnetic moment of a dipole[8]. For now, we can take F as a given function, by taking the spatial derivatives of which we obtain the magnetic induction components. It may easily be shown that definition (3.170) fulfills the Gauss's law.

Now, let us deal with the particle equation of motion in the cylindrical coordinates and we will be interested in the azimuthal component of it:

$$\left(m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_\varphi = q\left(v_z B_R - v_R B_z\right) = -\frac{q}{R}\left(v_z\frac{\partial F}{\partial z} + v_R\frac{\partial F}{\partial R}\right) = -\frac{q}{R}\frac{\mathrm{d}F}{\mathrm{d}t}, \tag{3.172}$$

where we used $v_z = \frac{\mathrm{d}z}{\mathrm{d}t}$ and $v_R = \frac{\mathrm{d}R}{\mathrm{d}t}$ and assumed that $\frac{\partial F}{\partial t} = \frac{\partial F}{\partial \varphi} = 0$. Due to the cylindrical symmetry, the remaining terms of the full time derivative $\frac{\mathrm{d}F}{\mathrm{d}t}$ related to the φ-component vanish.

Let us look at the coordinate system we have. We defined two linear coordinates R (radial coordinate, distance from the axis) and z (height) and one angular (the azimuthal angle) φ. The studied particle has a position vector $\boldsymbol{r}$. Obviously, the unit vector which is tangential to the azimuth φ at the given point is parallel to the cross-product of the position vector $\boldsymbol{r}$ and the unit vector in the z-direction, hence

$$\boldsymbol{e}_\varphi \propto \boldsymbol{e}_z \times \boldsymbol{r}. \tag{3.173}$$

But what is its amplitude? We can easily determine it from the amplitude of the cross product, which is

$$|z||r|\sin\vartheta = R, \tag{3.174}$$

where ϑ is an angle between $\boldsymbol{e}_z$ and $\boldsymbol{r}$ (also termed as co-latitude). Altogether we have

$$R\boldsymbol{e}_\varphi = \boldsymbol{e}_z \times \boldsymbol{r}. \tag{3.175}$$

Then

$$Rv_\varphi = R\boldsymbol{e}_\varphi \cdot \boldsymbol{v} = (\boldsymbol{e}_z \times \boldsymbol{r}) \cdot \boldsymbol{v}. \tag{3.176}$$

Now, we compute the time derivative of this expression. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}Rv_\varphi = (\boldsymbol{e}_z \times \boldsymbol{r}) \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + (\boldsymbol{e}_z \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}) \cdot \boldsymbol{v}. \tag{3.177}$$

[8]For instance, the dipole magnetic moment of the Earth is $\mathcal{M}_\mathrm{E} = 8 \times 10^{15}\ \mathrm{T\,m^3}$.

The second term vanishes as $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v}$ and the result of the cross product is perpendicular to $\boldsymbol{v}$. By using (3.175) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} R v_\varphi = R \boldsymbol{e}_\varphi \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = R \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \right)_\varphi . \tag{3.178}$$

Finally, let us investigate the expression

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(R v_\varphi + \frac{qF}{m} \right) = R \frac{\mathrm{d}v_\varphi}{\mathrm{d}t} + \frac{q}{m} \frac{\mathrm{d}F}{\mathrm{d}t} = 0, \tag{3.179}$$

where we used (3.178) and (3.172). Hence

$$R v_\varphi + \frac{qF}{m} = I_1 \tag{3.180}$$

is a motion integral.

Another motion integral is the total kinetic energy (we assumed that no dissipation takes place), which may be written in a simplistic form as

$$v_R^2 + v_\varphi^2 + v_z^2 = v^2 = I_2^2 \tag{3.181}$$

Certainly, $v_\varphi^2 \leq v^2$, as v_φ is only one component of $\boldsymbol{v}$.

Then

$$R|v_\varphi| \leq |v| R \quad \rightarrow \quad \left| I_1 - \frac{qF}{m} \right| \leq R I_2 . \tag{3.182}$$

We will not solve this inequation explicitly but will rather point out some general properties of it by drawing a graph (see Fig. 3.12). On the horizontal axis let us represent the radial coordinate R, on the vertical axis then the value of the expression $\frac{qF}{m}(R, \vartheta)$, where ϑ again represents the co-latitude (see Appendix C.2). The solutions for a given combination of q, F, and m will show as the hyperbola-like curves for varying ϑ in the canvas. The degenerate limiting $\vartheta = 0$ curve coincides with the coordinate axes.

The remaining terms in the expression will create a sector on the canvas, the apex of which will be at a coordinate $(0, I_1)$ and the opening angle will correspond to I_2 (the larger, the more is the sector opened). Particles with a given kinetic energy and the value of I_1 are now allowed to move in this diagram freely along their corresponding $\frac{qF}{m}$-line for a given ϑ, but only in the limits determined by the sector.

This simplistic approach gave us some important physical predictions. Particles with larger speeds are allowed to more within the larger range of radial distances. At lower co-latitudes (hence near the poles) the particles are in general located nearer the surface (lower R) and the extent of their allowed radial coordinate is lower (the belt is thinner) than near the equator. These predictions are fully consistent with the observations of the real van Allen radiation belts.

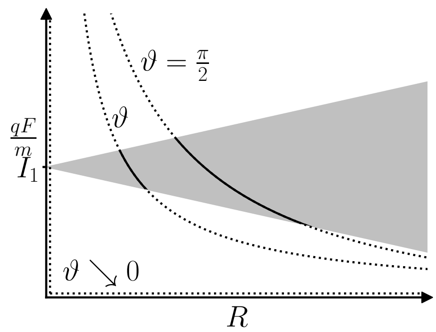

Figure 3.12: A graphical solution of the trapped particles in the axisymmetrical magnetosphere. In the shadowed regions, the particles are allowed due to the inequality (3.182). For a given ϑ, the particle may move along the given field line. Hence the total allowed region is given by the intersection of the two (denoted by the solid thick line). Such a region obviously depends on the co-latitude and the energy of the particle.

Chapter 4

Plasma as fluid

In Section 1.5, we derived the moments of the Boltzmann equation by neglecting the particle distribution function and by considering the physical parameters averaged over the ensemble of the particles. Plasma may be treated the same way, however, we have to bear in mind that the individual particles are charged and thus must follow an additional set of physical laws – the Maxwell equations.

4.1 Double-fluid model in physics of plasmas

When we consider electrons and ions non-interacting or at least weakly interacting, we may separate the particle distribution function to two terms, describing independently electrons and ions. Hence we have two independent “fluids” in the plasma, the electron fluid (described by n_e, $\boldsymbol{u}_\mathrm{e}$, p_e,...) and the ion fluid (fully described by n_i, $\boldsymbol{u}_\mathrm{i}$, p_i,...). For each type of particles α, where $\alpha \in \{\mathrm{e}, \mathrm{i}\}$, we have the equation of continuity, the Euler equation, and the equation of state, hence

$$\frac{\partial n_\alpha}{\partial t} + \boldsymbol{\nabla} \cdot (n_\alpha \boldsymbol{u}_\alpha) = 0 \tag{4.1}$$

$$m_\alpha n_\alpha \left[\frac{\partial \boldsymbol{u}_\alpha}{\partial t} + (\boldsymbol{u}_\alpha \cdot \boldsymbol{\nabla})\boldsymbol{u}_\alpha\right] = q_\alpha n_\alpha (\boldsymbol{E} + \boldsymbol{u}_\alpha \times \boldsymbol{B}) - \nabla p_\alpha \tag{4.2}$$

$$p_\alpha = C_\alpha (m_\alpha n_\alpha)^{\gamma_\alpha}, \tag{4.3}$$

where γ_α is the ratio of specific heats and C_α is a constant. This set of equations must be accompanied by the Maxwell equations

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = n_\mathrm{i} q_\mathrm{i} + n_\mathrm{e} q_\mathrm{e} \tag{4.4}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{4.5}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{4.6}$$

$$\frac{1}{\mu_0} \boldsymbol{\nabla} \times \boldsymbol{B} = n_\mathrm{i} q_\mathrm{i} \boldsymbol{u}_\mathrm{i} + n_\mathrm{e} q_\mathrm{e} \boldsymbol{u}_\mathrm{e} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}. \tag{4.7}$$

Note that on the general expression of the right-hand side of (4.4) for a mixture of different charges α stands $\sum_\alpha n_\alpha q_\alpha$. Similarly, the sum of the first two terms on the right-hand side of (4.7) for a general mixture of charges turns into $\sum_\alpha n_\alpha q_\alpha \boldsymbol{u}_\alpha$. Also, on the right-hand side of (4.5) additional force terms may be added, such as the action of the gravity force $m_\alpha n_\alpha \boldsymbol{g}$ or others.

The above given set of equations represents all together 18 equations (by considering the vector equations as a triplet) for 16 variables (for each electrons and ions we have n, u_x, u_y, u_z, and p and we must add components of the magnetic and electric fields). The disharmony is only apparent. Equations (4.4) and (4.6) play the role of boundary conditions and are not independent from (4.5) and (4.7). For instance, when we apply the operator of divergence to (4.5), we have

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{B}. \tag{4.8}$$

The left-hand side is a vanishing vector identity, hence

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \text{const}, \tag{4.9}$$

and our universe is set up so that this constant equals to zero. The physical meaning of this boundary condition is that there are no magnetic charges.

The equation of state (4.3) requires some commentary. Note that when dropping the index α, we define the mass density as $\rho = mn$. The *polytropic* equation of state in the new variable reads

$$p = C\rho^\gamma \quad \text{or} \quad \frac{\nabla p}{p} = \gamma \frac{\nabla \rho}{\rho}. \tag{4.10}$$

The value of γ varies with the type of the process in question. E.g. for the isothermic process, we have $\gamma = 1$. For the adiabatic process, $\gamma = (N+2)/N$, where N is the number of degrees of freedom. Note that $\gamma = 5/3$ for the adiabatic process of the ideal gas.

4.1.1 Plasma approximation

In solving the problems using the fluid approximation, we often use the *plasma approximation*. That essentially means that we neglect the possible concentration of charges occurring on small scales, and consider large scales only. Due to the requirement on the bulk neutrality of plasma, we must assume that the particle density of both fluids equals, hence $n_\mathrm{e} = n_\mathrm{i} = n$. In the process of deriving useful relations, we then do not use the Poisson equation (4.4), which would otherwise lead to a discrepancy, because in detail, $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$, which is not true due to the collective behaviour. We will show later that the plasma approximation is valid only when scales larger than Debye shielding length are considered.

4.2 Highly ionised plasmas

The need to capture the particle interactions on scales smaller than the Debye length leads naturally to additional term in the Euler equation. Such term is termed *collisional* *term*, even though strictly speaking it is not the same as the collisional term in the Boltzmann equation we derived earlier. The Euler equation for electrons then reads

$$m_\mathrm{e} n_\mathrm{e} \frac{\mathrm{d}\boldsymbol{u}_\mathrm{e}}{\mathrm{d}t} = -e n_\mathrm{e}(\boldsymbol{E} + \boldsymbol{u}_\mathrm{e} \times \boldsymbol{B}) - \nabla p_\mathrm{e} + \boldsymbol{P}_\mathrm{ei}. \tag{4.11}$$

The term P_ei represents the momentum change due to the collisions with ions. Due to the momentum conservation law, it is clear that

$$\boldsymbol{P}_\mathrm{ei} = -\boldsymbol{P}_\mathrm{ie}. \tag{4.12}$$

Heuristically, the collisional term may be written as

$$\boldsymbol{P}_\mathrm{ei} = \underbrace{\eta}_{⑤} \underbrace{e^2}_{①} \underbrace{n_\mathrm{e}}_{②} \underbrace{n_\mathrm{i} \underbrace{(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e})}_{③}}_{④} \sim \eta e^2 n^2 (\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}), \tag{4.13}$$

where we may notice the construction of the expression from the following

① The interaction is coulombic; hence, proportional to the multiplication of charges of both species.

② The rate of collisions is proportional to the electron flux.

③ Relative speed of the fluids.

④ Describes how often do the electrons meet ions.

⑤ The scaling parameter (describing the efficiency of the collisions), it has a physical meaning of the *specific resistivity.*

Alternatively,

$$\boldsymbol{P}_\mathrm{ei} = m_\mathrm{e} n (\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}) \nu_\mathrm{ei}, \tag{4.14}$$

where ν_ei is the collisional frequency, with which the electrons hit the ions. Such expression may also be viewed phenomenologically: the expression $m_\mathrm{e} n(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e})$ physically means the momentum exchanged due to the collisions, which is "normalised" by the collisional frequency (as the Euler equation evaluates the change of the momentum in time). Note that the particle density n does not hold the indication of species as the plasma approximation is assumed here. By comparing the two formulations of P_ei we obtain the expression for the collisional frequency as

$$\nu_\mathrm{ei} = \frac{ne^2}{m_\mathrm{e}} \eta = \omega_\mathrm{p}^2 \varepsilon_0 \eta, \tag{4.15}$$

where ω_p is a plasma frequency (to be defined in Section 5.4.1).

Let us estimate the specific resistivity η from a model of the electron squeezing around the ion and hence losing some portion of its momentum. See Fig. 4.1. The

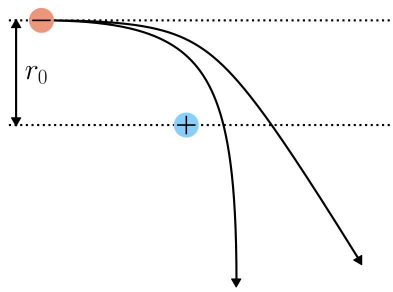

Figure 4.1: The geometry to estimate the specific resistivity

interaction is driven by the electric force, hence the force is

$$F = -\frac{e^2}{4\pi\varepsilon_0 r^2}, \tag{4.16}$$

where r is the distance between the ion (which is considered fixed in this coordinate system) and the passing electron. We roughly estimate that the interaction takes the total time Δt, where

$$\Delta t = \frac{r_0}{v}, \tag{4.17}$$

where r_0 is the impact parameter and v is the mutual speed of the electron and ion. This approximation assumes that the interaction is most effective for the short distances, we estimate the limit by r_0. Here we assumed that the essential interaction occurs on the scales of the impact parameter. During the interaction, the electron's momentum changes by

$$\Delta(m_\mathrm{e} v) \sim |F\,\Delta t| = \frac{e^2}{4\pi\varepsilon_0 r_0 v}. \tag{4.18}$$

Let us consider that during the interaction, the electron loses all of its momentum, hence $\Delta(m_\mathrm{e} v) = m_\mathrm{e} v$. This is the case when the electron is scattered to the direction perpendicular to the direction of the original motion.

$$\Delta(m_\mathrm{e} v) = m_\mathrm{e} v = \frac{e^2}{4\pi\varepsilon_0 r_0 v}, \tag{4.19}$$

which may be used to express the corresponding value of the impact parameter r_0. For this special case of the perpendicular scattering, we will denote this special value of the impact parameter as r_{90}:

$$r_{90} = \frac{e^2}{4\pi\varepsilon_0 m_\mathrm{e} v^2}. \tag{4.20}$$

Let's define the *effective cross-section* σ of the interaction as

$$\sigma = \pi r_{90}^2 = \frac{e^4}{16\pi\varepsilon_0^2 m_\mathrm{e}^2 v^4}. \tag{4.21}$$

Then the collisional frequency (how many collisions in a unit of time) may be expressed as

$$\nu_{\rm ei} = \sigma n v = \frac{ne^4}{16\pi\varepsilon_0^2 m_{\rm e}^2 v^3}. \tag{4.22}$$

If we further relate the typical speed of the electrons and their temperature by $m_{\rm e}v_{\rm e}^2 \sim K_{\rm B}T_{\rm e}$, we obtain for the specific resistivity

$$\eta = \frac{\nu_{\rm ei} m_{\rm e}}{ne^2} = \frac{e^2}{16\pi\varepsilon_0^2 m_{\rm e} v^3} = \frac{e^2 m_{\rm e}^{1/2}}{16\pi\varepsilon_0^2 (K_{\rm B}T_{\rm e})^{3/2}}. \tag{4.23}$$

So far we have considered only the perpendicular scattering. To account for all impact parameters, we must integrate over all of them, $\int_0^\infty 1/r\,{\rm d}r$. Unfortunately, such function diverges, and the workaround is to limit the integration by the lower and upper bounds from the physical view. The lower bound is obviously the perpendicular scattering. The upper bound is the Debye length, as on larger scales, the plasma is neutral. Then we have

$$\int\limits_{r_{90}}^{\lambda_{\rm D}} \frac{1}{r}{\rm d}r = [\ln r]_{r_{90}}^{\lambda_{\rm D}} = \ln\frac{\lambda_{\rm D}}{r_{90}} = \ln\Lambda, \tag{4.24}$$

where Λ is the coulombic logarithm. This brings a correction to what we have previously derived, hence the full expression of the specific resistivity reads

$$\eta = \frac{e^2 m_{\rm e}^{1/2}}{16\pi\varepsilon_0^2 (K_{\rm B}T_{\rm e})^{3/2}} \ln\Lambda. \tag{4.25}$$

Note that

- η does not depend on density. The larger density of the charge carriers is balanced by the higher rate of collisions.
- $\eta \propto T^{-3/2}$, hence for large temperatures the specific resistivity is very small. As a consequence, if we want to heat the plasma via the Joule heating from the electric current flowing through the plasma, the processes get very inefficient when large temperatures are reached, hence one cannot obtain the temperatures necessary for thermonuclear fusion using such energy pumping. The different mechanism must be used.
- For $K_{\rm B}T_{\rm e} \sim 100$ eV the plasma specific resistivity is 5×10^{-7} Ωm, which is of the same order as the resistivity of metals such as copper.

Also, note that

$$\boldsymbol{P}_{\rm ei} = \eta e^2 n^2 (\boldsymbol{u}_{\rm i} - \boldsymbol{u}_{\rm e}) = en_{\rm e}\boldsymbol{E}. \tag{4.26}$$

Then

$$\boldsymbol{E} = \eta e n (\boldsymbol{u}_{\rm i} - \boldsymbol{u}_{\rm e}) = \eta \boldsymbol{j}, \tag{4.27}$$

which is the Ohm's law.

4.3 Single-fluid model

Even a rougher approximation exists when we stop considering two charged species in the plasma – a *single-fluid approximation.* The two species are electrons (having the particle mass m) and ions (having mass M). The governing equations of the single-fluid approximation are obtained by linearly combining the equations for double-fluid approximation. These equations are replenished with the collision terms $P_{\rm ei} = \eta e^2 n^2 (\boldsymbol{u}_{\rm i} - \boldsymbol{u}_{\rm e})$, where η is a specific resistivity, and an ad-hoc term of the gravity force. We further neglect the convective derivative $(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}$ by assuming the speeds of fluids are generally small and hence this is the quadratic term. This term is also negligible when velocity does not change much along the streamlines, which is a reasonable approximation. We further define the bulk density

$$\rho \equiv n_{\rm i} M + n_{\rm e} m \sim n(M+m) \sim nM, \tag{4.28}$$

the bulk velocity

$$\boldsymbol{u} \equiv \frac{1}{\rho}(n_{\rm i} M \boldsymbol{u}_{\rm i} + n_{\rm e} m \boldsymbol{u}_{\rm e}) \sim \frac{n(M\boldsymbol{u}_{\rm i} + m\boldsymbol{u}_{\rm e})}{n(M+m)} = \frac{M\boldsymbol{u}_{\rm i} + m\boldsymbol{u}_{\rm e}}{M+m}, \tag{4.29}$$

the bulk current density

$$\boldsymbol{j} \equiv e(n_{\rm i}\boldsymbol{u}_{\rm i} - n_{\rm e}\boldsymbol{u}_{\rm e}) \sim ne(\boldsymbol{u}_{\rm i} - \boldsymbol{u}_{\rm e}), \tag{4.30}$$

and the bulk pressure

$$p = p_{\rm i} + p_{\rm e}. \tag{4.31}$$

Let's consider the sum of Euler equations for both fluids, where we *ad-hoc* added the term of the acceleration by gravity:

$$Mn\frac{\partial \boldsymbol{u}_{\rm i}}{\partial t} = en(\boldsymbol{E} + \boldsymbol{u}_{\rm i} \times \boldsymbol{B}) - \nabla p_{\rm i} + Mn\boldsymbol{g} + \boldsymbol{P}_{\rm ie}, \tag{4.32}$$

$$mn\frac{\partial \boldsymbol{u}_{\rm e}}{\partial t} = -en(\boldsymbol{E} + \boldsymbol{u}_{\rm e} \times \boldsymbol{B}) - \nabla p_{\rm e} + mn\boldsymbol{g} + \boldsymbol{P}_{\rm ei} \tag{4.33}$$

by using the symmetry $P_{\rm ie} = -P_{\rm ei}$ and definitions above we have

$$n(M+m)\frac{\partial}{\partial t}\left(\frac{M\boldsymbol{u}_{\rm i} + m\boldsymbol{u}_{\rm e}}{M+m}\right) = n\frac{\partial}{\partial t}(M\boldsymbol{u}_{\rm i} + m\boldsymbol{u}_{\rm e}) = en(\boldsymbol{u}_{\rm i} - \boldsymbol{u}_{\rm e}) \times \boldsymbol{B} - \nabla p + n(M+m)\boldsymbol{g}. \tag{4.34}$$

By using definitions of the bulk density and the bulk current density we have

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p + \rho\boldsymbol{g}, \tag{4.35}$$

which is the equation of motion for single-fluid plasma. Note that in general $\boldsymbol{j} \nparallel \boldsymbol{u}$ because the bulk velocity $\boldsymbol{u}$ is weighted by masses, whereas the current density $\boldsymbol{j}$ is weighted by charges. In principle, discrepancies may appear.

Let's consider a different combination of the same equations. Let's multiply (4.32) by m and (4.33) by M

$$mMn\frac{\partial \boldsymbol{u}_\mathrm{i}}{\partial t} = emn(\boldsymbol{E} + \boldsymbol{u}_\mathrm{i} \times \boldsymbol{B}) - m\nabla p_\mathrm{i} + mMn\boldsymbol{g} + m\boldsymbol{P}_\mathrm{ie}, \tag{4.36}$$

$$mMn\frac{\partial \boldsymbol{u}_\mathrm{e}}{\partial t} = -eMn(\boldsymbol{E} + \boldsymbol{u}_\mathrm{e} \times \boldsymbol{B}) - M\nabla p_\mathrm{e} + mMn\boldsymbol{g} + M\boldsymbol{P}_\mathrm{ei}. \tag{4.37}$$

Now subtract the two. We have

$$\begin{aligned} nmM\frac{\partial}{\partial t}(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}) = en(M+m)\boldsymbol{E} + en(m\boldsymbol{u}_\mathrm{i} + M\boldsymbol{u}_\mathrm{e}) \times \boldsymbol{B} - \\ - m\nabla p_\mathrm{i} + M\nabla p_\mathrm{e} - (M+m)\boldsymbol{P}_\mathrm{ei}. \end{aligned} \tag{4.38}$$

Now $\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e} = \boldsymbol{j}/(ne)$ may be expressed using the bulk electric current and

$$m\boldsymbol{u}_\mathrm{i} + M\boldsymbol{u}_\mathrm{e} = M\boldsymbol{u}_\mathrm{i} + m\boldsymbol{u}_\mathrm{e} + M(\boldsymbol{u}_\mathrm{e} - \boldsymbol{u}_\mathrm{i}) + m(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}) = \frac{\rho}{n}\boldsymbol{u} - (M-m)\frac{\boldsymbol{j}}{ne}, \tag{4.39}$$

and also

$$P_\mathrm{ei} = \eta e^2 n^2 (\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}) = \eta en\boldsymbol{j}. \tag{4.40}$$

By further dividing (4.38) by $en(M+m) = e\rho$ we have

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} - \eta\boldsymbol{j} = \frac{1}{e\rho}\left[\frac{Mmn}{e}\frac{\partial}{\partial t}\frac{\boldsymbol{j}}{n} + (M-m)\boldsymbol{j} \times \boldsymbol{B} + m\nabla p_\mathrm{i} - M\nabla p_\mathrm{e}\right]. \tag{4.41}$$

For $m/M \to 0$ and when

$$\frac{Mmn}{e}\frac{\partial}{\partial t}\frac{\boldsymbol{j}}{n} = MnB\frac{m}{eB}\frac{\partial}{\partial t}\frac{\boldsymbol{j}}{n} = MnB\omega_\mathrm{c}^{-1}\frac{\partial}{\partial t}\frac{\boldsymbol{j}}{n} \ll (M-m)\boldsymbol{j} \times \boldsymbol{B}, \tag{4.42}$$

which is fullfilled when the time variations of the current density are much slower than the cyclotron frequency. In that case

$$\frac{1}{\omega_\mathrm{c}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{j}}{n}\right) \ll \frac{\boldsymbol{j}}{n} \tag{4.43}$$

and we obtain

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} - \eta\boldsymbol{j} = \frac{1}{en}\left(\boldsymbol{j} \times \boldsymbol{B} - \nabla p_\mathrm{e}\right), \tag{4.44}$$

which is the *generalised Ohm's law.* The first term on the right-hand side is the expression for Hall's current. In various applications, the right-hand side is often completely neglected.

Similarly (and we leave the derivation to the reader) one may obtain the continuity equation for density and charge density $\rho_\mathrm{e} = e(n_\mathrm{i} - n_\mathrm{e})$ for single-fluid plasma by summing and subtracting the continuity equations for both fluids:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho\boldsymbol{u} = 0 \tag{4.45}$$

and

$$\frac{\partial \rho_{\mathrm{e}}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0. \tag{4.46}$$

Hence the full set of equations for a sigle-fluid plasma reads

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p + \rho \boldsymbol{g}, \tag{4.47}$$

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j}, \tag{4.48}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \boldsymbol{u} = 0, \tag{4.49}$$

$$\frac{\partial \rho_{\mathrm{e}}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0. \tag{4.50}$$

This system must be accompanied by Maxwell equations to fully describe the plasma in equilibrium, where the definitions for ρ_{e} and $\boldsymbol{j}$ are simply used in Equations (4.4)-(4.7). The single-fluid approximation is used usually when resistivity play a non-negligible role in the plasma and is often used also in astrophysics. The system of single-fluid equations is most often understood under the term *magnetohydrodynamic equations* or *MHD* equations.

4.4 Approximations to plasma fluid

Additional approximations to plasma fluid approximation are used in some cases.

4.4.1 Force-free and ideal MHD

The *force-free approximation* is valid for systems, where locally the plasma does not "feel" the Lorentz force, i.e.

$$f_{\mathrm{Lorentz}} = \rho_e \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} = 0. \tag{4.51}$$

Note that the above given equation represents an "effective" of "bulk" Lorentz force acting on both species in the mixture.

This approximation is also used in the case when the Lorentz force is negligible compared to other considered forces. Another view to this approximation may also be formulated in such a way that the electromagnetic field induced by the motion of plasma fluid exactly and immediately cancels out the external forces.

For E small, the force-free approximation turns to

$$\boldsymbol{j} \times \boldsymbol{B} = 0. \tag{4.52}$$

The current density is related to the curl of the magnetic induction via Maxwell equations by

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu \boldsymbol{j}. \tag{4.53}$$

Combining the two we have

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = 0, \tag{4.54}$$

which indicates that the curl of the magnetic induction is parallel to the magnetic induction, i.e.,

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha \boldsymbol{B}. \tag{4.55}$$

This approximation is usually referred to as *linear force free*. Note that the constant of proportionality α may change with position in space.

The *ideal MHD approximation* is somewhat different. It uses the fact that in the comoving frame the electric field disappears. I.e.

$$\boldsymbol{E'} = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{0}. \tag{4.56}$$

By using the generalised Ohm's law it turns out that the ideal MHD approximation equals to the case when the plasma is ideally conductive, therefore when the resistivity is zero. This may be immediately seen from

$$\boldsymbol{E'} = \eta \boldsymbol{j}, \tag{4.57}$$

where the left-hand side is zero because of (4.56) and this relation may be valid in a trivial case when $\boldsymbol{j} = 0$ or when $\eta = 0$ (which represents an infinite conductivity).

The two approximations mentioned in this subsection are usable in the case when the electromagnetic field is degenerated, i.e. when $\boldsymbol{E} \cdot \boldsymbol{B} = 0$. The two coincide when $\boldsymbol{j} = \rho_e \boldsymbol{u}$.

Frozen field

The consequence of the ideal MHD approximation is something called *frozen field*. It is a description of observed behaviour of highly-conductive plasmas, where a strong coupling between the plasma and the field exists. It was found responsible for strengthening the magnetic field in the stars or temporal changes of the Earth's magnetosphere under the influence of the varying solar wind.

Let us start from taking the curl of the Ampère's law:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \nabla \boldsymbol{\nabla} \cdot \boldsymbol{B} - \triangle \boldsymbol{B} = \boldsymbol{\nabla} \times \left(\mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right). \tag{4.58}$$

The term $\nabla \boldsymbol{\nabla} \cdot \boldsymbol{B}$ vanishes due to the Gauss's law for magnetism. By using the ideal MHD approximation (4.56), Eq. (4.57), and the definition of $\sigma \equiv 1/\eta$ we have

$$\sigma \mu_0 \boldsymbol{\nabla} \times (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) + \mu_0 \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{E}}{\partial t} = - \triangle \boldsymbol{B}. \tag{4.59}$$

For $\sigma \to \infty$ must $\boldsymbol{\nabla} \times (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) = 0$ in order to $\triangle \boldsymbol{B}$ be finite. By additionally using the Farraday's law we then have

$$-\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = 0 \quad \to \quad \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = \frac{\partial \boldsymbol{B}}{\partial t}. \tag{4.60}$$

Let us note at this point that $\boldsymbol{\nabla} \times (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) = 0$ does not imply $\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = 0$. A mathematical object having zero value does not necessarily have to have vanishing derivatives!

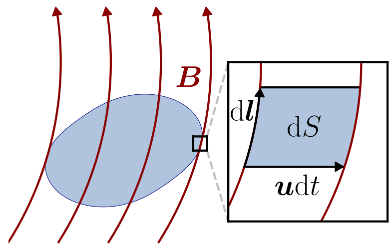

Figure 4.2: Magnetic flux through a closed curve in the frozen-field approximation.

The frozen-field approximation may also be understood as a condition when the magnetic flux through a closed curve bound to the plasma does not change with time. Let us prove this statement. Let's start by defining the system as in Fig. 4.2. By zooming in it is obvious that the change of the surface vector element is given by a cross-product of the plasma velocity and the elementary length along the closed curve, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{S} = \boldsymbol{u} \times \mathrm{d}\boldsymbol{l}$.

We will now derive the change of the magnetic flux Ψ with time:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = \int \boldsymbol{B}\cdot\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} + \int \frac{\partial \boldsymbol{B}}{\partial t}\cdot \mathrm{d}\boldsymbol{S}. \tag{4.61}$$

We apply the relation between the surface and length elements from above and after using the cyclic exchange in a mixed product we have

$$\oint_l \boldsymbol{B}\cdot(\boldsymbol{u}\times \mathrm{d}\boldsymbol{l}) + \int \frac{\partial \boldsymbol{B}}{\partial t}\cdot \mathrm{d}\boldsymbol{S} = -\oint_l (\boldsymbol{u}\times\boldsymbol{B})\cdot \mathrm{d}\boldsymbol{l} + \int \frac{\partial \boldsymbol{B}}{\partial t}\cdot \mathrm{d}\boldsymbol{S}. \tag{4.62}$$

We further use the Stokes mathematical theorem and obtain

$$-\int \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})\cdot \mathrm{d}\boldsymbol{S} + \int \frac{\partial \boldsymbol{B}}{\partial t}\cdot \mathrm{d}\boldsymbol{S} = -\int \left[\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) - \frac{\partial \boldsymbol{B}}{\partial t}\right]\cdot \mathrm{d}\boldsymbol{S} = \boldsymbol{0}, \tag{4.63}$$

where we used (4.60). Q.E.D.

4.4.2 Axisymmetrical systems, stream functions

Another class of approximations comes with the symmetries and/or degenerations of the system. Let us consider the axially symmetric stationary system represented in the cylindrical coordinates (R, φ, z). I.e. $\frac{\partial}{\partial \varphi} = 0$ and $\frac{\partial}{\partial t} = 0$. The curl of $\boldsymbol{E}$ that has components

$$\boldsymbol{\nabla}\times\boldsymbol{E} = \left(\frac{1}{R}\frac{\partial E_z}{\partial\varphi} - \frac{\partial E_R}{\partial z}, \frac{\partial E_R}{\partial z} - \frac{\partial E_z}{\partial R}, \frac{1}{R}\frac{\partial}{\partial R}RE_\varphi - \frac{1}{R}\frac{\partial E_R}{\partial \varphi}\right) = \boldsymbol{0}. \tag{4.64}$$

From the third component it turns out that $E_\varphi = 0$.

By using the ideal MHD approximation, we further have

$$E_\varphi = -(\boldsymbol{u} \times \boldsymbol{B})_\varphi = -u_R B_z + u_z B_R = 0, \tag{4.65}$$

and therefore

$$\frac{u_R}{u_z} = \frac{B_R}{B_z}. \tag{4.66}$$

Thus the poloidal components are bound with an unknown scalar function ξ

$$\boldsymbol{u}_\mathrm{p} = \xi(R, z)\boldsymbol{B}_\mathrm{p}, \tag{4.67}$$

which is a consequence of the axial symmetry, stationarity, and the used ideal MHD approximation. Such a result is therefore not valid generally!

Let us only remind that the poloidal component of a vector $\vec{a}$ is simply a combination of the radial and height components, i.e., $\boldsymbol{a}_\mathrm{p} = a_R \boldsymbol{e}_R + a_z \boldsymbol{e}_z$.

Because $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, where $\boldsymbol{A}$ is a magnetic vector-potential, we may write

$$\Psi(R, z) = R A_\varphi, \tag{4.68}$$

where Ψ is a scalar magnetic flux function. Then simply

$$B_R = -\frac{1}{R}\frac{\partial \Psi}{\partial z} \quad \text{and} \quad B_z = \frac{1}{R}\frac{\partial \Psi}{\partial R}. \tag{4.69}$$

By using the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}_\mathrm{p}) + \frac{\partial \rho v_\varphi}{\partial \varphi} = \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}_\mathrm{p}) = 0, \tag{4.70}$$

we have

$$\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}_\mathrm{p}) = \boldsymbol{\nabla} \cdot (\rho \xi \boldsymbol{B}_\mathrm{p}) = \nabla \rho \xi \cdot \boldsymbol{B}_\mathrm{p} + \rho \xi \boldsymbol{\nabla} \cdot \boldsymbol{B}_\mathrm{p} = 0. \tag{4.71}$$

The last term vanishes due to the Gauss's law for magnetism. By decomposing the divergence operator into the non-vanishing components and using (4.69)

$$\boldsymbol{\nabla} \cdot (\rho \xi \boldsymbol{B}_\mathrm{p}) = \nabla_R(\rho\xi) B_R + \nabla_z(\rho\xi) B_z = -\frac{1}{R}\nabla_R(\rho\xi)\nabla_z \Psi + \frac{1}{R}\nabla_z(\rho\xi)\nabla_R \Psi = 0, \tag{4.72}$$

we see that this is the φ component of an expression $\frac{1}{R}\nabla(\rho\xi) \times \nabla\Psi$. Due to the assumptions the arguments of the gradient operators do not depend on φ. The gradient therefore has only the component perpendicular to both $\boldsymbol{e}_R$ and $\boldsymbol{e}_z$ non-trivial. That is the φ-component, which also vanished as we proved in (4.72). Thus

$$\nabla(\rho\xi) \times \nabla\Psi = \boldsymbol{0}. \tag{4.73}$$

It may be interpretted as $\nabla(\rho\xi) \parallel \nabla\Psi$ and therefore the scalar function ξ may be related to Ψ as

$$4\pi\rho\xi = F_1(\Psi). \tag{4.74}$$

Function F_1 is termed the *stream function*, which is obviously constant on a constant surface of Ψ. Its particular mathematical expression may be determined from the boundary conditions. The use of the stream function is the reduction of variables to be solved for the given system as it introduces a bound between $\boldsymbol{u}$ and $\boldsymbol{B}$.

Similarly, using the poloidal and azimuthal (toroidal) decomposition and equations (4.67) and (4.69) we have

$$\begin{aligned}\boldsymbol{u}\times\boldsymbol{B} &= u_\varphi \boldsymbol{e}_\varphi \times \boldsymbol{B}_\mathrm{p} + \boldsymbol{u}_\mathrm{p}\times B_\varphi \boldsymbol{e}_\varphi = u_\varphi \boldsymbol{e}_\varphi \times \boldsymbol{B}_\mathrm{p} + \xi \boldsymbol{B}_\mathrm{p}\times B_\varphi \boldsymbol{e}_\varphi \\ &= (u_\varphi - \xi B_\varphi)(B_z - B_R)\boldsymbol{e}_z = (u_\varphi - \xi B_\varphi)\frac{1}{R}\left(\frac{\partial\Psi}{\partial R} - \frac{\partial \Psi}{\partial z}\right)\boldsymbol{e}_z \\ &= \frac{u_\varphi - \xi B_\varphi}{R}\nabla\Psi. \end{aligned} \tag{4.75}$$

In a stationary frozen-field approximation we have

$$\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) = \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}, \tag{4.76}$$

and therefore

$$\nabla\times\left(\frac{u_\varphi - \xi B_\varphi}{R}\nabla\Psi\right) = \nabla\left(\frac{u_\varphi - \xi B_\varphi}{R}\right)\times\nabla\Psi = \boldsymbol{0}, \tag{4.77}$$

where we proceeded similarly to the previous case. Again, the surfaces of the arguments of the gradient operators must coalign, and hence we may define a second stream function as

$$\frac{u_\varphi - \xi B_\varphi}{R} = F_2(\Psi). \tag{4.78}$$

By taking the projection of the Euler equation to $\boldsymbol{e}_\varphi$, we can show that

$$B_\varphi \boldsymbol{e}_\varphi \cdot \nabla(RB_\varphi - F_1 R v_\varphi) = 0, \tag{4.79}$$

and hence define the third stream function as

$$RB_\varphi - F_1(\Psi)Rv_\varphi = F_3(\Psi). \tag{4.80}$$

And finally, by taking the projection of the Euler equation onto $\boldsymbol{B}_\mathrm{p}$, we obtain a definition for a fourth stream function as

$$\frac{1}{2}u^2 + \int_{\Psi=\mathrm{const}} \frac{\mathrm{d}P}{\rho} + \Phi - Ru_\varphi F_3(\Psi) = F_4(\Psi). \tag{4.81}$$

Altogether the stream function provide a “technology” for seaking the solution of the system in a given geometry.

4.5 Drifts in the plasma fluid

Let’s discuss the drifts in the plasma fluid.

4.5.1 Drifts perpendicular to $\boldsymbol{B}$

When investigating the drifts in the direction perpendicular to $\boldsymbol{B}$, we solve the perpendicular component of the Euler equation, which for collision-less plasma reads:

$$mn\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = qn(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - \nabla p \tag{4.82}$$

Let us consider systems, where the left-hand side is negligible to the right-hand side.[1] Hence we solve a much simpler equation

$$0 = qn(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - \nabla p \tag{4.83}$$

by applying the vector multiplication by $\boldsymbol{B}$:

$$\begin{aligned} 0 &= qn[\boldsymbol{E}\times\boldsymbol{B} + (\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B}] - \nabla p\times\boldsymbol{B} = \\ &= qn(\boldsymbol{E}\times\boldsymbol{B} - \boldsymbol{u}B^2 + \boldsymbol{B}\boldsymbol{u}\cdot\boldsymbol{B}) - \nabla p\times\boldsymbol{B}. \end{aligned} \tag{4.84}$$

The perpendicular component of this equation is

$$qn(\boldsymbol{E}\times\boldsymbol{B} - \boldsymbol{u}_\perp B^2) - \nabla p\times\boldsymbol{B} = 0 \tag{4.85}$$

and thus

$$\boldsymbol{u}_\perp = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} - \frac{\nabla p\times\boldsymbol{B}}{qnB^2} = \boldsymbol{v}_\mathrm{E} + \boldsymbol{v}_\mathrm{D}. \tag{4.86}$$

We found the expression for the perpendicular drift in the plasma fluid. The first term resembles the already known E-B drift, the second term is new and represents the *diamagnetic drift* $\boldsymbol{v}_\mathrm{D}$. The direction of $\boldsymbol{v}_\mathrm{D}$ is perpendicular to ∇p, hence the assumption of $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ being negligible holds only if $\boldsymbol{E} = 0$. The assumption further holds if $\nabla\phi$ and ∇p are parallel, then $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ is still zero for $\boldsymbol{E} = -\nabla\phi$. In the other cases the situation is more complicated and the solution may involve the necessity to deal with the $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ term.

4.5.2 Drifts parallel to $\boldsymbol{B}$

Now let's look at the remaining component of the drifts in a plasma fluid. Let's consider the coordinate system so that the axis $\boldsymbol{e}_z$ is parallel to the magnetic field. Then we study the z-component of the equation of motion

$$mn\left(\frac{\partial u_z}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_z\right) = qnE_z - \frac{\partial p}{\partial z} \tag{4.87}$$

with equation of state in a form

$$\frac{\nabla p}{p} = \gamma\frac{\nabla n}{n}, \tag{4.88}$$

[1] To estimate this, let's use the Fourier transform of Euler equation, for now blindly applying the rules (5.11) to (5.13). By simply waving the hands we see that the left-hand side is proportional to $\mathrm{i}\omega + \mathrm{i}\frac{1}{L}$, where ω is a mode frequency and L is a characteristic scale of spatial change, while the right-hand side is proportional to $\mathrm{i}\omega_\mathrm{c} \propto \frac{1}{r_\mathrm{L}}$. Usually, $\omega \ll \omega_\mathrm{c}$ and $r_\mathrm{L} \ll L$, hence the left-hand side is negligible.

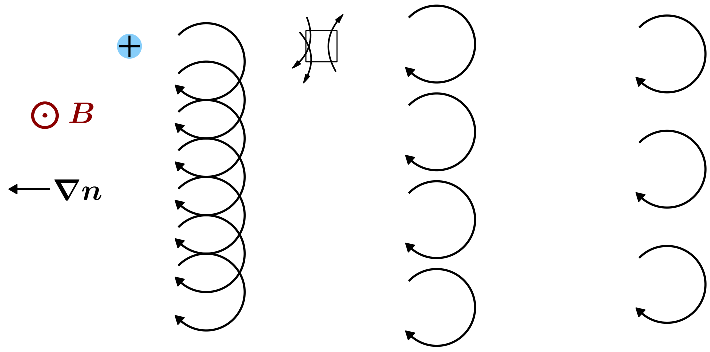

Figure 4.3: A scheme of the origin of the diamagnetic drift. In the "zooming" square one sees that on the side with a larger density, the arrows pointing down are more abundant than on the opposite side. Thus, on average, the motion downwards prevails which is represented by the drift.

where we further use $\gamma = 1$, because we will solve the equation only for electrons that are fast and may be considered isothermic. Additional assumptions apply: $(\boldsymbol{u} \cdot \nabla) u_z \to 0$, u_z homogeneous, and $\nabla p = K_\mathrm{B} T \nabla n$. Then

$$mn \frac{\partial u_z}{\partial t} = qnE_z - K_\mathrm{B} T \frac{\partial n}{\partial z}. \tag{4.89}$$

For electrons we have $q = -e$, $m \to 0$, and $T \to T_\mathrm{e}$. By using the Poisson equation $E_z = -\partial \phi / \partial z$ we have

$$e \frac{\partial \phi}{\partial z} = \frac{K_\mathrm{B} T_\mathrm{e}}{n} \frac{\partial n}{\partial z} \tag{4.90}$$

and hence

$$e\phi = K_\mathrm{B} T_\mathrm{e} \ln n + \mathcal{C} \tag{4.91}$$

and finally

$$n = n_0 \exp \left[\frac{e\phi}{K_\mathrm{B} T_\mathrm{e}} \right], \tag{4.92}$$

which is the *Boltzmann relation*. The Boltzmann relation describes the spatial distribution of electrons around the random concentration of ions. In the local concentration of ions, the pressure gradient pushes electrons away from this region. The consecutive charge separation gives rise to the local electric field, which acts against the pressure gradient. In the equilibrium, both the pressure and electrostatic forces are balanced. The Boltzmann relation is a very useful relation when we further solve the system of equations for ions. Due to the mass difference between the two species the spatial distribution of electrons may be approximated by the Boltzmann relation.

Chapter 5

Waves in plasmas

5.1 Linear waves

It is not trivial to comprehensively describe solutions supporting waves in plasmas. We will investigate a few examples only of *linear waves*. In practice, we will decompose the variables describing both plasma and magnetic and electric fields into two components: a *background component* (having index 0) and a *fluctuating component* (having index 1), e.g., $\rho = \rho_0 + \rho_1$. In (silently) applying this decomposition, we will assume that:

1. The background components automatically fulfill the full set of magnetohydrodynamic equations. The background is the zeroth order of the smallness variable.

2. The fluctuating components are small compared to the background (e.g., $\rho_1/\rho_0 \ll 1$). When averaging the fluctuating component (both in space and/or time), the averaged value is zero (e.g., $\langle\rho_1\rangle = 0$). Implicitly, we assume the same about both temporal and spatial derivatives of the fluctuating component. The fluctuating part is the first order of the smallness variable.

We will solve the set (or subset) of magnetohydrodynamic equations to the first order.

Let's stop for a while and make this decomposition crystal clear forever. It might get confusing especially in the case of velocity. When we computed moments of the Boltzmann equation, we performed also a decomposition having a fluctuating part. Let's compare the two:

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}, \tag{5.1}$$

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1. \tag{5.2}$$

These two are crucially different. While in Eq. (5.1) a *particle* velocity $\boldsymbol{v}$ is decomposed into a *mean* (or bulk) velocity $\boldsymbol{u}$ and the component $\boldsymbol{w}$ fluctuating about this mean value, where the amplitude of the fluctuations may be (and typically is) much larger than the amplitude of the bulk motion, in Eq. (5.2) the *bulk* velocity $\boldsymbol{u}$ is decomposed into the background fulfilling the MHD equations $\boldsymbol{u}_0$ and a small correction to this background value. Hence $\boldsymbol{u}$ from Eq. (5.2) indeed equals to $\boldsymbol{u}$ from Eq. (5.1).

5.2 Example: Acoustic waves in fluids

To illustrate the linearisation approach, let us derive the dispersion relation for waves in a fluid, fully described by the Euler equation in the form

$$\rho\left[\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right]=-\nabla p=-\frac{\gamma p}{\rho}\nabla\rho, \tag{5.3}$$

where on the right-hand side we assumed the adiabatic equation of state $p\rho^{-\gamma}=\text{const}$, and a continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})=0. \tag{5.4}$$

We will assume the following:

Pressure waves in fluids

- Strictly hydrodynamical case (no fields)
- No viscosity
- No bulk motion of the background, $\boldsymbol{u}_0=0$
- The background is homogeneous

By introducing the fluctuating part to the velocity and density, i.e., $\boldsymbol{u}=\boldsymbol{u}_0+\boldsymbol{u}_1$, $p=p_0+p_1$, and $\rho=\rho_0+\rho_1$, equations (5.3) and (5.4) take the form of

$$(\rho_0+\rho_1)\left\{\frac{\partial \boldsymbol{u}_0}{\partial t}+\frac{\partial \boldsymbol{u}_1}{\partial t}+\left[(\boldsymbol{u}_0+\boldsymbol{u}_1)\cdot\boldsymbol{\nabla}\right](\boldsymbol{u}_0+\boldsymbol{u}_1)\right\}=-\frac{\gamma(p_0+p_1)}{(\rho_0+\rho_1)}\nabla(\rho_0+\rho_1) \tag{5.5}$$

$$\frac{\partial(\rho_0+\rho_1)}{\partial t}+\nabla\cdot\left[(\rho_0+\rho_1)(\boldsymbol{u}_0+\boldsymbol{u}_1)\right]=0. \tag{5.6}$$

Now, let's take the continuity equation (5.6) as an example. It can be explicitly written as

$$\overset{①}{\frac{\partial\rho_0}{\partial t}}+\overset{②}{\frac{\partial\rho_1}{\partial t}}+\overset{③}{\nabla\cdot(\rho_0\boldsymbol{u}_0)}+\overset{④}{\rho_0\nabla\cdot\boldsymbol{u}_1}+\overset{⑤}{\boldsymbol{u}_1\cdot\nabla\rho_0}+\overset{⑥}{\nabla\cdot(\rho_1\boldsymbol{u}_0)}+\overset{⑦}{\nabla\cdot(\rho_1\boldsymbol{u}_1)}=0. \tag{5.7}$$

Let's discuss the terms in the above equation:

- Terms ① and ③ automatically fulfill the continuity equation for the background variables: $\frac{\partial\rho_0}{\partial t}+\nabla\cdot(\rho_0\boldsymbol{u}_0)=0$.
- Term ⑤ is zero because we assumed homogeneous background, i.e. $\nabla\rho_0=0$.
- Term ⑥ is zero because we assumed $\boldsymbol{u}_0=0$.

- Term ⑦ is a second order term, thus it can be neglected.

Thus only relation

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_1 = 0 \tag{5.8}$$

remains. This is the *linearised continuity equation.*

Similarly[1], we obtain the *linearised Euler equation*:

$$\rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} = -\gamma \frac{p_0}{\rho_0} \nabla \rho_1. \tag{5.9}$$

Equations (5.8) and (5.9) need to be solved. We are looking for a solution supporting waves, thus, we may search for a solution in a form

$$A_1 = \tilde{A}_1 \exp\left[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)\right], \tag{5.10}$$

where $\tilde{A}_1$ is the Fourier image of A_1 or the amplitude, $\boldsymbol{k}$ is a wave vector in the Fourier space and ω is a frequency. There are two possibilities for how to see this approach:

1. We solve these equations in the Fourier space. At this step, a careful reader should open the mathematical textbook and repeat all the assumptions of the Fourier transform.
2. We view the solution as a superposition of modes, each mode[2] is described by a unique combination of $\boldsymbol{k}$ and ω. Thus we solve the equations for one general mode and a complete solution is a superposition of all modes with various amplitudes.

Either way, the beauty of the formulation of the equations in the Fourier space is that the differential operators turn into the multiplication. One can easily prove from Eq. (5.10) that

$$\frac{\partial \bullet}{\partial t} \to -\mathrm{i}\omega \bullet \tag{5.11}$$

$$\nabla \bullet \to \mathrm{i}\boldsymbol{k} \bullet \tag{5.12}$$

$$\triangle \bullet \to -k^2 \bullet \tag{5.13}$$

Our goal is to find a *dispersion relation*, thus, a relation which couples $\boldsymbol{k}$ and ω. Dispersion relation $D(\boldsymbol{k}, \omega) = 0$ fully describes the given mode of the wave. Equations (5.8) and (5.9) written in the Fourier space take the following form:

$$-\mathrm{i}\omega \rho_1 + \mathrm{i}\rho_0 \boldsymbol{k} \cdot \boldsymbol{u}_1 = 0, \tag{5.14}$$

$$-\mathrm{i}\omega \rho_0 \boldsymbol{u}_1 + \mathrm{i}\gamma \frac{p_0}{\rho_0} \boldsymbol{k} \rho_1 = 0. \tag{5.15}$$

[1] For clarity, let's handle the fraction on the right-hand side explicitly:

$$\frac{p_0 + p_1}{\rho_0 + \rho_1} = \frac{\frac{p_0 + p_1}{\rho_0}}{\frac{\rho_0 + \rho_1}{\rho_0}} = \frac{\frac{p_0}{\rho_0} + \frac{p_1}{\rho_0}}{1 + \frac{\rho_1}{\rho_0}} = \frac{p_0}{\rho_0},$$

because $\frac{p_1}{\rho_0}$ and $\frac{\rho_1}{\rho_0}$ are negligible terms.

[2] A *mode* is defined either as any of the wave-like and/or oscillation-like plasma motions or as one possible configuration of the travelling or steady wave. Both definitions are equivalent.

For simplicity let's define the coordinate system so that the wave propagates in the main direction, thus, wave vector $\boldsymbol{k}$ may be replaced by its scalar equivalent wave number k. Then by solving the equations above one obtains a *dispersion relation*

$$\frac{\omega}{k} = \sqrt{\frac{\gamma p_0}{\rho_0}} \equiv c_\mathrm{s}, \tag{5.16}$$

where c_s is a speed of sound.
A few remarks to what we just obtained:

- The dispersion relation exists, thus, the waves propagate. The propagation is fully described by the wave vector $\boldsymbol{k}$. Possible combinations of the wave vector and frequency of the mode are bound by the dispersion relation. The waves propagate in a form of pressure fluctuations.
- The wavenumber of the wave is inversely proportional to its *wavelength* $\lambda = (2\pi)/k$.
- The *phase speed* v_φ is a speed with which places with the same phase propagate through the medium. The phase speed is given by $v_\varphi = \omega/k$. In our case, $v_\varphi = c_\mathrm{s}$.
- On the other hand, the *group speed* v_g is a speed with which the modulation information propagates. It also relates the wavenumber and the frequency of the mode by $v_\mathrm{g} = \mathrm{d}\omega/\mathrm{d}k$. In our case, again, $v_\mathrm{g} = c_\mathrm{s}$ and $v_\mathrm{g} = v_\varphi$. As we will show later, this does not have to always be the case. It can be illustrated in a very simple example: we have one mode only with a constant amplitude in time. A constant "beep" does not carry any information. Its dispersion relation reads: $\omega = \mathrm{const.}$ Thus its phase speed is not defined (it might be viewed as infinite as well), but the group speed is zero. We just showed an additional important property: phase speed does not have to be limited (again, it does not carry any information), while the group speed is limited (from the top) by the speed of light c.
- The term *dispersion relation* indicates that the wave *disperses*, changes the waveform during the propagation. That indeed is the case, when the wave is *dispersive*, i.e., when the propagation phase speed depends on wave frequency or wave number. Then the amplitude spectrum of the wave packet changes as the packet passes through the medium – individual frequencies exhibit various phase shifts. Then the wave-packet propagation is not a simple translation parallel with the direction of propagation but it is more complex and additionally to the movement of the wave packet the packet itself changes its shape. Dispersive waves usually have group speed different from the phase speed. Sound waves in fluids are non-dispersive. A simplistic illustration of the dispersive and non-dispersive waves is given in Fig. 5.1.

- Note that the approach we will use is equivalent to *approximation by the geometrical optics* a.k.a. *ray approximation*, which ignores the effects of the finite wavelength. Such an approach is justified when the wavelength is much shorter than the typical spatial scale of the changes in the medium. In the other case, the effects may be observed, which are not captured by the ray approximation. Finite wavelength effects are captured by the *scattering theory*, which are not easy to be solved both analytically and numerically. Approximations to the scattering theory, such as the Born approximation, capture some of the most important finite wavelength effects quite well.

Let us see what happens when we relax an assumption of $\boldsymbol{u}_0 = 0$. Then the Fourier image of the continuity equation will read:

$$-\mathrm{i}\omega\rho_1 + \mathrm{i}\rho_1\boldsymbol{k}\cdot\boldsymbol{u}_0 + \mathrm{i}\rho_0\boldsymbol{k}\cdot\boldsymbol{u}_1 = 0, \tag{5.17}$$

which may be written as

$$-\mathrm{i}\rho_1(\omega - \boldsymbol{k}\cdot\boldsymbol{u}_0) + \mathrm{i}\rho_0\boldsymbol{k}\cdot\boldsymbol{u}_1 = 0. \tag{5.18}$$

Hence by introducing $\omega' = \omega - \boldsymbol{k}\cdot\boldsymbol{u}_0$ we change the equation to the previous case. Similar operation may be done to the Euler equation[3]. Hence the final dispersion relation will be

$$\omega - \boldsymbol{k}\cdot\boldsymbol{u}_0 = kc_\mathrm{s}, \tag{5.21}$$

and the motion of the background fluid with respect to the observer's frame introduces a *Doppler shift* of the frequencies.

5.3 Types of plasma waves

To study the waves in plasma, we will use the fluid approximation. It is fine enough to capture all essential properties to investigate the possible propagation of waves but is also simple enough to be able to obtain some analytical solutions. There is the whole spectrum of waves propagating through the plasma, which may be decomposed into several modes with different dispersion relations. We will study only a handful of the possible modes, some of which may have important physical consequences.

Plasma waves are usually excited by means of instabilities (this is not surprising, e.g. sound waves propagating through a convective envelope of Sun-like stars are excited by the convective instability below the photosphere of the star) or they may be excited

[3] Due to $\boldsymbol{u}_0 \neq 0$ we must keep the "convective derivative" term in the equation, leading to the linearised form

$$\rho_0\frac{\partial\boldsymbol{u}_1}{\partial t} + \rho_0(\boldsymbol{u}_0\cdot\boldsymbol{\nabla}\boldsymbol{u}_1) = -\gamma\frac{p_0}{\rho_0}\boldsymbol{\nabla}\rho_1, \tag{5.19}$$

which has a Fourier form of

$$-\mathrm{i}\rho_0(\omega - \boldsymbol{k}\cdot\boldsymbol{u}_0)\boldsymbol{u}_1 + \mathrm{i}k\gamma\frac{p_0}{\rho_0}\rho_1 = 0, \tag{5.20}$$

where a possible substitution $\omega' = \omega - \boldsymbol{k}\cdot\boldsymbol{u}_0$ is immediately visible.

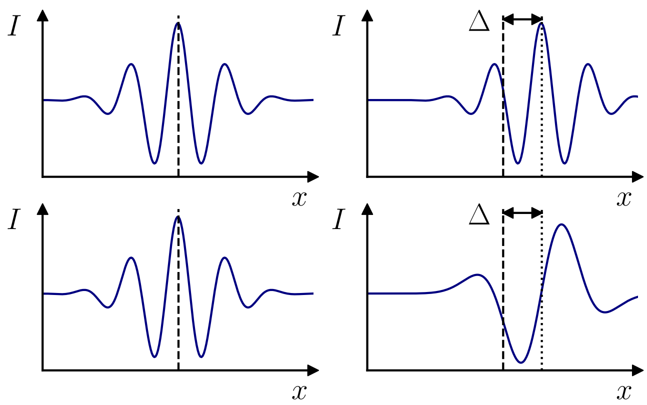

Figure 5.1: Difference between dispersive and non-dispersive waves. In the case of the non-dispersive waves, the time evolution of the waveform consists of the translation in space only. In the case of the dispersive waves, various modes propagate with different phase speeds, hence the waveform is not only translated but also corrugated.

artificially. The dispersion relation does not say anything about the excitation process. Basic types of plasma waves can be distinguished by giving one of the two values to the following three properties:

1. In the wave, either electrons or ions oscillate. Thus we have *electron* or *ion* wave. Usually, when we solve for an electron wave, the ions are approximated as a non-movable background, because ions are at least three orders of magnitude heavier than electrons.

2. The wave can be *longitudinal* or *electrostatic* when $\boldsymbol{k} \parallel \boldsymbol{E}_1$ or *tranversal* or *electromagnetic*, when $\boldsymbol{k} \perp \boldsymbol{E}_1$, where $\boldsymbol{E}_1$ is a fluctuating component of the electric field.

3. The wave is *parallel* when $\boldsymbol{k} \parallel \boldsymbol{B}_0$ or *perpendicular* when $\boldsymbol{k} \perp \boldsymbol{B}_0$, where $\boldsymbol{B}_0$ is a background magnetic field.

Plasma waves we will discuss in the following Sections will always be distinguishable by the three above described tags.

The approach we will take is that we will prescribe the expected geometry of the wave (in terms of the type, direction of propagation and which of the components it contains) and search for the solution using the linearised fluid and Maxwell equations. If the solution in the form of a dispersion relation exists, the plasma supports this kind of wave. Our solutions will not give the answer if this particular wave really gets excited under given conditions and what is its amplitude (for this one would need to state the initial or boundary conditions), not even saying anything about whether this particular mode is dominant or completely marginal at given conditions.

The linearisation approach explicitly states that we introduce some plasma background, which fulfills the fluid equations and remains unchanged by the wave propagation. It is the perturbations in density, velocity, pressure, electric or magnetic field, which oscillates and possibly propagates from place to place as a wave. We will see that the type of the propagating wave usually depends on the direction with respect to the background magnetic field.

Let's make a simplified example, which we will in detail investigate in Section 5.6.3. Let us consider a stationary and homogeneous plasma cloud in a background magnetic field. Somewhere inside this cloud we shortly blick with a light-source, a bulb if you want. In normal transparent neutral gas, such impulse would cause the spherical light-wave to propagate from the point of initiation with a speed of light in all directions. In plasma, it is more difficult, as the light waves are known to be variations of the electric and magnetic field. In plasma, these small electric and magnetic fields, oscillating with the wave, interact with the charged particles by displacing them slightly, which causes the modification of the electric and magnetic field in the wave. The motion of particles and the electric and magnetic fields are bound via the set of (linearised) fluid equations, which in general depends on the direction to the background magnetic field. Hence the waveform gets modified depending on the angle of the propagation with respect to the background magnetic field. We will see that along the magnetic field two circularly polarised modes propagating with different speeds appear, accompanied by incoherent local oscillations of plasma. In the direction perpendicular to the magnetic field two linearly polarised modes, again with different propagation speeds, appear. In a general direction of propagation, we will find a combination of all five modes. Electromagnetic waves do not cause the variations in plasma density, hence only the electric and magnetic fields are affected and inherently also the motions of individual particles. Another kind of impulse, such as a sudden local increase of plasma density, will excite another type of waves, but not the complex of electromagnetic ones, only if such impulse is not accompanied by some emission mechanism.

5.4 Electrostatic waves without background magnetic field

5.4.1 Plasma oscillations

The very basic oscillatory movement registered in plasmas takes the form of *plasma oscillations*. Let's derive the dispersion relation first. We search for a solution under the following assumptions:

Plasma oscillations

- No background fields, i.e., $\boldsymbol{E}_0 = \boldsymbol{B}_0 = 0$, we do not consider induced perturbation to the magnetic field, i.e., $\boldsymbol{B}_1 = 0$. Thus from the set of Maxwell equations, only the Poisson equation must be retained
- No thermal motion, plasma is cold

- Ions are fixed in space and have a uniform distribution
- Background density of ions and electrons is the same, i.e., $n_{\mathrm{e}0} = n_{\mathrm{i}0} = n_0$
- Background is stationary and homogeneous, $\boldsymbol{u}_{\mathrm{e}0} = 0$

Thus we solve the following set of linearised equations:

$$mn_0 \frac{\partial \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = -en_0 \boldsymbol{E}_1, \tag{5.22}$$

$$\frac{\partial n_{\mathrm{e}1}}{\partial t} + \boldsymbol{\nabla} \cdot [n_0 \boldsymbol{u}_{\mathrm{e}1}] = 0, \tag{5.23}$$

$$\boldsymbol{E}_1 = -\nabla \phi_1, \tag{5.24}$$

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}_1 = -en_{\mathrm{e}1}. \tag{5.25}$$

Now we express $\boldsymbol{E}_1$ from (5.24) and insert it into appropriate terms in (5.22) and (5.25), and apply a divergence operator to (5.22) to obtain:

$$\varepsilon_0 \triangle \phi_1 = n_{\mathrm{e}1} e, \tag{5.26}$$

$$mn_0 \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = en_0 \triangle \phi_1, \tag{5.27}$$

$$\frac{\partial n_{\mathrm{e}1}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e}1} = 0. \tag{5.28}$$

Now we express term $\triangle \phi_1$ from (5.26) and insert it into (5.27). We do a simple algebraic adjustment to (5.28) by leaving only $\boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e}1}$ on the left-hand side of equation:

$$mn_0 \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = \frac{e^2 n_0}{\varepsilon_0} n_{\mathrm{e}1}, \tag{5.29}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e}1} = -\frac{1}{n_0} \frac{\partial n_{\mathrm{e}1}}{\partial t}. \tag{5.30}$$

By combining these two equations we finally obtain

$$-m \frac{\partial^2 n_{\mathrm{e}1}}{\partial t^2} = \frac{e^2 n_0}{\varepsilon_0} n_{\mathrm{e}1}. \tag{5.31}$$

Now we take the Fourier transform of this equation, practically by applying rules (5.11) and (5.12):

$$\omega^2 n_{\mathrm{e}1} = \frac{e^2 n_0}{\varepsilon_0 m} n_{\mathrm{e}1}. \tag{5.32}$$

Thus we obtained a *dispersion relation for plasma oscillations*

$$\omega_{\mathrm{p}}^2 = \frac{n_0 e^2}{\varepsilon_0 m}. \tag{5.33}$$

This result requires a little discussion

- The dispersion relation shows that there exist oscillations with this characteristic *plasma frequency* ω_{p}, but there is no wave propagating since there is no wave vector $\boldsymbol{k}$ in the dispersion relation. The physical interpretation is the following: electrons may be displaced from their equilibrium positions determined by stationary background ions by some random fluctuation. The induced electric field acts as a restoring force to return them back to their original positions. As they move back, they have some inertia and overshoot their equilibrium positions. They are displaced again and the electric field again acts as a restoring force. This random movement is periodic. The motion of individual electrons is independent, their mutual phase shifts are random. In fact, the idea of individual electrons is very simplifying – we should rather speak about metaparticles, "bunch of electrons".

- The phase speed of this motion, $v_\varphi = \omega/k$, is not defined. This is not surprising in the frame that we just said. Since the mutual phase shifts of individual electrons are random, a formal phase speed takes *any* value.

- The group speed, $v_{\mathrm{g}} = \mathrm{d}\omega/\mathrm{d}k = 0$. No signal propagates.

- The plasma frequency is essentially the function of the background density only.

- Despite the assumption that both the ions and electrons are cold, this condition is not realistic. It is enough to assume that the temperature of electrons is larger than the temperature of ions and we further assume that electrons are isothermal (that is that T_{e} is uniform). Debye length relates to plasma frequency as $\lambda_{\mathrm{D}}^2 = \frac{\varepsilon_0 K_{\mathrm{B}} T_{\mathrm{e}}}{n_0 e^2} = \frac{K_{\mathrm{B}} T_{\mathrm{e}}}{m} \frac{1}{\omega_{\mathrm{p}}^2} \sim \frac{1}{2} v_{\mathrm{T}}^2 \frac{1}{\omega_{\mathrm{p}}^2}$, where we approximated the thermal speed $v_{\mathrm{T}} \sim \sqrt{\frac{2K_{\mathrm{B}}T_{\mathrm{e}}}{m}}$. Obviously, the fluctuations induced by plasma oscillations are shielded on the length scales comparable to the Debye length, thus, plasma oscillations do not induce any macroscopic charge.

5.4.2 Electron plasma wave

In the previous section, we showed that plasma supports the oscillatory motion, however, we did not derive the dispersion relation of the wave. We need to keep the additional physical effects. The thermal motion may cause the oscillations to propagate.

Longitudinal electron wave

- No background fields, i.e., $\boldsymbol{E}_0 = \boldsymbol{B}_0 = 0$, we do not consider induced perturbation to the magnetic field, i.e., $\boldsymbol{B}_1 = 0$. Thus from the set of Maxwell equations, only the Poisson equation must be retained
- Ions are cold, electrons are subjects of thermal motion
- Ions are fixed in space and have a uniform distribution
- Background density of ions and electrons is the same, i.e., $n_{\mathrm{e}0} = n_{\mathrm{i}0} = n_0$
- Background is stationary and homogeneous, $\boldsymbol{u}_{\mathrm{e}0} = 0$

To include the effect of the thermal motion, we need to keep the pressure term in the equation of motion. The gradient of pressure is coupled with the thermal motion through the equation of state

$$\nabla p_{\mathrm{e}} = \nabla(3K_{\mathrm{B}}T_{\mathrm{e}}n_{\mathrm{e}}) = 3K_{\mathrm{B}}T_{\mathrm{e}}\nabla(n_0 + n_{\mathrm{e}1}), \tag{5.34}$$

where number 3 indicates that there is only one degree of freedom, thus, we (silently) assume the propagation in one direction only. The modified system of linearised equations to be solved then reads:

$$mn_0\frac{\partial \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = -en_0\boldsymbol{E}_1 - 3K_{\mathrm{B}}T_{\mathrm{e}}\nabla n_{\mathrm{e}1}, \tag{5.35}$$

$$\frac{\partial n_{\mathrm{e}1}}{\partial t} + n_0\boldsymbol{\nabla}\cdot\boldsymbol{u}_{\mathrm{e}1} = 0, \tag{5.36}$$

$$\boldsymbol{E}_1 = -\nabla\phi_1, \tag{5.37}$$

$$\varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}_1 = -en_{\mathrm{e}1}. \tag{5.38}$$

The sequence of algebraic steps performed is identical to those in Section 5.4.1, thus we state only the final step here

$$-m\frac{\partial^2 n_{\mathrm{e}1}}{\partial t^2} = \frac{e^2 n_0}{\varepsilon_0}n_{\mathrm{e}1} - 3K_{\mathrm{B}}T_{\mathrm{e}}\,\triangle n_{\mathrm{e}1}. \tag{5.39}$$

By taking the Fourier transform we obtain

$$\omega^2 n_{\mathrm{e}1} - \omega_{\mathrm{p}}^2 n_{\mathrm{e}1} - 3k^2\frac{K_{\mathrm{B}}T_{\mathrm{e}}}{m}n_{\mathrm{e}1} = 0 \tag{5.40}$$

and finally a dispersion relation for longitudinal electron waves in plasmas

$$\omega^2 = \omega_{\mathrm{p}}^2 + \frac{3}{2}k^2 v_{\mathrm{T}}^2, \tag{5.41}$$

where $v_\mathrm{T} = \sqrt{\frac{2K_\mathrm{B}T_\mathrm{e}}{m}}$ is the thermal speed of electrons.

The dispersion relation seemingly differs only slightly from the dispersion relation for electron oscillations, however, it implies qualitatively different properties:

- This relation describes a propagating wave (it contains wave vector $\boldsymbol{k}$). It propagates only for frequencies larger than the plasma frequency, for frequencies smaller, the resulting wave vector takes imaginary values, thus the spatial part causes the exponential in (5.10) to decay. Excited waves with frequencies smaller than plasma frequency are rapidly attenuated. The wave propagates only in plasmas, the dispersion relation degenerates to $\omega = 0$ in a vacuum.

- The phase speed is given by $v_\varphi = \frac{\sqrt{\omega_\mathrm{p}^2 + 3/2k^2v_\mathrm{T}^2}}{k}$. It depends on the wavenumber, thus the waves are dispersive (different modes travel with a different phase speed).

- The group speed $v_\mathrm{g} = \frac{3}{2}\frac{v_\mathrm{T}^2 k}{\omega} = \frac{3}{2}\frac{v_\mathrm{T}^2 k}{\sqrt{\omega_\mathrm{p}^2+3/2k^2v_\mathrm{T}^2}}$ is not zero, thus the waves are able to carry information. Asymptotically for large k, $\lim\limits_{k\to\infty} v_\mathrm{g} = \frac{3}{2}\lim\limits_{k\to\infty}\frac{v_\mathrm{T}^2 k}{\sqrt{\omega_\mathrm{p}^2+3/2k^2v_\mathrm{T}^2}} \sim \frac{3}{2}\frac{v_\mathrm{T}^2 k}{\sqrt{3/2k^2v_\mathrm{T}^2}} = \sqrt{\frac{3}{2}}v_\mathrm{T}$, the waves propagate essentially with the thermal speed.

We solved the dispersion relation for perturbation in density of electrons n_e1. The definition of the Fourier transform (5.10) allows to introduce a phase shift, so that the variable fulfill the system of equations. Let's assume that the full solution for n_e1 has a form of

$$n_\mathrm{e1} = \overline{n_\mathrm{e1}}\cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right) \tag{5.42}$$

and we allow for a phase shift δ_E for $\boldsymbol{E}_1$:

$$\boldsymbol{E}_1 = \overline{\boldsymbol{E}_1}\cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta_E\right). \tag{5.43}$$

Both variables $\boldsymbol{E}_1$ and n_e1 are coupled via Poisson equation $\varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}_1 = -en_\mathrm{e1}$. Thus

$$-\varepsilon_0\boldsymbol{k}\cdot\overline{\boldsymbol{E}_1}\sin\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta_E\right) = -e\overline{n_\mathrm{e1}}\cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right). \tag{5.44}$$

By expressing the left-hand side we obtain

$$\sin\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right)\cos\delta_E + \cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right)\sin\delta_E = \frac{e}{\varepsilon_0}\frac{\overline{n_\mathrm{e1}}}{\overline{\boldsymbol{E}_1}\cdot\boldsymbol{k}}\cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right). \tag{5.45}$$

This equation has a solution when we take for example $\delta_E = \frac{\pi}{2}$.

The perturbation to electron density and the electric field induced by the wave are shifted in phase by a quarter of the period. How about the direction of the vector variables?

The Fourier image of the Lorentz force is

$$-\mathrm{i}\omega m\boldsymbol{u}_\mathrm{e1} = e(\boldsymbol{E}_1 + \boldsymbol{u}_\mathrm{e1}\times\boldsymbol{B}_0). \tag{5.46}$$

Since in our case $\boldsymbol{B}_0 = 0$, the relation simplifies to

$$\boldsymbol{u}_{e1} = -\frac{e}{\omega}\boldsymbol{E}_1, \tag{5.47}$$

thus $\boldsymbol{u}_{e1} \parallel \boldsymbol{E}_1$. Similarly Farraday's law written in the Fourier space is

$$\boldsymbol{k} \times \boldsymbol{E}_1 = \omega \boldsymbol{B}_1, \tag{5.48}$$

but $\boldsymbol{B}_1 = 0$, hence $\boldsymbol{k} \parallel \boldsymbol{E}_1$. The waves are longitudinal.

This result may also simply be obtained by application of the curl operator to a Poisson equation $\boldsymbol{E}_1 = -\nabla\phi_1$ and using the vector identity $\boldsymbol{\nabla} \times \nabla\phi_1 \equiv 0$.

5.4.3 Ion waves

When it comes to ion waves, the situation becomes more complicated, since we can't consider ions an immovable background anymore. In section 5.2 we derived the dispersion relation for the acoustic waves in the non-magnetised neutral fluid. Will the solution change, when the fluid is composed of charged particles? We search for the wave, assuming:

Ion acoustic wave

- No background fields, i.e., $\boldsymbol{E}_0 = \boldsymbol{B}_0 = 0$, we do not consider induced perturbation to the magnetic field, i.e., $\boldsymbol{B}_1 = 0$. Thus from the set of Maxwell equations only the Poisson equation must be retained
- Both ions and electrons are subjects of thermal motion, their temperatures may be different
- Background density of ions and electrons is the same, i.e., $n_{e0} = n_{i0} = n_0$
- Background is stationary and homogeneous, $\boldsymbol{u}_{e0} = \boldsymbol{u}_{i0} = 0$

Then we solve a system of linearised equations

$$mn_0\frac{\partial \boldsymbol{u}_{e1}}{\partial t} = -en_0\boldsymbol{E}_1 - 3K_B T_e \nabla n_{e1}, \tag{5.49}$$

$$\frac{\partial n_{e1}}{\partial t} + n_0 \boldsymbol{\nabla}\cdot\boldsymbol{u}_{e1} = 0, \tag{5.50}$$

$$Mn_0\frac{\partial \boldsymbol{u}_{i1}}{\partial t} = en_0\boldsymbol{E}_1 - 3K_B T_i \nabla n_{i1}, \tag{5.51}$$

$$\frac{\partial n_{i1}}{\partial t} + n_0 \boldsymbol{\nabla}\cdot\boldsymbol{u}_{i1} = 0, \tag{5.52}$$

$$\boldsymbol{E}_1 = -\nabla\phi_1, \tag{5.53}$$

$$\varepsilon_0 \boldsymbol{\nabla}\cdot\boldsymbol{E}_1 = e(n_{i1} - n_{e1}). \tag{5.54}$$

Ions are much heavier than electrons, thus, we may assume that electrons will simply follow the ions and organize themselves so that they shield possible charge concentrations. Therefore, the perturbation to the electron density $n_{\mathrm{e}1}$ may be estimated by the Boltzmann relation (4.92), thus

$$n_{\mathrm{e}1} = n_{\mathrm{e}} - n_{\mathrm{e}0} = n_0 \left(\exp \frac{e\phi_1}{K_{\mathrm{B}}T_{\mathrm{e}}} - 1 \right) \sim n_0 \frac{e\phi_1}{K_{\mathrm{B}}T_{\mathrm{e}}} \tag{5.55}$$

and we may consider (5.49) and (5.50) solved.

Now we take $\boldsymbol{E}_1$ from (5.53), insert it into (5.51) and use (5.55) in (5.54). Additionally, we apply a divergence operator to (5.51). We have

$$Mn_0 \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}1}}{\partial t} = -en_0 \,\triangle\phi_1 - 3K_{\mathrm{B}}T_{\mathrm{i}} \,\triangle n_{\mathrm{i}1}, \tag{5.56}$$

$$\frac{\partial n_{\mathrm{i}1}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}1} = 0, \tag{5.57}$$

$$-\varepsilon_0 \,\triangle\phi_1 = en_{\mathrm{i}1} - en_0 \frac{e\phi_1}{K_{\mathrm{B}}T_{\mathrm{e}}}. \tag{5.58}$$

Now we express $\boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}1}$ from (5.57), insert into (5.56) and take a Fourier transform of the two remaining equations. We have

$$\omega^2 M n_{\mathrm{i}1} = en_0 k^2 \phi_1 + 3K_{\mathrm{B}}T_{\mathrm{i}} k^2 n_{\mathrm{i}1}, \tag{5.59}$$

$$\varepsilon_0 k^2 \phi_1 = en_{\mathrm{i}1} - e^2 n_0 \frac{\phi_1}{K_{\mathrm{B}}T_{\mathrm{e}}}. \tag{5.60}$$

Finally, we take ϕ_1 from (5.60), insert into (5.59) and solve for $n_{\mathrm{i}1}$ to get

$$\omega^2 = \frac{e^2 n_0 k^2}{M \left(\varepsilon_0 k^2 + \frac{e^2 n_0}{K_{\mathrm{B}}T_{\mathrm{e}}} \right)} + \frac{3K_{\mathrm{B}}T_{\mathrm{i}}k^2}{M}. \tag{5.61}$$

Let $\Omega_{\mathrm{p}} = \sqrt{\frac{e^2 n_0}{M\varepsilon_0}}$ be an *ion plasma frequency.* If we remind Debye length $\lambda_{\mathrm{D}} = \sqrt{\frac{\varepsilon_0 K_{\mathrm{B}}T_{\mathrm{e}}}{n_0 e^2}}$ and ion thermal speed $v_{\mathrm{Ti}} = \sqrt{\frac{2K_{\mathrm{B}}T_{\mathrm{i}}}{M}}$, we finally obtain a *dispersion relation of ion acoustic waves*

$$\omega^2 = \frac{\Omega_{\mathrm{p}}^2 \lambda_{\mathrm{D}}^2 k^2}{\lambda_{\mathrm{D}}^2 k^2 + 1} + \frac{3}{2} v_{\mathrm{Ti}}^2 k^2. \tag{5.62}$$

For further discussion, let's take one more approximation, the *plasma approximation.* In practice we focus to large-scale effects, thus we set $k\lambda_{\mathrm{D}} \ll 1$. That is essentially equivalent of setting $n_{\mathrm{i}} = n_{\mathrm{e}}$.[4] Under this approximation, the dispersion relation simplifies to:

$$\omega^2 = k^2 \left(\frac{K_{\mathrm{B}}T_{\mathrm{e}}}{M} + 3\frac{K_{\mathrm{B}}T_{\mathrm{i}}}{M} \right). \tag{5.63}$$

[4] Note that by assuming the plasma approximation, we consider the bulk densities of ions and electrons to be equal. This does not imply either $n_{\mathrm{i}1} = n_{\mathrm{e}1}$ or $n_{\mathrm{i}0} = n_{\mathrm{e}0}$!

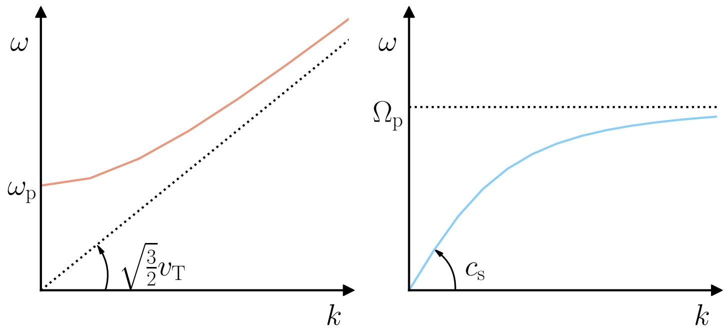

Figure 5.2: Asymptotic solutions for longitudinal electron and ion waves

- Phase speed $v_\varphi = \sqrt{\frac{K_\mathrm{B}T_\mathrm{e}+3K_\mathrm{B}T_\mathrm{i}}{M}} \equiv c_\mathrm{s}$ is a speed of sound in plasma. Under the plasma approximation, these waves are not dispersive. Dispersivity shows up as the effect of Debye shielding, which is mode-dependent.

- Group speed equals to the phase speed, thus $v_\mathrm{g} = v_\varphi = c_\mathrm{s}$. Ion waves are waves with a constant speed of propagation.

- In laboratory plasmas, $T_\mathrm{i} \ll T_\mathrm{e}$, thus $c_\mathrm{s} \sim \sqrt{\frac{K_\mathrm{B}T_\mathrm{e}}{M}}$. The propagation speed is essentially set by the *mass of ions* and *temperature of electrons*.

- The waves are longitudinal. We keep the proof to the reader.

- The ion acoustic wave exists only if the thermal motion of at least electrons is possible. Electrons are pulled by ions to balance the charge concentrations, hence $v_\mathrm{g} = v_\varphi$. However, the shielding is not perfect, there exist microscopic accumulations of ions, which propagate as the wave by through a $\boldsymbol{E}_1$ field.

- By assuming that $T_\mathrm{i} \ll T_\mathrm{e}$, the asymptotic dispersion relations read: $\omega = 0$ for $k \to 0$ and $\omega = \Omega_\mathrm{p}$ for $k \to \infty$. The asymptotic solution for small scales (large k) is fundamentally different from longitudinal *electron* waves, as also illustrated in Fig. 5.2.

5.5 Electrostatic waves in background magnetic field

5.5.1 Electron waves

Electron electrostatic perpendicual waves

- Background magnetic field, no background electric field, i.e., $\boldsymbol{B}_0 \neq 0$, $\boldsymbol{E}_0 = 0$

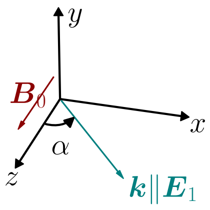

Figure 5.3: The coordinate system used in the derivation of the electrostatic electron waves in the presence of the background magnetic field.

- Ions stationary, thermal motion of electrons is being neglected
- Background density of ions and electrons is the same, i.e., $n_{e0} = n_{i0} = n_0$
- Background is stationary and homogeneous, $\boldsymbol{u}_{e0} = 0$
- $\boldsymbol{k} \parallel \boldsymbol{E}_1$

Let's write the Fourier images of fluid equations for electrons:

$$-\mathrm{i}\omega m \boldsymbol{u}_{e1} = -e(\boldsymbol{E}_1 + \boldsymbol{u}_{e1} \times \boldsymbol{B}_0), \tag{5.64}$$

$$-\mathrm{i}\omega n_{e1} + \mathrm{i}n_0 \boldsymbol{k} \cdot \boldsymbol{u}_{e1} = 0, \tag{5.65}$$

$$\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{E}_1 = -\frac{e}{\varepsilon_0} n_{e1}, \tag{5.66}$$

We express $\boldsymbol{E}_1$ from (5.64) and combine together (5.65) and (5.66) through n_{e1}.

$$\boldsymbol{E}_1 = \frac{\mathrm{i}\omega m}{e} \boldsymbol{u}_{e1} - \boldsymbol{u}_{e1} \times \boldsymbol{B}_0, \tag{5.67}$$

$$\boldsymbol{k} \cdot \boldsymbol{E}_1 = \frac{\mathrm{i}e n_0}{\varepsilon_0 \omega} \boldsymbol{k} \cdot \boldsymbol{u}_{e1}. \tag{5.68}$$

Since we are considering longitudinal waves, $\boldsymbol{k} \parallel \boldsymbol{E}_1$, hence $\boldsymbol{k} \cdot \boldsymbol{E}_1 = kE_1$. Thus equation (5.68) allows to express the amplitude $E_1 = |\boldsymbol{E}_1|$ of electric field:

$$E_1 = \frac{\mathrm{i}e n_0}{\varepsilon_0 \omega k} \boldsymbol{k} \cdot \boldsymbol{u}_{e1}. \tag{5.69}$$

Let us consider the Cartesian coordinate system (Fig. 5.3), so that $\boldsymbol{e}_z \parallel \boldsymbol{B}_0$ and $\boldsymbol{E}_1$ lies in the $x - z$ plane. Hence $\boldsymbol{B}_0 = (0, 0, B_0)$ and $\boldsymbol{E}_1 = (E_1 \sin\alpha, 0, E_1 \cos\alpha)$, and similarly $\boldsymbol{k} = (k \sin\alpha, 0, k \cos\alpha)$, where α is the angle between $\boldsymbol{B}_0$ and $\boldsymbol{k}$. We may then write (5.67) in components

$$\begin{pmatrix} E_1 \sin\alpha \\ 0 \\ E_1 \cos\alpha \end{pmatrix} = \frac{\mathrm{i}e n_0}{\varepsilon_0 \omega} (k u_{e1,x} \sin\alpha + k u_{e1,z} \cos\alpha) \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix} = \tag{5.70}$$

$$= \frac{\mathrm{i}\omega m}{e} \begin{pmatrix} u_{e1,x} \\ u_{e1,y} \\ u_{e1,z} \end{pmatrix} - B_0 \begin{pmatrix} u_{e1,y} \\ -u_{e1,x} \\ 0 \end{pmatrix}. \tag{5.71}$$

Then using (5.69), multiplying the whole equation by $\frac{1}{\mathrm{i}}\frac{e}{\omega m}$ and using $\omega_\mathrm{p}^2 = \frac{e^2 n_0}{\varepsilon_0 m}$ and $\omega_\mathrm{c} = \frac{eB}{m}$, and reorganising everything to the left-hand side of the equation we have

$$\begin{pmatrix} u_{\mathrm{e}1,x} \\ u_{\mathrm{e}1,y} \\ u_{\mathrm{e}1,z} \end{pmatrix} - \frac{\omega_\mathrm{p}^2}{\omega^2}\left(u_{\mathrm{e}1,x}\sin\alpha + u_{\mathrm{e}1,z}\cos\alpha\right)\begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix} + \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}\begin{pmatrix} u_{\mathrm{e}1,y} \\ -u_{\mathrm{e}1,x} \\ 0 \end{pmatrix} = 0. \quad (5.72)$$

Such equation may be written as

$$M_u \cdot \boldsymbol{u}_{\mathrm{e}1} = 0, \quad (5.73)$$

with matrix M_u being

$$M_u = \begin{bmatrix} 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\sin^2\alpha & \mathrm{i}\frac{\omega_\mathrm{c}}{\omega} & -\frac{\omega_\mathrm{p}^2}{\omega^2}\sin\alpha\cos\alpha \\ -\mathrm{i}\frac{\omega_\mathrm{c}}{\omega} & 1 & 0 \\ -\frac{\omega_\mathrm{p}^2}{\omega^2}\sin\alpha\cos\alpha & 0 & 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\cos^2\alpha \end{bmatrix} \quad (5.74)$$

Nontrivial solution of equation (5.73) exists only if $\det M_u = 0$. Let's evaluate $\det M_u$ using the Laplace expansion using the third row:

$$\det M_u = -\frac{\omega_\mathrm{p}^4}{\omega^4}\sin^2\alpha\cos^2\alpha + \left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\cos^2\alpha\right)\left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\sin^2\alpha - \frac{\omega_\mathrm{c}^2}{\omega^2}\right) = 0\,, \quad (5.75)$$

which is the dispersion relation for electrostatic waves in the presence of the background magnetic field. Note that according to this dispersion relation no wave propagates, only oscillations appear in plasma.

Let's discuss two extreme cases. For $\alpha = 0$ hence for oscillatory motions along the magnetic field, we have

$$\left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\right)\left(1 - \frac{\omega_\mathrm{c}^2}{\omega^2}\right) = 0, \quad (5.76)$$

which has two physical solutions

$$\omega^2 = \omega_\mathrm{p}^2, \quad \text{and} \quad \omega^2 = \omega_\mathrm{c}^2. \quad (5.77)$$

Therefore, in the direction of the magnetic field, we have ordinary plasma oscillations and the Larmor rotation.

For $\alpha = \pi/2$, i.e., the oscillatory motions perpendicular to the background magnetic field we have

$$1 - \frac{\omega_\mathrm{p}^2}{\omega^2} - \frac{\omega_\mathrm{c}^2}{\omega^2} = 0, \quad (5.78)$$

which may be written as

$$\omega^2 = \omega_\mathrm{p}^2 + \omega_\mathrm{c}^2 \equiv \omega_\mathrm{h}^2. \tag{5.79}$$

The symbol ω_h indicates the upper hybrid frequency, the only possible frequency for electron oscillations perpendicular to $\boldsymbol{B}_0$.

5.5.2 Ion waves

Ion electrostatic perpendicular waves

- Background magnetic field, no background electric field, i.e., $\boldsymbol{B}_0 \neq 0$, $\boldsymbol{E}_0 = 0$
- Electrons are subject of thermal motion if they are allowed to, ions are cold
- Background density of ions and electrons is the same, i.e., $n_\mathrm{e0} = n_\mathrm{i0} = n_0$
- Background is stationary and homogeneous, $\boldsymbol{u}_\mathrm{e0} = \boldsymbol{u}_\mathrm{i0} = 0$
- $\boldsymbol{k} \parallel \boldsymbol{E}_1$

In the case of the ion waves perpendicular to the background magnetic field, we need to strictly distinguish the two cases: when the waves are *nearly* perpendicular, and when they are *strictly* perpendicular (Fig. 5.4). The difference is fundamental. In the case of the nearly perpendicular configuration, there is some, however small, component of the background magnetic field parallel to the wave vector. It is important since in this approximation the electrons follow ions and in such case, they are still able to shield the concentrations of charges via the mechanism of Debye shielding. They are free to move along the magnetic field lines and the shielding is effective. In the case of a strictly perpendicular magnetic field, there is no option for electrons to shield anymore, since they can't move around. Let's investigate the two cases separately.

$\boldsymbol{k}$ nearly perpendicular to $\boldsymbol{B}_0$

In this case we consider the effect of the Debye shielding and thus we can use the Boltzmann relation as a solution to the problem of electrons. We further solve the Euler equation for ions having the form of

$$-\mathrm{i}\omega M \boldsymbol{u}_\mathrm{i1} = -e\mathrm{i}\boldsymbol{k}\phi_1 + e\boldsymbol{u}_\mathrm{i1} \times \boldsymbol{B}_0 \tag{5.80}$$

in the Fourier space. In components we have (again, assuming the Cartesian coordinate system, where $\boldsymbol{k} = k\boldsymbol{e}_x$ and $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$)

$$-\mathrm{i}\omega M u_{\mathrm{i1},x} = -e\mathrm{i}k\phi_1 + e u_{\mathrm{i1},y} B_0, \tag{5.81}$$

$$-\mathrm{i}\omega M u_{\mathrm{i1},y} = -e u_{\mathrm{i1},x} B_0, \tag{5.82}$$

which gives the x-component of the perturbed velocity

$$u_{\mathrm{i}1,x} = \frac{ek\phi_1}{\omega M}\left(1 - \frac{\Omega_{\mathrm{c}}^2}{\omega^2}\right)^{-1}. \tag{5.83}$$

The continuity equation gives as usually

$$n_{\mathrm{i}1} = n_0 \frac{k}{\omega} u_{\mathrm{i}1,x}, \tag{5.84}$$

and the electron density perturbation is solved by the Boltzmann relation

$$n_{\mathrm{e}1} = n_0 \frac{e\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}}}. \tag{5.85}$$

By assuming the plasma approximation $n_{\mathrm{i}} = n_{\mathrm{e}}$, we have $n_{\mathrm{i}1} = n_{\mathrm{e}1}$ and hence

$$u_{\mathrm{i}1,x} = \frac{\omega e\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}} k}, \tag{5.86}$$

which may be compared to (5.83) to obtain the dispersion relation

$$\omega^2 - \Omega_{\mathrm{c}}^2 = \frac{K_{\mathrm{B}} T_{\mathrm{e}}}{M} k^2. \tag{5.87}$$

Let us remind that the definition of the speed of sound in the plasma reads

$$c_{\mathrm{s}}^2 = \left(\frac{K_{\mathrm{B}} T_{\mathrm{e}}}{M} + \frac{3K_{\mathrm{B}} T_{\mathrm{i}}}{M}\right) \tag{5.88}$$

which has a simpler form due to the assumption of $T_{\mathrm{i}} \to 0$ (or equivalently, $T_{\mathrm{i}} \ll T_{\mathrm{e}}$). Hence the dispersion relation (5.87) may be rewritten in a form

$$\omega^2 = \Omega_{\mathrm{c}}^2 + k^2 c_{\mathrm{s}}^2. \tag{5.89}$$

This case is possible only if the limiting angle $\theta = \frac{\pi}{2} - \alpha$, where α is the angle between $\boldsymbol{k}$ and $\boldsymbol{B}_0$, is larger than the ratio of ion and electron speed in the direction of $\boldsymbol{B}_0$, hence roughly for $\theta > (m/M)^{1/2}$. In this case, the electrons have kinetic energy (from the thermal motion) large enough to chase the ions and also move around them to shield the originating concentrations of charges. For angles smaller when electrons do not have thermal speed large enough to shield the ion, we need to use a different formalism.

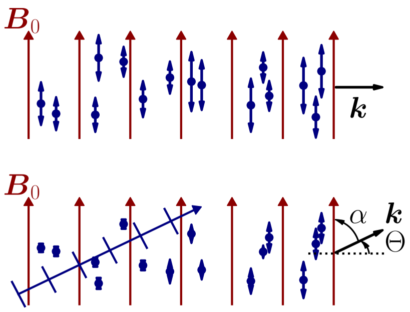

Figure 5.4: Comparison of the cases with $\boldsymbol{k}$ strictly (top) and nearly (bottom) perpendicular to $\boldsymbol{B}_0$. The two-headed arrows indicate the possible motions of the electrons, which are allowed to move only along the magnetic field lines. In the first case of an assument strictly perpendicular propagation, they are not able to shield, because their allowed motion does not have a component in the direction of the assument wave propagation.

$\boldsymbol{k} \perp \boldsymbol{B}_0$

For $\boldsymbol{k}$ exactly perpendicular to $\boldsymbol{B}_0$ there is no component of $\boldsymbol{k}$ parallel to $\boldsymbol{B}_0$, thus electrons are not allowed to move along the $\boldsymbol{B}_0$ and thus they cannot keep the neutrality. Hence we must solve fluid equations for both electrons and ions. The system of equations is closed by using the plasma approximation. Hence we have simply used (5.83) for describing the motions of ions in the direction of the wave vector, and equivalently construct the corresponding equation for electrons just by replacing $e \to -e$ and $M \to m$. Thus

$$u_{\mathrm{e}1,x} = -\frac{ek\phi_1}{\omega m}\left(1 - \frac{\omega_{\mathrm{c}}^2}{\omega^2}\right)^{-1}. \tag{5.90}$$

Similarly for the continuity equation,

$$n_{\mathrm{i}1} = n_0 \frac{k}{\omega} u_{\mathrm{i}1,x}, \tag{5.91}$$

$$n_{\mathrm{e}1} = n_0 \frac{k}{\omega} u_{\mathrm{e}1,x}. \tag{5.92}$$

By applying the plasma approximation approach we have $n_{\mathrm{e}1} = n_{\mathrm{i}1}$ and hence $u_{\mathrm{e}1,x} = u_{\mathrm{i}1,x}$. Thus

$$M\left(1 - \frac{\Omega_{\mathrm{c}}^2}{\omega^2}\right) = -m\left(1 - \frac{\omega_{\mathrm{c}}^2}{\omega^2}\right), \tag{5.93}$$

and further

$$\omega^2(M+m) = m\omega_{\mathrm{c}}^2 + M\Omega_{\mathrm{c}}^2 = e^2B_0^2\left(\frac{1}{m} + \frac{1}{M}\right) = e^2B_0^2\left(\frac{m+M}{mM}\right) \tag{5.94}$$

and thus

$$\omega^2 = \frac{eB_0}{m}\frac{eB_0}{M} = \omega_{\mathrm{c}}\Omega_{\mathrm{c}} \equiv \omega_{\mathrm{d}}^2. \tag{5.95}$$

In this case, the oscillations are allowed to have the only one possible frequency, the *lower hybrid frequency*.

5.6 Electromagnetic waves

When discussing the electromagnetic waves, we automatically assume that the resulting solution will fulfill the assumption $\boldsymbol{k} \perp \boldsymbol{E}_1$. This assumption should always be verified at the end. In this text, we will keep the verification to the reader. Since they are electromagnetic, we cannot neglect the magnetic component of the wave, thus $\boldsymbol{B}_1 \neq 0$.

5.6.1 In vacuum: Light waves

Electromagnetic waves in vacuum degenerate to light waves. Thus we need only the set of Maxwell equations.

Electromagnetic waves in vacuum

- No background magnetic or electric field, i.e., $\boldsymbol{B}_0 = \boldsymbol{E}_0 = 0$
- No plasma, thus $n_\mathrm{e} = n_\mathrm{i} = n_0 = 0$, $\boldsymbol{u}_\mathrm{e} = \boldsymbol{u}_\mathrm{i} = 0$
- $\boldsymbol{k} \perp \boldsymbol{E}_1$

$$\nabla \times \boldsymbol{E}_1 = -\frac{\partial \boldsymbol{B}_1}{\partial t}, \tag{5.96}$$

$$\frac{1}{\mu_0}\nabla \times \boldsymbol{B}_1 = \varepsilon_0 \frac{\partial \boldsymbol{E}_1}{\partial t}, \tag{5.97}$$

$$\nabla \cdot \boldsymbol{B}_1 = 0. \tag{5.98}$$

Now we take a time derivative of (5.96) and apply the curl operator to (5.97). We have

$$\nabla \times \frac{\partial \boldsymbol{E}_1}{\partial t} = -\frac{\partial^2 \boldsymbol{B}_1}{\partial t^2}, \tag{5.99}$$

$$\frac{1}{\mu_0 \varepsilon_0}\nabla \times \nabla \times \boldsymbol{B}_1 = \nabla \times \frac{\partial \boldsymbol{E}_1}{\partial t}, \tag{5.100}$$

$$\nabla \cdot \boldsymbol{B}_1 = 0. \tag{5.101}$$

Both (5.99) and (5.100) contain the term $\nabla \times \frac{\partial \boldsymbol{E}_1}{\partial t}$. Using this term we combine these two equations and take a Fourier transform of the system of the remaining two equations. We have

$$-c^2 \boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{B}_1) = \omega^2 \boldsymbol{B}_1, \tag{5.102}$$

$$\boldsymbol{k} \cdot \boldsymbol{B}_1 = 0, \tag{5.103}$$

where we used the definition of the *speed of light* $c = \sqrt{(\mu_0 \varepsilon_0)^{-1}}$. The double vector multiplication may be written using two terms. We have

$$-c^2 \left[(\boldsymbol{k} \cdot \boldsymbol{B}_1)\boldsymbol{k} - k^2 \boldsymbol{B}_1 \right] = \omega^2 \boldsymbol{B}_1, \tag{5.104}$$

$$\boldsymbol{k} \cdot \boldsymbol{B}_1 = 0. \tag{5.105}$$

Now we finally use (5.105) in (5.104) and have

$$c^2k^2\boldsymbol{B}_1 = \omega^2\boldsymbol{B}_1, \tag{5.106}$$

which simplifies to

$$\omega^2 = c^2k^2. \tag{5.107}$$

This is the dispersion relation for electromagnetic waves in vacuum, describing the propagation of light. Obviously, the light waves are non-dispersive and both phase and group speeds equal to the speed of light in vacuum.

5.6.2 Electromagnetic waves in plasma

In plasma, the dispersion relation for electromagnetic waves gets modified due to the charged particles.

Electromagnetic waves in plasma

- No background magnetic or electric field, i.e., $\boldsymbol{B}_0 = \boldsymbol{E}_0 = 0$
- Background stationary and homogeneous
- We expect large frequencies ($\omega \gg \Omega_\mathrm{p}$), thus heavy ions remain stationary, only electrons respond
- Both ions and electrons are cold (no thermal motion)
- $\boldsymbol{k} \perp \boldsymbol{E}_1$

We proceed similarly to the case of electromagnetic waves in vacuum, we only need to retain the electric current $\boldsymbol{j}_1$ in the Ampére's law.

$$\nabla \times \boldsymbol{E}_1 = -\frac{\partial \boldsymbol{B}_1}{\partial t}, \tag{5.108}$$

$$\frac{1}{\mu_0}\nabla \times \boldsymbol{B}_1 = \varepsilon_0\frac{\partial \boldsymbol{E}_1}{\partial t} + \boldsymbol{j}_1. \tag{5.109}$$

Similarly to the previous section, we take a time derivative of (5.109) and apply a curl operator to (5.108) and combine the two using the term $\nabla \times \frac{\partial \boldsymbol{B}_1}{\partial t}$, which is contained in both equations. We have

$$\nabla \times \nabla \times \boldsymbol{E}_1 = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}_1}{\partial t^2} - \frac{1}{\varepsilon_0 c^2}\frac{\partial \boldsymbol{j}_1}{\partial t}. \tag{5.110}$$

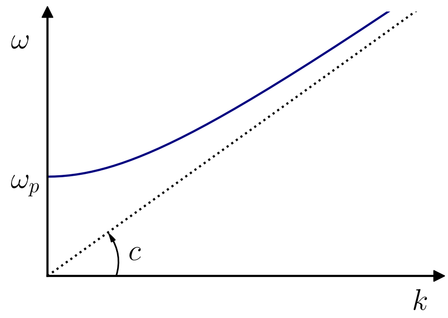

Figure 5.5: Dispersion diagram of electromagnetic waves in plasma.

Again, we take a Fourier transform of the above given equation and split the double vector multiplication into two terms:

$$-(\boldsymbol{k}\cdot\boldsymbol{E}_1)\boldsymbol{k} + k^2\boldsymbol{E}_1 = \frac{\mathrm{i}\omega}{\varepsilon_0 c^2}\boldsymbol{j}_1 + \frac{\omega^2}{c^2}\boldsymbol{E}_1. \tag{5.111}$$

From the beginning, we search for the dispersion relation of electromagnetic waves, thus we limit ourselves to the case when $\boldsymbol{k} \perp \boldsymbol{E}_1$, hence $\boldsymbol{k}\cdot\boldsymbol{E}_1 = 0$. Thus (5.111) simplifies to

$$(\omega^2 - c^2k^2)\boldsymbol{E}_1 = -\frac{\mathrm{i}\omega}{\varepsilon_0}\boldsymbol{j}_1. \tag{5.112}$$

In the solution, we expect frequencies much larger than ion plasma frequency, thus the perturbed component of the electric current is composed of electrons only, ions are considered as fixed in space in this approximation. Thus

$$\boldsymbol{j}_1 = -n_0 e\boldsymbol{u}_{\mathrm{e}1}. \tag{5.113}$$

The fluctuating component of electron velocity $\boldsymbol{u}_{\mathrm{e}1}$ may be obtained from Euler equation for electrons, which for cold plasma with no background magnetic field simplifies to

$$m\frac{\partial \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = -e\boldsymbol{E}_1. \tag{5.114}$$

Hence, the current density $\boldsymbol{j}_1$ is

$$\boldsymbol{j}_1 = -\frac{n_0e^2}{\mathrm{i}\omega m}\boldsymbol{E}_1, \tag{5.115}$$

where we expressed the Euler equation for electrons in the Fourier space. Now we continue to modify (5.112):

$$(\omega^2 - c^2k^2)\boldsymbol{E}_1 = \frac{\mathrm{i}\omega}{\varepsilon_0}\frac{n_0e^2}{\mathrm{i}\omega m}\boldsymbol{E}_1 = \frac{n_0e^2}{\varepsilon_0 m}\boldsymbol{E}_1 = \omega_{\mathrm{p}}^2\boldsymbol{E}_1. \tag{5.116}$$

Hence the *dispersion relation for electromagnetic waves in plasma* reads

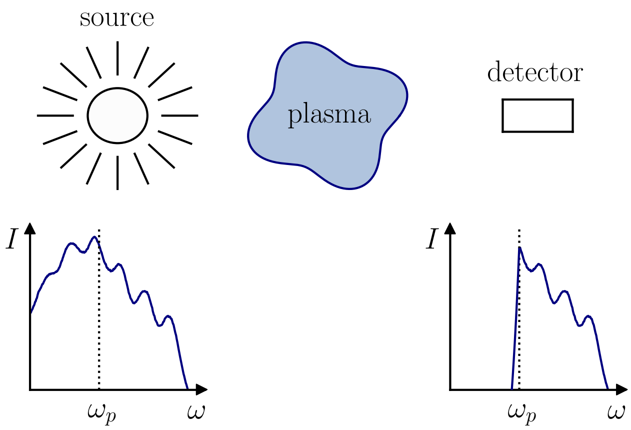

Figure 5.6: The scheme of the influence of cut-off frequency on the measured spectrum.

$$\omega^2 = \omega_{\rm p}^2 + c^2k^2. \tag{5.117}$$

Let's make a few comments

- The dispersion diagram is displayed in Fig. 5.5.
- Phase velocity: $v_\varphi^2 = \left(\frac{\omega}{k}\right)^2 = c^2 + \frac{\omega_{\rm p}^2}{k^2} > c^2$. Phase velocity of these waves is larger than the speed of light. However, it is the phase velocity, thus it does not carry any signal, and the principles of the special theory of relativity are not violated.
- The dispersion relation describes propagating waves that are for large k or in the high-frequency regime only weakly dispersive.
- Group velocity: $v_{\rm g} = {\rm d}\omega/{\rm d}k = \frac{2c^2k}{2\sqrt{\omega_{\rm p}^2+c^2k^2}} = \frac{c^2}{v_\varphi} < c$ for $v_\varphi > c$. The group speed is smaller than the speed of light.
- The waves propagate only for large frequencies, when $\omega > \omega_{\rm p}$. The electron plasma frequency $\omega_{\rm p}$ serves as a *cut-off frequency*. For frequencies smaller than $\omega_{\rm p}$ the waves do not propagate (the wave vector $\boldsymbol{k}$ is imaginary). Then we have $ck = (\omega^2 - \omega_{\rm p}^2)^{1/2} = {\rm i}\sqrt{\omega_{\rm p}^2 - \omega^2}$. Taking the spatial part of the Fourier transform (5.10), we find the physical variables to vary as $\exp {\rm i}\boldsymbol{k}\cdot\boldsymbol{r} \sim \exp {\rm i}kx = \exp\left[-x\frac{\sqrt{\omega_{\rm p}^2-\omega^2}}{c}\right] = \exp\left[-\frac{x}{\delta}\right]$, where $\delta = \frac{c}{\sqrt{\omega_{\rm p}^2-\omega^2}}$ is the attenuation length.

The electromagnetic waves propagating through plasma may serve as a primitive mean to estimate the plasma density. The process is illustrated in Fig. 5.6. From measuring the spectrum we find the cut-off frequency, which depends on the plasma density n_0

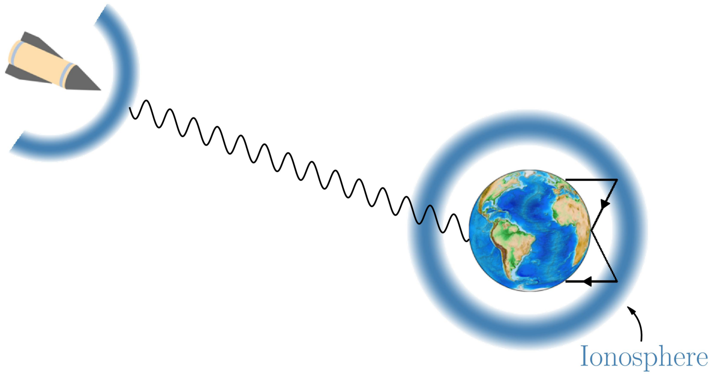

Figure 5.7: The radio trasmission from the spacecraft returning to the Earth is blocked by surrounding plasma, when the spacecraft undergoes the atmospheric breaking maneuvre. The breaking heats up and ionises the air in the close vicinity, thereby introducing a plasma-frequency cut-off to the radio communication.

only. This is being used in astrophysics for example in estimating the plasma density of the solar atmosphere. Imagine that we have an ionised plasma cloud hurled during the solar flare by the magnetic reconnection, which rises in the atmosphere, and acts as a source for different kinds of plasma waves, including the electromagnetic ones, usually in the band of radio waves. As the cloud rises, the density of ambient plasma decreases, therefore, also the cut-off frequency decreases and thus waves at lower frequencies are allowed to escape the region and propagate toward the observer. By measuring the cut-off frequency as a function of time and estimating the rising speed of the plasma blob, we perform the density scanning of the solar atmosphere. The spacecraft returning from space to Earth are unable to communicate directly with the control for about three minutes during the atmospheric breaking maneuvre (Fig. 5.7)

(Add a story of measuring the density of the interstellar matter from Voyager 1, must confirm what kind of waves were involved and perform the calculation)

5.6.3 Complex of electromagnetic waves in presence of $\boldsymbol{B}_0$

Electromagnetic waves in plasmas interact mostly with electrons, the heavy ions can't adjust to the large frequencies of the electromagnetic waves. Hence we will solve the system of equations

$$-\mathrm{i}\omega m\boldsymbol{u}_{\mathrm{e}1} = -e(\boldsymbol{E}_1 + \boldsymbol{u}_{\mathrm{e}1} \times \boldsymbol{B}_0), \tag{5.118}$$

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{E}_1 = \mathrm{i}\omega\boldsymbol{B}_1, \tag{5.119}$$

$$\frac{\mathrm{i}}{\mu_0}\boldsymbol{k} \times \boldsymbol{B}_1 = -\varepsilon_0\mathrm{i}\omega\boldsymbol{E}_1 - en_0\boldsymbol{u}_{\mathrm{e}1}. \tag{5.120}$$

The equations are already written in the form of Fourier images. Note that we did not consider the continuity equation, which over-determines the system. That is because the density perturbation $n_{\mathrm{e}1}$ does appear in the continuity equation only. Now we express $\boldsymbol{u}_{\mathrm{e}1}$ from (5.118) and combine (5.119) and (5.120) via $\boldsymbol{B}_1$, where in the combined

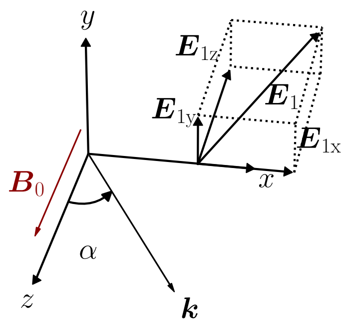

Figure 5.8: Coordinate system: electromagnetic waves in presence of $\boldsymbol{B}_0$.

equation we keep solely $\boldsymbol{u}_{\mathrm{e}1}$ on the left-hand side:

$$\boldsymbol{u}_{\mathrm{e}1} = -\mathrm{i}\frac{e}{m\omega}\boldsymbol{E}_1 - \mathrm{i}\frac{eB_0}{m\omega}\boldsymbol{u}_{\mathrm{e}1} \times \boldsymbol{b}_0, \tag{5.121}$$

$$\boldsymbol{u}_{\mathrm{e}1} = -\mathrm{i}\frac{\varepsilon_0\omega}{en_0}\boldsymbol{E}_1 - \mathrm{i}\frac{c^2\varepsilon_0\omega}{\omega^2 en_0}\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_1), \tag{5.122}$$

where we used the definition of the speed of light $c^2 = (\varepsilon_0\mu_0)^{-1}$ and defined the unity vector $\boldsymbol{b}_0$ in the direction of the magnetic field, so that $\boldsymbol{B}_0 = B_0\boldsymbol{b}_0$. Now we insert (5.122) into (5.121). One needs to be careful here, since we find $\boldsymbol{u}_{\mathrm{e}1}$ in (5.121) on both sides of the equation. We further use $\omega_{\mathrm{c}} = eB_0/m$ to obtain

$$\begin{aligned} -\mathrm{i}\frac{\varepsilon_0\omega}{en_0}\boldsymbol{E}_1 - \mathrm{i}\frac{c^2\varepsilon_0}{\omega en_0}\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_1) = \\ -\mathrm{i}\frac{e}{m\omega}\boldsymbol{E}_1 - \mathrm{i}\frac{\omega_{\mathrm{c}}}{\omega}\left[-\mathrm{i}\frac{\varepsilon_0\omega}{en_0}\boldsymbol{E}_1 - \mathrm{i}\frac{c^2\varepsilon_0}{\omega en_0}\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_1)\right] \times \boldsymbol{b}_0. \end{aligned} \tag{5.123}$$

Now we express the double-cross products and multiply the whole equation by $-\frac{en_0\omega}{\mathrm{i}\varepsilon_0}$. We have

$$\begin{aligned} \omega^2\boldsymbol{E}_1 + c^2(\boldsymbol{k}\cdot\boldsymbol{E}_1)\boldsymbol{k} - c^2k^2\boldsymbol{E}_1 = \\ = \frac{e^2n_0}{\varepsilon_0 m}\boldsymbol{E}_1 - \mathrm{i}\omega_{\mathrm{c}}\omega\boldsymbol{E}_1 \times \boldsymbol{b}_0 - \mathrm{i}\frac{\omega_{\mathrm{c}}}{\omega}c^2(\boldsymbol{k}\cdot\boldsymbol{E}_1)(\boldsymbol{k}\times\boldsymbol{b}_0) + \mathrm{i}\frac{\omega_{\mathrm{c}}}{\omega}c^2k^2(\boldsymbol{E}_1\times\boldsymbol{b}_0). \end{aligned} \tag{5.124}$$

By reordering and using $\omega_{\mathrm{p}}^2 = \frac{e^2n_0}{\varepsilon_0 m}$ we have

$$(\omega^2 - \omega_{\mathrm{p}}^2 - c^2k^2)\boldsymbol{E}_1 + \mathrm{i}\frac{\omega_{\mathrm{c}}}{\omega}(\omega^2 - c^2k^2)(\boldsymbol{E}_1\times\boldsymbol{b}_0) + c^2(\boldsymbol{k}\cdot\boldsymbol{E}_1)\boldsymbol{k} + \mathrm{i}\frac{\omega_{\mathrm{c}}}{\omega}c^2(\boldsymbol{k}\cdot\boldsymbol{E}_1)(\boldsymbol{k}\times\boldsymbol{b}_0) = 0. \tag{5.125}$$

To solve this equation, we introduce the Cartesian coordinate system (Fig. 5.8), so that $\boldsymbol{B}_0 \parallel \boldsymbol{e}_z$ and $\boldsymbol{k}$ lies in the x–z plane. Therefore $\boldsymbol{b}_0 = (0, 0, 1)$ and $\boldsymbol{k} = (k\sin\alpha, 0, k\cos\alpha)$, where α is the angle between $\boldsymbol{B}_0$ and $\boldsymbol{k}$. The electric intensity vector has all components $\boldsymbol{E}_1 = (E_{1x}, E_{1y}, E_{1z})$. In this particular representation (5.125) turns into

$$(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2)\begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \end{pmatrix} + \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2)\begin{pmatrix} E_{1y} \\ -E_{1x} \\ 0 \end{pmatrix} +$$
$$+ c^2(kE_{1x}\sin\alpha + kE_{1z}\cos\alpha)\begin{pmatrix} k\sin\alpha \\ 0 \\ k\cos\alpha \end{pmatrix} +$$
$$+ \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}c^2(E_{1x}k\sin\alpha + kE_{1z}\cos\alpha)\begin{pmatrix} 0 \\ -k\sin\alpha \\ 0 \end{pmatrix} = 0. \tag{5.126}$$

This is a set of three equations for three components of perturbation of electric intensity, which may be written in the matrix form as

$$M_{\boldsymbol{E}_1} \cdot \boldsymbol{E}_1 = 0 \tag{5.127}$$

with system matrix

$$M_{\boldsymbol{E}_1} = \begin{pmatrix} \omega^2 - \omega_\mathrm{p}^2 - c^2k^2\cos^2\alpha & \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2) & c^2k^2\sin\alpha\cos\alpha \\ -\mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2\cos^2\alpha) & \omega^2 - \omega_\mathrm{p}^2 - c^2k^2 & -\mathrm{i}\frac{\omega_\mathrm{c}}{\omega}c^2k^2\cos\alpha\sin\alpha \\ c^2k^2\sin\alpha\cos\alpha & 0 & \omega^2 - \omega_\mathrm{p}^2 - c^2k^2\sin^2\alpha \end{pmatrix}. \tag{5.128}$$

Equation (5.127) has the nontrivial solution when $\det M_{\boldsymbol{E}_1} = 0$. We expand $\det M_{\boldsymbol{E}_1}$ around the third row to have

$$\det M_{\boldsymbol{E}_1} = c^4k^4\sin^2\alpha\cos^2\alpha\left[\frac{\omega_\mathrm{c}^2}{\omega^2}(\omega^2 - c^2k^2) - (\omega^2 - \omega_\mathrm{p}^2 - c^2k^2)\right] +$$
$$+ (\omega^2 - \omega_\mathrm{p}^2 - c^2k^2\sin^2\alpha)\Big[(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2\cos^2\alpha)(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2) -$$
$$- \frac{\omega_\mathrm{c}^2}{\omega^2}(\omega^2 - c^2k^2)(\omega^2 - c^2k^2\cos^2\alpha)\Big] = 0. \tag{5.129}$$

This is the dispersion relation for a *complex of electromagnetic waves in plasmas*. We will now study two special cases.

Electromagnetic waves parallel to $\boldsymbol{B}_0$

Waves parallel to $\boldsymbol{B}_0$ have $\boldsymbol{k} \parallel \boldsymbol{B}_0$ and thus $\alpha = 0$. The dispersion relation (5.129) simplifies to

$$(\omega^2 - \omega_\mathrm{p}^2)\left[(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2)^2 - \frac{\omega_\mathrm{c}^2}{\omega^2}(\omega^2 - c^2k^2)^2\right] = 0. \tag{5.130}$$

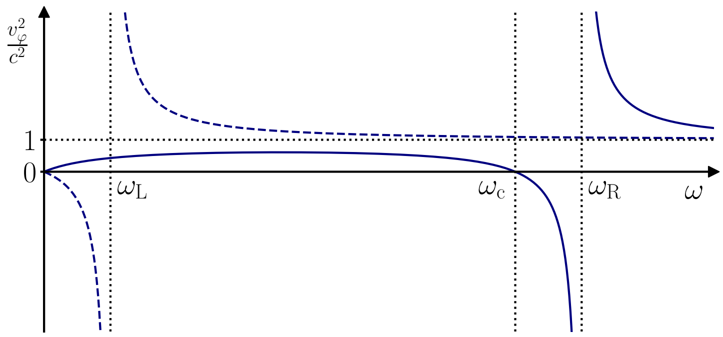

Figure 5.9: The dispersion diagram of the L- (dashed) and R- (solid) waves. The figure illustrates the cut-offs and resonances.

This equation has four roots, one of which is not physical. By setting the first parenthesis equal to zero one obtains

$$\omega^2 = \omega_{\rm p}^2, \tag{5.131}$$

therefore, along the magnetic field, the first mode of oscillatory motions is composed of ordinary plasma oscillations.

By setting the second parenthesis equal to zero we obtain other two roots:

$$\omega^2 - \omega_{\rm p}^2 - c^2k^2 = \pm\frac{\omega_{\rm c}}{\omega}(\omega^2 - c^2k^2) \tag{5.132}$$

which may be written as

$$\omega^2 - c^2k^2 = \frac{\omega_{\rm p}^2}{1 - \frac{\omega_{\rm c}}{\omega}} \tag{5.133}$$

and

$$\omega^2 - c^2k^2 = \frac{\omega_{\rm p}^2}{1 + \frac{\omega_{\rm c}}{\omega}}. \tag{5.134}$$

The wave described by (5.133) carries the name *R-wave*, while the other one described by (5.134) is named the *L-wave*. Their names have the origin in their circular polarisation, which we will study in a moment.

- The dispersion diagrams in two forms are given in Figs. 5.9 and 5.10.
- Both waves are dispersive

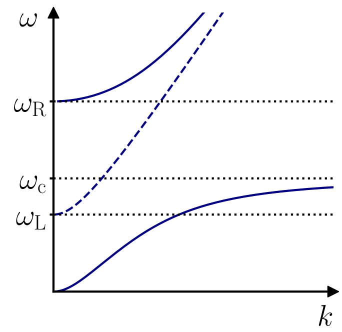

Figure 5.10: The dispersion $(k-\omega)$ diagram of L- (dashed) and R- (solid) waves.

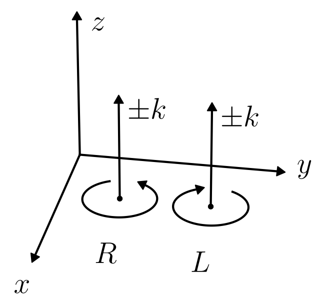

Figure 5.11: The coordinate system to demonstrate the polarisation and propagation of the R- and L-modes.

- We keep the derivation of phase and group speed to the reader, however already from the dispersion relation we see that at the same frequency, the propagation speed of the R-wave is larger.

- Let's investigate the polarisation of these two waves. We start from (5.127) with $\alpha = 0$. We have

$$\begin{pmatrix} \omega^2 - \omega_\mathrm{p}^2 - c^2k^2 & \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2) & 0 \\ -\mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2) & \omega^2 - \omega_\mathrm{p}^2 - c^2k^2 & 0 \\ 0 & 0 & \omega^2 - \omega_\mathrm{p}^2 \end{pmatrix} \cdot \begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \end{pmatrix} = 0, \tag{5.135}$$

from which we immediately see that $E_{1z} = 0$. The relation between the two other components is

$$(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2)E_{1x} + \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2)E_{1y} = 0. \tag{5.136}$$

We further use the dispersion relation in a form $c^2k^2 = \omega^2 - \omega_\mathrm{p}^2\left(1 \mp \frac{\omega_\mathrm{c}}{\omega}\right)^{-1}$ to finally obtain

$$E_{1y} = \pm\mathrm{i}E_{1x}. \tag{5.137}$$

Thus we have $E_x = -\mathrm{i}E_y$ for the R-wave (the upper sign is valid) and $E_x = \mathrm{i}E_y$ for the L-wave. We immediately see that the waves are circularly polarised (the amplitudes of both components are equal and both components are shifted in phase by quarter of circle). We explicitly express the spatio-temporal dependence of the electric intensity,

$$\boldsymbol{E}_1 = \overline{\boldsymbol{E}_1}\exp\left[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\right] = \overline{\boldsymbol{E}_1}\left[\cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right) + \mathrm{i}\sin\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right)\right]. \tag{5.138}$$

If we introduce a phase shift δ_E between the components and define

$$E_x = \Re(\boldsymbol{E}_1) = \overline{\boldsymbol{E}_1}\cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right) \tag{5.139}$$

and

$$E_y = \Im(\boldsymbol{E}_1) = \overline{\boldsymbol{E}_1}\sin\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta_E\right), \tag{5.140}$$

we see that for the R-wave, $\delta_E = \pi/2$ and thus E_y preceeds E_x, the wave's polarisation is right-handed (see Fig. 5.11). Similarly, in the case of the L-wave, E_y is retarded by $\delta_E = -\pi/2$ and the wave's polarisation is left-handed. Note that the polarisation *does not* depend on the propagation direction, only on the direction of the background magnetic field.

- The R-wave has a cut-off at $\omega_R = \frac{1}{2}\left[\omega_\mathrm{c} + (\omega_\mathrm{c}^2 + 4\omega_\mathrm{p}^2)^{1/2}\right]$ (hence the designation of this frequency) and a resonance at ω_c. The L(-)wave has a cut-off at $\omega_L = \frac{1}{2}\left[-\omega_\mathrm{c} + (\omega_\mathrm{c}^2 + 4\omega_\mathrm{p}^2)^{1/2}\right]$ and no resonance.

Application: Farraday rotation. The detection of R- and L-waves in the radio spectrum of astrophysical objects allows to estimate either the distance of these objects

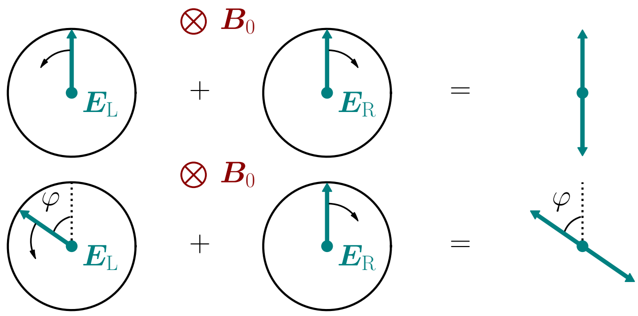

Figure 5.12: The superposition of L- and R- circularly polarised waves with the same amplitude yields the wave with a linear polarisation. The tilt of the polarisation plane is a consequence of the Faraday rotation effect.

or the plasma frequency of the interstellar environment. Assume that we have a plane wave composed of both R- and L-waves, which propagates along interstellar $\boldsymbol{B}_0$. As we pointed out, the R-wave propagates faster.

Let us write the vector of electric fields of both considered R- and L-wave in component with respect to some Cartesian coordinate system

$$\boldsymbol{E}_\mathrm{R} = E_0 \exp[\mathrm{i}(\boldsymbol{k}_\mathrm{R} \cdot \boldsymbol{r} - \omega t)] \left[\boldsymbol{e}_x + \mathrm{i}\boldsymbol{e}_y\right], \tag{5.141}$$

$$\boldsymbol{E}_\mathrm{L} = E_0 \exp[\mathrm{i}(\boldsymbol{k}_\mathrm{L} \cdot \boldsymbol{r} - \omega t)] \left[\boldsymbol{e}_x - \mathrm{i}\boldsymbol{e}_y\right]. \tag{5.142}$$

The components are written in agreement with (5.137) and $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are unity vectors in x and y directions.

The two waves superpose to form one detected wave (Fig. 5.12):

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{E}_\mathrm{R} + \boldsymbol{E}_\mathrm{L}) = \frac{1}{2} E_0 \mathrm{e}^{-\mathrm{i}\omega t} \left[\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{R}\cdot\boldsymbol{r}}(\boldsymbol{e}_x + \mathrm{i}\boldsymbol{e}_y) + \mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{L}\cdot\boldsymbol{r}}(\boldsymbol{e}_x - \mathrm{i}\boldsymbol{e}_y)\right]. \tag{5.143}$$

Should $\boldsymbol{k}_\mathrm{R} = \boldsymbol{k}_\mathrm{L} = \boldsymbol{k}$ then

$$\boldsymbol{E} = \frac{1}{2} E_0 \mathrm{e}^{-\mathrm{i}\omega t} \left[2\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{e}_x\right] = E_0 \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\boldsymbol{e}_x, \tag{5.144}$$

which is the linearly polarised wave in the x-direction.

In our case, however, $\boldsymbol{k}_\mathrm{R} \neq \boldsymbol{k}_\mathrm{L}$,

$$k_\mathrm{R,L} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_\mathrm{p}^2/\omega^2}{1 \mp \omega_\mathrm{c}/\omega}}. \tag{5.145}$$

In the high-frequency regime, for $\omega \gg \omega_\mathrm{c}, \omega_\mathrm{p}$ by using Taylor series $\sqrt{1-x} \sim 1 - \frac{x}{2}$ and $(1+x)^{-1} \sim 1 - x$ we obtain

$$k_\mathrm{R,L} \sim \frac{\omega}{c}\left(1 - \frac{1}{2}\frac{\omega_\mathrm{p}^2}{\omega^2}\frac{1}{1 \mp \omega_\mathrm{c}/\omega}\right) \sim \frac{\omega}{c}\left[1 - \frac{1}{2}\frac{\omega_\mathrm{p}^2}{\omega^2}\left(1 \pm \frac{\omega_\mathrm{c}}{\omega}\right)\right] \tag{5.146}$$

Hence when we use a decomposition of both wavenumbers as

$$k_\mathrm{R,L} = k \pm \Delta k, \tag{5.147}$$

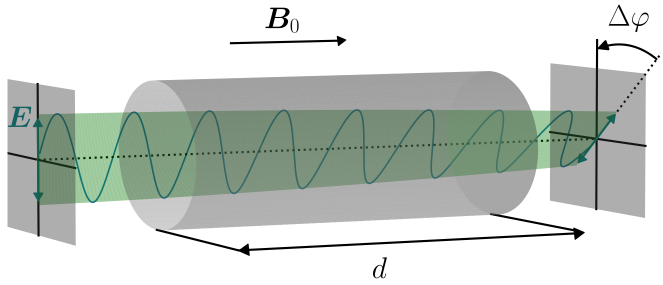

Figure 5.13: The sketch illustrating the Farraday's rotation. The combination of the R- and L-modes creates a linearly polarised wave in the plane before entering the plasma cylinder. In the plasma, the L-mode lags behind the R-mode, which causes the rotation of the polarisation axis. In the plane after leaving the plasma cylinder the linear polarisation depict the phase shift $\Delta\varphi$.

we have

$$k = \frac{\omega}{c}\left[1 - \frac{1}{2}\frac{\omega_{\rm p}^2}{\omega^2}\right] \tag{5.148}$$

and

$$\Delta k = -\frac{1}{2}\frac{\omega_{\rm p}^2\omega_{\rm c}}{c\omega^2}. \tag{5.149}$$

By inserting (5.147) into (5.143) we have

$$\begin{aligned} \boldsymbol{E} &= \frac{1}{2}E_0 {\rm e}^{{\rm i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\left[{\rm e}^{{\rm i}\Delta\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{e}_x + {\rm i}\boldsymbol{e}_y) + {\rm e}^{-{\rm i}\Delta\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{e}_x - {\rm i}\boldsymbol{e}_y)\right] \\ &= E_0 {\rm e}^{{\rm i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\left[\cos(-\Delta\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{e}_x + \sin(-\Delta\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{e}_y\right]. \end{aligned} \tag{5.150}$$

Obviously, the difference in the phase speeds of the R- and L-wave causes the rotation of the polarisation plane of the resulting linearly polarised wave. When adjusting the coordinate system so that $-\Delta\boldsymbol{k}\cdot\boldsymbol{r} = -\Delta k z$ the angle of the polarisation φ (it equals to the argument of the goniometric functions) plane fulfills

$$\frac{{\rm d}\varphi}{{\rm d}z} = -\Delta k. \tag{5.151}$$

To obtain the total rotation of the polarisation plane for the waves travelling the distance d (see Fig. 5.13) we must integrate (5.151):

$$\begin{aligned} \varphi &= \varphi_0 + \int_0^d \frac{{\rm d}\varphi}{{\rm d}z}{\rm d}z = \varphi_0 + \int_0^d -\Delta k {\rm d}z = \varphi_0 + \frac{1}{2c\omega^2}\int_0^d \omega_{\rm p}^2(z)\omega_{\rm c}(z){\rm d}z = \\ &= \varphi_0 + \frac{e^3}{2m^2\varepsilon_0 c\omega^2}\int_0^d n_0(z)B_0(z){\rm d}z. \end{aligned} \tag{5.152}$$

When we approximate both the density and magnetic field by constants, we simply obtain

$$\varphi = \varphi_0 + \frac{e^3}{2m^2\varepsilon_0 c}\frac{1}{\omega^2}n_0 B_0 d. \tag{5.153}$$

By measuring the angle of the polarisation plane for various frequencies one may fit this relation and obtain the information about either the properties of the interstellar space or about the distance of the radiating object.

Electromagnetic waves perpendicular to $\boldsymbol{B}_0$

Let us go back to the general dispersion relation of the complex of electromagnetic waves in plasmas (5.129) and discuss the other special case when $\alpha = \pi/2$. The dispersion relation simplifies to

$$(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2)\left[(\omega^2 - \omega_\mathrm{p}^2)(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2) - \omega_\mathrm{c}^2(\omega^2 - c^2k^2)\right] = 0. \tag{5.154}$$

This equation again has two solutions. By zeroing the first parenthesis we obtain

$$\omega^2 = \omega_\mathrm{p}^2 + c^2k^2, \tag{5.155}$$

which is the dispersion relation identical to the electromagnetic waves in non-magnetised plasmas. Due to this correspondence, this wave is called *ordinary* or the *O-mode*. It is linearly polarized with $\boldsymbol{E}_1 \parallel \boldsymbol{B}_0$. It has a cut-off at the plasma frequency.

By zeroing the second parenthesis we have

$$(\omega^2 - \omega_\mathrm{p}^2)(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2) = \omega_\mathrm{c}^2(\omega^2 - c^2k^2), \tag{5.156}$$

which by expressing $\omega^2 - c^2k^2$ from both sides and rearranging becomes

$$\omega^2 - c^2k^2 = \omega_\mathrm{p}^2\frac{\omega^2 - \omega_\mathrm{p}^2}{\omega^2 - \omega_\mathrm{h}^2}, \tag{5.157}$$

where we used the definition of the upper hybrid frequency $\omega_\mathrm{h}^2 = \omega_\mathrm{p}^2 + \omega_\mathrm{c}^2$. Such wave is termed the *extraordinary wave* or the *X-mode*. It is partly transversal and partly longitudinal, it propagates perpendicularly to $\boldsymbol{B}_0$. As the density is increased, the phase velocity rises from c until the cut-off at ω_R is reached. As the density is further increased, the wave is evanescent until the resonance at the upper hybrid frequency ω_h. Then it can propagate again until the second cut-off at ω_L. The cut-off frequencies are given by: $\omega_R = \frac{1}{2}\left[\omega_\mathrm{c} + (\omega_\mathrm{c}^2 + 4\omega_\mathrm{p}^2)^{1/2}\right]$ and $\omega_L = \frac{1}{2}\left[-\omega_\mathrm{c} + (\omega_\mathrm{c}^2 + 4\omega_\mathrm{p}^2)^{1/2}\right]$.

For the O- and X-modes the plasma acts as a birefringent environment. The dispersion relations in two forms are given in Figs. 5.14, 5.15, and 5.16.

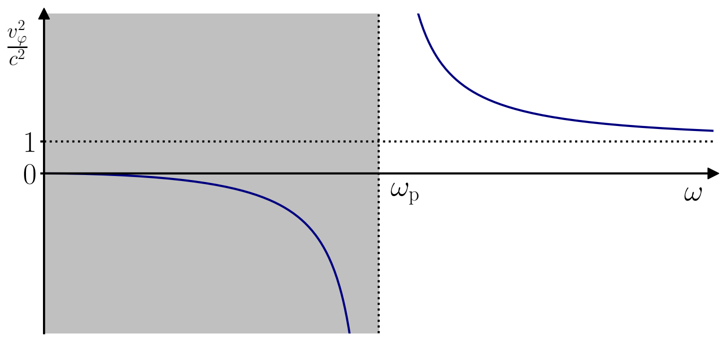

Figure 5.14: Dispersion relation of the O-wave. The shaded region indicates the region of attenuation (square of the phase velocity is negative).

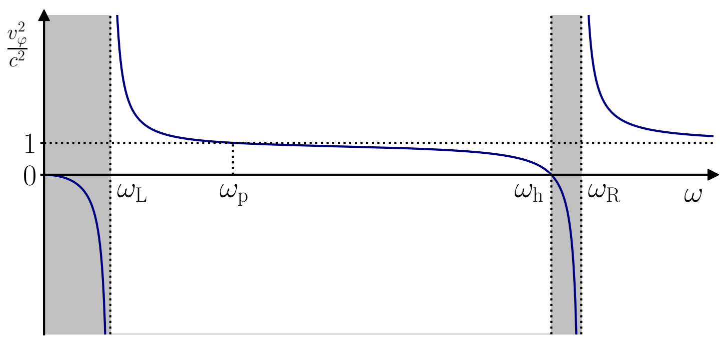

Figure 5.15: (X waves)

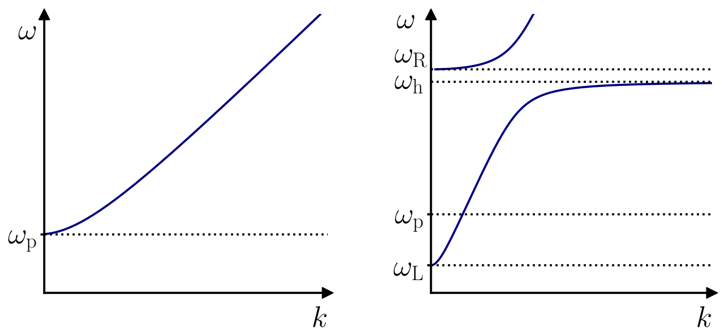

Figure 5.16: The dispersion $k - \omega$ diagram for the O-waves (left) and X-waves (right)

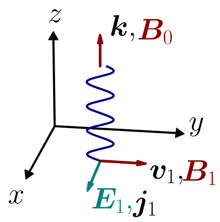

Figure 5.17: Geometry of an Alfvén wave propagating along $\boldsymbol{B}_0$.

5.7 MHD waves

By MHD waves (a.k.a. hydromagnetic waves) we understand low frequency ion oscillations in the presence of the magnetic field. A special case is the Alfvén wave.

5.7.1 Alfvén wave

Alfvén waves

- Background magnetic field, no background electric field, i.e., $\boldsymbol{B}_0 \neq 0$, $\boldsymbol{E}_0 = 0$
- Background stationary and homogeneous
- We solve equations of motion for both ions and electrons
- Both ions and electrons are cold (no thermal motion)

We introduce the Cartesian coordinate system $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$, so that:

- $\boldsymbol{B}_0 = B_0 \boldsymbol{e}_z$,
- $\boldsymbol{E}_1 \parallel \boldsymbol{j}_1 \parallel \boldsymbol{e}_x$, $\boldsymbol{E}_1 \perp \boldsymbol{B}_0$,
- $\boldsymbol{B}_1 \parallel \boldsymbol{u}_1 \parallel \boldsymbol{e}_y$,
- $\boldsymbol{k} \parallel \boldsymbol{B}_0$, $\boldsymbol{B}_1 \perp \boldsymbol{E}_1$.

This system and the wave setup is also indicated graphically in Fig. 5.17.

Again, we use the solution of the system of Maxwell equations (5.108) and (5.109) from the previous section by directly using (5.112):

$$(\omega^2 - c^2 k^2) \boldsymbol{E}_1 = -\frac{\mathrm{i}\omega}{\varepsilon_0} \boldsymbol{j}_1. \tag{5.158}$$

We do not assume that the ions create a stationary background, thus the current density $\boldsymbol{j}_1$ is proportional to the difference between fluctuation speed of ions and electrons. Hence

$$\boldsymbol{j}_1 = j_1 \boldsymbol{e}_x = n_0 e (u_{\mathrm{i}1,x} - u_{\mathrm{e}1,x}) \boldsymbol{e}_x. \tag{5.159}$$

Expression for both velocity fluctuations must be obtained from the Euler equations. For ions we have

$$M \frac{\partial \boldsymbol{u}_{\mathrm{i}1}}{\partial t} = e(\boldsymbol{E}_1 + \boldsymbol{u}_{\mathrm{i}1} \times \boldsymbol{B}_0), \tag{5.160}$$

which in components and in the Fourier space gives

$$-\mathrm{i}\omega M u_{\mathrm{i}1,x} = eE_1 + eu_{\mathrm{i}1,y} B_0, \tag{5.161}$$

$$-\mathrm{i}\omega M u_{\mathrm{i}1,y} = -eu_{\mathrm{i}1,x} B_0. \tag{5.162}$$

By combining these two equations we have

$$-\mathrm{i}\omega M u_{\mathrm{i}1,x} = eE_1 + eB_0 \frac{eB_0}{\mathrm{i}\omega M} u_{\mathrm{i}1,x} \tag{5.163}$$

and after some algebra we obtain

$$u_{\mathrm{i}1,x} = \frac{\mathrm{i}e}{\omega M} \left(1 - \frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1} E_1. \tag{5.164}$$

Similarly [e.g. by inserting (5.164) into (5.162)], we obtain for $u_{\mathrm{i}y}$:

$$u_{\mathrm{i}1,y} = \frac{e}{\omega M} \frac{\Omega_\mathrm{c}}{\omega} \left(1 - \frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1} E_1. \tag{5.165}$$

We solve the Euler's equation for electrons by analogy (formally by application of the following substitutions: $M \to m$, $e \to -e$, and $\Omega_\mathrm{c} \to -\omega_\mathrm{c}$), to obtain:

$$u_{\mathrm{e}1,x} = -\frac{\mathrm{i}e}{\omega m} \left(1 - \frac{\omega_\mathrm{c}^2}{\omega^2}\right)^{-1} E_1, \tag{5.166}$$

$$u_{\mathrm{e}1,y} = -\frac{e}{\omega m} \frac{-\omega_\mathrm{c}}{\omega} \left(1 - \frac{\omega_\mathrm{c}^2}{\omega^2}\right)^{-1} E_1. \tag{5.167}$$

We expect low-frequency waves. Thus we are allowed to use the approximation $\omega_\mathrm{c} \gg \omega$. Under this approximation (5.166) and (5.167) becomes:

$$u_{\mathrm{e}1,x} = -\frac{\mathrm{i}e}{\omega m} \left(\frac{\omega^2 - \omega_\mathrm{c}^2}{\omega^2}\right)^{-1} E_1 \sim \frac{\mathrm{i}e}{\omega m} \frac{\omega^2}{\omega_\mathrm{c}^2} E_1 \to 0 \tag{5.168}$$

and

$$u_{\mathrm{e}1,y} = \frac{e}{\omega m} \frac{\omega_\mathrm{c}}{\omega} \left(\frac{\omega^2 - \omega_\mathrm{c}^2}{\omega^2}\right)^{-1} E_1 \sim -\frac{e}{\omega m} \frac{\omega_\mathrm{c}}{\omega} \frac{\omega^2}{\omega_\mathrm{c}^2} E_1 = -\frac{e}{\omega_\mathrm{c} m} E_1 = -\frac{E_1}{B_0}. \tag{5.169}$$

Now we use the estimates obtained for the fluctuating speeds (5.164) and (5.168) and use them in (5.159), which we use in (5.158).

$$(\omega^2 - c^2k^2)E_1 = -\frac{\mathrm{i}\omega}{\varepsilon_0}n_0 e\left[\frac{\mathrm{i}e}{\omega M}\left(1-\frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1}E_1 - 0\right] =$$
$$= \frac{n_0 e^2}{\varepsilon_0 M}\left(1-\frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1}E_1 = \Omega_\mathrm{p}^2\left(1-\frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1}E_1. \tag{5.170}$$

Hence the dispersion relation reads

$$\omega^2 - c^2k^2 = \Omega_\mathrm{p}^2\left(1-\frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1}. \tag{5.171}$$

If we limit ourselves to the case, when $\omega \ll \Omega_\mathrm{c}$, which is a very strong assumption, we basically say that we are interested only in the very low-frequency waves, the dispersion relation simplifies to

$$\omega^2 - c^2k^2 = \Omega_\mathrm{p}^2\left(\frac{\omega^2-\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1} \sim -\omega^2\frac{\Omega_\mathrm{p}^2}{\Omega_\mathrm{c}^2} = -\omega^2\frac{n_0 M}{\varepsilon_0 B_0^2} = -\omega^2\frac{\rho_0}{\varepsilon_0 B_0^2}. \tag{5.172}$$

Hence

$$\frac{\omega^2}{k^2} = \frac{c^2}{1+\frac{\rho_0}{\varepsilon_0 B_0^2}}. \tag{5.173}$$

In plasma, the expression $1+\frac{\rho_0}{\varepsilon_0 B_0^2}$ has a meaning of the square-root of relative permitivity ε_r, which in plasmas is usually much larger than unity. Hence $\frac{\rho_0}{\varepsilon_0 B_0^2} \gg 1$ and we have

$$\frac{\omega^2}{k^2} = \frac{c^2}{\frac{\rho\mu_0 c^2}{B_0^2}} = \frac{B_0^2}{\mu_0\rho_0}. \tag{5.174}$$

Thus the *dispersion relation for Alvén waves* reads

$$\frac{\omega}{k} = \frac{B_0}{\sqrt{\mu_0\rho_0}} \equiv c_\mathrm{A}. \tag{5.175}$$

c_A is the *Alfvén speed.*
Notes:

- The relations among the oscillating quantities are displayed in Fig. 5.18.
- Magnetic component of the wave, B_y, is in the y direction and looks like sinusoid stringing the field lines of the background magnetic field
- We have a drift $v_y = E_1\boldsymbol{e}_x \times \boldsymbol{B}_0/B_0^2$, which is the same for both ions and electrons

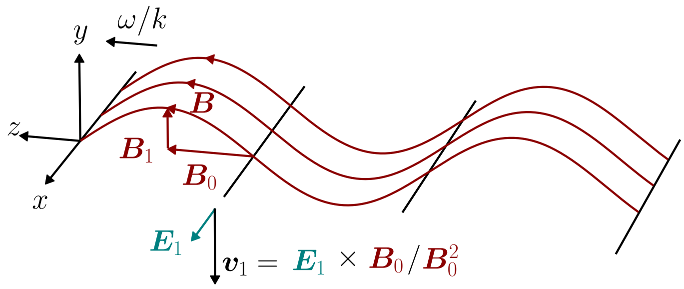

Figure 5.18: Relations among the oscillating quantities in an Alfvén wave and the distortion of the lines of force.

- The phase speed of the perturbations to the field lines equals to the speed of both ions and electrons. To prove it, let's take the Fourier image of the induction equation without the dissipative term $\omega B_1 = k v_{y,B} B_0$ and the Farraday's law $kE_1 = -\omega B_1$. Combining the two we obtain $v_{y,B} = -\frac{\omega}{k}\frac{k}{\omega}\frac{E_1}{B_0} = -\frac{E_1}{B_0}$, which is the expression for the speed of particles. Therefore, both the fluid and the perturbed magnetic field oscillate together. The plasma is frozen in the magnetic field.

- The results we obtained are valid only if there is no background electric field parallel to the magnetic.

- Alfvén speed is a characteristic speed of propagation of perturbations to the magnetic field. This may yield important estimates. For example, sunspots are believed to be caused by the flux tube, which was perturbed at the base of the solar convection zone and rises up to the photosphere (see the schematic illustration in Fig. 5.19). The characteristic time of rise may be roughly estimated when using the Alfvén speed as the estimate for the rising speed. For the Sun, assuming the depth of the convection zone to be 200 Mm, the average bulk density in the convection zone 200 $\mathrm{kg\,m^{-3}}$, and the initial field strength in the perturbed flux tube to be 10 kG=1 T. Then we have $\tau \sim \frac{d}{c_\mathrm{A}} \sim \frac{200\times10^6}{\frac{1}{\sqrt{4\pi\times10^{-7}\cdot200}}}$ s $\sim$ 36 days. This estimate is not far from the more appropriate modelling, which includes a detailed balance of the rising flux tube and the ambient solar plasma. The characteristic rising time is larger than the solar rotation period, thus one should expect the deflection of the rising flux tube due to the Coriolis force.

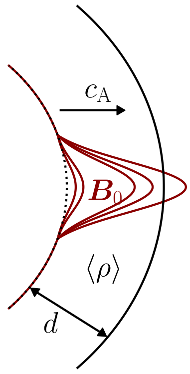

Figure 5.19: Illustration of the emerging magnetic field from the bottom of the convective zone of Sun-like stars. A series of snapshots of the perturbed flux tube rising towards solar surface is indicated in the figure.

5.7.2 General MHD waves

As the final case, let us study the general case of magnetohydrodynamic waves. Hence, we will solve the linearised version of the following system of equations:

$$\frac{\partial \rho}{\partial t} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{5.176}$$

$$\rho \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\nabla p + (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}/\mu_0, \tag{5.177}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho^\gamma}\right) = 0, \tag{5.178}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{5.179}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{5.180}$$

Note that:

- We assume a homogeneous and stationary background and no background motion ($\boldsymbol{u}_0 = 0$).
- We collapse the set of Maxwell equations into the induction equation with no dissipative terms, to the Gauss's law for magnetism and explicitly use Ampére's law in the expression for the Lorentz force; the frozen-field approximation is implied.
- We assume the adiabatic approximation to the equation of state.
- We assume that the flow is non-convecting, that is the terms where the operator $(\boldsymbol{u} \cdot \boldsymbol{\nabla})$ is applied are neglected.

Except for the equation of state, the linearisation is trivial and can be done from the top of anyone's head:

$$\frac{\partial \rho_1}{\partial t} + \rho_0(\boldsymbol{\nabla}\cdot\boldsymbol{u}_1) = 0, \tag{5.181}$$

$$\rho_0\frac{\partial \boldsymbol{u}_1}{\partial t} = -\nabla p_1 + (\boldsymbol{\nabla}\times\boldsymbol{B}_1)\times\boldsymbol{B}_0/\mu_0, \tag{5.182}$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}_1\times\boldsymbol{B}_0), \tag{5.183}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}_1 = 0. \tag{5.184}$$

The equation of the state needs to be done more carefully. Let us do a slow derivation, in each step we immediately neglect the second and higher-order terms:

$$0 = \left\{\frac{\partial}{\partial t} + \boldsymbol{u}_1\cdot\boldsymbol{\nabla}\right\}\left[\frac{p_0+p_1}{(\rho_0+\rho_1)^\gamma}\right] =$$
$$= \frac{1}{(\rho_0+\rho_1)^\gamma}\left[\frac{\partial p_1}{\partial t} + (\boldsymbol{u}_1\cdot\boldsymbol{\nabla})p_0\right] + (p_0+p_1)\left\{\frac{-\gamma}{(\rho_0+\rho_1)^{\gamma+1}}\left[\frac{\partial \rho_1}{\partial t} + (\boldsymbol{u}_1\cdot\boldsymbol{\nabla})\rho_0\right]\right\}. \tag{5.185}$$

Now we apply the assumption of the homogeneous background, that is we remove the terms where derivatives of the background quantities appear.

$$0 = \frac{1}{(\rho_0+\rho_1)^\gamma}\left[\frac{\partial p_1}{\partial t} - \gamma\frac{p_0+p_1}{\rho_0+\rho_1}\frac{\partial \rho_1}{\partial t}\right] = \frac{1}{(\rho_0+\rho_1)^\gamma}\left\{\frac{\partial p_1}{\partial t} - c_\mathrm{s}^2\frac{\partial \rho_1}{\partial t}\right\}, \tag{5.186}$$

where we used the definition of the speed of sound $c_\mathrm{s}^2 = \gamma p_0/\rho_0$ and we evaluated term $\frac{p_0+p_1}{\rho_0+\rho_1}$ already in Section 5.2. Since the term $\frac{1}{(\rho_0+\rho_1)^\gamma}$ is some non-zero number, the expression in curly parentheses must be zero, which is our linearised equation of state (by further taking into account that the background is homogeneous):

$$\frac{\partial p_1}{\partial t} - c_\mathrm{s}^2\frac{\partial \rho_1}{\partial t} = 0. \tag{5.187}$$

Thus we combine (5.181), (5.182), (5.183), (5.184), and (5.187) to obtain the dispersion relation. We start from taking a time derivative of (5.182):

$$\frac{\partial^2 \boldsymbol{u}_1}{\partial t^2} = -\frac{1}{\rho_0}\nabla\frac{\partial p_1}{\partial t} + (\boldsymbol{\nabla}\times\frac{\partial \boldsymbol{B}_1}{\partial t})\times\frac{\boldsymbol{B}_0}{\rho_0\mu_0}. \tag{5.188}$$

Now we use (5.187) and have

$$\frac{\partial^2 \boldsymbol{u}_1}{\partial t^2} = -\frac{1}{\rho_0}\nabla\left(c_\mathrm{s}^2\frac{\partial \rho_1}{\partial t}\right) + (\boldsymbol{\nabla}\times\frac{\partial \boldsymbol{B}_1}{\partial t})\times\frac{\boldsymbol{B}_0}{\rho_0\mu_0} \tag{5.189}$$

and further by using (5.181) and (5.183) we have

$$\frac{\partial^2 \boldsymbol{u}_1}{\partial t^2} = c_\mathrm{s}^2\nabla\left(\nabla\cdot\boldsymbol{u}_1\right) + \{\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times(\boldsymbol{u}_1\times\boldsymbol{B}_0)]\}\times\frac{\boldsymbol{B}_0}{\rho_0\mu_0}. \tag{5.190}$$

The equation is being solved in the Fourier space again:

$$\omega^2 \boldsymbol{u}_1 = c_\mathrm{s}^2 \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{u}_1) + \{\boldsymbol{k}\times[\boldsymbol{k}\times(\boldsymbol{u}_1\times\boldsymbol{B}_0)]\}\times\frac{\boldsymbol{B}_0}{\mu_0\rho_0}$$
$$= c_\mathrm{s}^2 \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{u}_1) + \{\boldsymbol{k}\times[\boldsymbol{k}\times(\boldsymbol{u}_1\times\boldsymbol{b}_0)]\}\times\boldsymbol{b}_0 c_\mathrm{A}^2, \tag{5.191}$$

where we used $\boldsymbol{b}_0 = \boldsymbol{B}_0/|\boldsymbol{B}_0|$ and $c_\mathrm{A}^2 = B_0^2/(\mu_0\rho_0)$ is the Alfvén speed. The complicated multiplication of the right-hand side is

$$\{\boldsymbol{k}\times[\boldsymbol{k}\times(\boldsymbol{u}_1\times\boldsymbol{b}_0)]\}\times\boldsymbol{b}_0 = \{\boldsymbol{k}\times[\boldsymbol{u}_1(\boldsymbol{k}\cdot\boldsymbol{b}_0) - \boldsymbol{b}_0(\boldsymbol{k}\cdot\boldsymbol{u}_1)]\}\times\boldsymbol{b}_0 =$$
$$= (\boldsymbol{k}\cdot\boldsymbol{b}_0)\left[(\boldsymbol{k}\times\boldsymbol{u}_1)\times\boldsymbol{b}_0\right] - (\boldsymbol{k}\cdot\boldsymbol{u}_1)\left[(\boldsymbol{k}\times\boldsymbol{b}_0)\times\boldsymbol{b}_0\right] =$$
$$= -(\boldsymbol{k}\cdot\boldsymbol{b}_0)\left[\boldsymbol{k}(\boldsymbol{u}_1\cdot\boldsymbol{b}_0) - \boldsymbol{u}_1(\boldsymbol{k}\cdot\boldsymbol{b}_0)\right] + (\boldsymbol{k}\cdot\boldsymbol{u}_1)\left[\boldsymbol{k}b_0^2 - \boldsymbol{b}_0(\boldsymbol{k}\cdot\boldsymbol{b}_0)\right] =$$
$$= (\boldsymbol{k}\cdot\boldsymbol{b}_0)^2\boldsymbol{u}_1 - (\boldsymbol{k}\cdot\boldsymbol{u}_1)(\boldsymbol{k}\cdot\boldsymbol{b}_0)\boldsymbol{b}_0 + \boldsymbol{k}\left[\boldsymbol{k}\cdot\boldsymbol{u}_1 - (\boldsymbol{k}\cdot\boldsymbol{b}_0)(\boldsymbol{u}_1\cdot\boldsymbol{b}_0)\right]. \tag{5.192}$$

Now we use $\boldsymbol{k}\cdot\boldsymbol{b}_0 = k\cos\alpha$, where α is the field inclination. We continue evaluating (5.191):

$$\omega^2\boldsymbol{u}_1 = c_\mathrm{s}^2\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{u}_1) + c_\mathrm{A}^2\left\{(\boldsymbol{k}\cdot\boldsymbol{b}_0)^2\boldsymbol{u}_1 - (\boldsymbol{k}\cdot\boldsymbol{u}_1)(\boldsymbol{k}\cdot\boldsymbol{b}_0)\boldsymbol{b}_0 + \boldsymbol{k}\left[\boldsymbol{k}\cdot\boldsymbol{u}_1 - (\boldsymbol{k}\cdot\boldsymbol{b}_0)(\boldsymbol{u}_1\cdot\boldsymbol{b}_0)\right]\right\}, \tag{5.193}$$

thus

$$\frac{\omega^2\boldsymbol{u}_1}{c_\mathrm{A}^2} = k^2\cos^2\alpha\boldsymbol{u}_1 - (\boldsymbol{k}\cdot\boldsymbol{u}_1)k\cos\alpha\boldsymbol{b}_0 + \boldsymbol{k}\left[\left(1+\frac{c_\mathrm{s}^2}{c_\mathrm{A}^2}\right)(\boldsymbol{k}\cdot\boldsymbol{u}_1) - k\cos\alpha(\boldsymbol{u}_1\cdot\boldsymbol{b}_0)\right]. \tag{5.194}$$

We take the projections of (5.194) first into the direction of $\boldsymbol{k}$

$$\omega^2(\boldsymbol{k}\cdot\boldsymbol{u}_1) = k^2c_\mathrm{A}^2\cos^2\alpha(\boldsymbol{u}_1\cdot\boldsymbol{k}) - c_\mathrm{A}^2k^2\cos^2\alpha(\boldsymbol{k}\cdot\boldsymbol{u}_1) +$$
$$+ c_\mathrm{A}^2(\boldsymbol{k}\cdot\boldsymbol{u}_1)k^2 + c_\mathrm{s}^2(\boldsymbol{k}\cdot\boldsymbol{u}_1)k^2 - c_\mathrm{A}^2k^3\cos\alpha(\boldsymbol{u}_1\cdot\boldsymbol{b}_0). \tag{5.195}$$

Hence

$$\left[-\omega^2 + c_\mathrm{A}^2k^2 + c_\mathrm{s}^2k^2\right](\boldsymbol{u}_1\cdot\boldsymbol{k}) = c_\mathrm{A}^2k^3\cos\alpha(\boldsymbol{b}_0\cdot\boldsymbol{u}_1). \tag{5.196}$$

Then we project (5.194) into the direction of $\boldsymbol{b}_0$.

$$\omega^2(\boldsymbol{b}_0\cdot\boldsymbol{u}_1) = c_\mathrm{A}^2k^2\cos^2\alpha(\boldsymbol{u}_1\cdot\boldsymbol{b}_0) - c_\mathrm{A}^2(\boldsymbol{k}\cdot\boldsymbol{u}_1)k\cos\alpha +$$
$$+ c_\mathrm{A}^2(\boldsymbol{k}\cdot\boldsymbol{u}_1)k\cos\alpha + c_\mathrm{s}^2(\boldsymbol{k}\cdot\boldsymbol{u}_1)k\cos\alpha - c_\mathrm{A}^2k^2\cos^2\alpha(\boldsymbol{u}_1\cdot\boldsymbol{b}_0), \tag{5.197}$$

hence

$$\omega^2(\boldsymbol{b}_0\cdot\boldsymbol{u}_1) = c_\mathrm{s}^2(\boldsymbol{k}\cdot\boldsymbol{u}_1)k\cos\alpha. \tag{5.198}$$

Equation (5.196) may further be played with:

$$\frac{\boldsymbol{b}_0\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{k}} = \frac{-\omega^2 + c_\mathrm{A}^2k^2 + c_\mathrm{s}^2k^2}{c_\mathrm{A}^2k^3\cos\alpha} \tag{5.199}$$

and similarly (5.198) becomes

$$\frac{\boldsymbol{b}_0 \cdot \boldsymbol{u}_1}{\boldsymbol{u}_1 \cdot \boldsymbol{k}} = \frac{c_\mathrm{s}^2 k \cos\alpha}{\omega^2}. \tag{5.200}$$

These two equations equal, thus we have

$$\frac{-\omega^2 + c_\mathrm{A}^2 k^2 + c_\mathrm{s}^2 k^2}{c_\mathrm{A}^2 k^3 \cos\alpha} = \frac{c_\mathrm{s}^2 k \cos\alpha}{\omega^2} \tag{5.201}$$

and therefore

$$\omega^4 - \omega^2 k^2 (c_\mathrm{A}^2 + c_\mathrm{s}^2) + c_\mathrm{s}^2 c_\mathrm{A}^2 k^4 \cos^2\alpha = 0. \tag{5.202}$$

By solving the quadratic equation in $(\omega/k)^2$ we have a solution:

$$\left(\frac{\omega}{k}\right)^2 = \left[\frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}\sqrt{c_\mathrm{s}^4 + c_\mathrm{A}^4 - 4c_\mathrm{s}^2 c_\mathrm{A}^2 \cos^2\alpha + 2c_\mathrm{A}^2 c_\mathrm{s}^2}\right]. \tag{5.203}$$

This is the general dispersion relation for the *magnetoacoustic wave*. The one with the plus sign is called a *fast wave*, while the one with the minus sign is called the *slow wave*. Obviously, the speed of propagation depends on the direction of propagation with respect to the vector of the background magnetic field.

For waves travelling *along* the magnetic field $\alpha = 0$ we have:

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}\sqrt{c_\mathrm{s}^4 + c_\mathrm{A}^4 - 2c_\mathrm{A}^2 c_\mathrm{s}^2} = \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}|c_\mathrm{s}^2 - c_\mathrm{A}^2| = \left\{ \begin{array}{c} c_\mathrm{s}^2 \\ c_\mathrm{A}^2 \end{array} \right. , \tag{5.204}$$

for $c_\mathrm{s} > c_\mathrm{A}$ and oppositely for $c_\mathrm{A} > c_\mathrm{s}$ (note that we still keep the notation that the upper solution represents the fast wave, whereas the lower represents the slow-wave mode). To summarise the two cases in a general solution, we may write that the *fast mode* has a dispersion relation $\omega/k = \max(c_\mathrm{s}, c_\mathrm{A})$, whereas the *slow mode* disperses according to $\omega/k = \min(c_\mathrm{s}, c_\mathrm{A})$. Note that as a special case of waves propagating along the field, we obtained the dispersion relation for the Alfvén wave. Also the *slow wave*, which is purely acoustic, propagates. For waves travelling across the field for $\alpha = \pi/2$ we have

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}\sqrt{c_\mathrm{s}^4 + c_\mathrm{A}^4 + 2c_\mathrm{A}^2 c_\mathrm{s}^2} = \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) = \left\{ \begin{array}{c} c_\mathrm{s}^2 + c_\mathrm{A}^2 \\ 0 \end{array} \right. , \tag{5.205}$$

therefore, only the *fast wave* propagates. In a special case of no background field, i.e. $\boldsymbol{B}_0 = 0$ we simply have $c_\mathrm{A} = 0$ and thus

$$\left(\frac{\omega}{k}\right)^2 = \left\{ \begin{array}{c} 0 \\ c_\mathrm{s}^2 \end{array} \right. , \tag{5.206}$$

and the wave just degenerates to the ordinary acoustic wave.

5.7.3 Mode conversion

In the stratified medium, we should supply additional terms in our equations (namely the gravitation term) and drop the requirement of the homogeneous background. In some cases, we may find surfaces, where the sound speed equals the Alfvén speed, the *equipartition surface* (prove that at this surface, the plasma β equals one). Here all the dispersion relations collapse and the waves are resonant. The use of the simple ray approximation (which we were using all the time) is not appropriate in these regions, finite-wavelength effects play an important role. Using the higher-order theory one may find[5] that a *mode conversion* may occur. Essential in the equipartition layers the waves convert types, purely acoustic waves may become magnetoacoustic fast waves etc. The efficiency of the conversion depends on an incident angle to the magnetic field.

The mode conversion is an effect which essentially prohibits reliable helioseismic tomography of the Sun beneath sunspots. The sunspots are regions of the strong field, thus, there definitely is the equipartition layer below the surface of the Sun. The (purely acoustic) seismic waves excited by the vigorous surface convection propagate through the convection zone (in the stratified medium, thus they do not propagate straight but rather bend as the local sound speed varies), go through the equipartition layer and partly convert to the magnetoacoustic modes. They undergo one more conversion after total reflection from the surface. As a consequence, the total power of the acoustic waves, which is measured and analysed by helioseismology, is lower in the regions of the strong field, leading to unknown large systematic errors in analyses.

Similarly, the acoustic waves in the solar atmosphere are totally reflected at the temperature minimum region (lower chromosphere, $T \sim 4200$ K), where the acoustic cut-off frequency is around 5.2 mHz (the cut-off frequency is a function of the local plasma state parameters, $\omega_{\text{cut-off}} = \frac{\gamma g}{2c_s}$, where g is the local gravity acceleration). However, in the magnetised regions, the magnetoacoustic waves are allowed to propagate along the magnetic field. Thus when the conversion takes place, the originally acoustic waves may leak to the higher atmosphere as converted magnetoacoustic and Alfvén waves. Numerical simulations revealed that the inclined magnetic field acts as a factor lowering the cut-off frequency, which is then modified to $\omega_{\text{cut-off}} = \frac{\gamma g \cos\alpha}{2c_s}$, where α is the magnetic field inclination to the local vertical. In the magnetised regions with inclined magnetic field the sound waves excited by the solar convection may carry some non-negligible acoustic flux to the chromosphere and thus contributing to its heating[6].

[5] e.g. Cally, P. 2006: *Dispersion relations, rays and ray splitting in magnetohelioseismology*, Royal Society of London Transactions Series A 364, p.333–349

[6] e.g. Sobotka, M., Švanda, M., Jurčák, J. et al. 2013: *Dynamics of the solar atmosphere above a pore with a light bridge*, Astronomy&Astrophysics, in press, `ArXiv:1309.7790S`

Chapter 6

Diffusion

6.1 Diffusion in weakly ionised plasmas

In Section 4.2 we introduced an additional term to the Euler equation accounting for collisions between particles. We defined the particle flux Γ

$$\mathbf{\Gamma} = n\boldsymbol{u}, \tag{6.1}$$

where n was the particle density and $\boldsymbol{u}$ their speed.

The formalism used here is similar. For simplicity, let us assume that the particles flow along the x-axis of the Cartesian coordinate system. By collisions occurring along with the spatial interval $\mathrm{d}x$, the fraction of the flux $\mathrm{d}\Gamma$ is lost (see Fig. 6.1). We may write

$$\mathrm{d}\Gamma = -n_\mathrm{n}\sigma\Gamma\mathrm{d}x, \tag{6.2}$$

where n_n is the density of neutral particles because we expect the collisions to be dominated by neutral particles. σ is the effective cross-section of the collisions. From the effective cross-section, we may define useful quantities $\lambda_s = 1/n_\mathrm{n}\sigma$ to be the mean free path between the consecutive collisions and similarly the mean collisional frequency

$$\nu = n_\mathrm{n}\langle\sigma v\rangle \approx n_\mathrm{n}\sigma\boldsymbol{u}, \tag{6.3}$$

where the averaging is performed over the velocity space. The averaging is performed because the quantum mechanics teaches us that the effective cross-section may be a function of velocity. The mean period between the consecutive collisions is then $\tau = 1/\nu$.

By taking the time derivative of (6.1) and (6.2) when for simplicity neglecting the possible time dependence of other variables but the coordinates and velocity and using the definition of Γ we have

$$\frac{\mathrm{d}\mathbf{\Gamma}}{\mathrm{d}t} = n\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -n_\mathrm{n}\sigma n\boldsymbol{u}\frac{\mathrm{d}x}{\mathrm{d}t} = -n_\mathrm{n}\sigma nu\boldsymbol{u} = -n\nu\boldsymbol{u}, \tag{6.4}$$

Hence

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\nu\boldsymbol{u}, \tag{6.5}$$

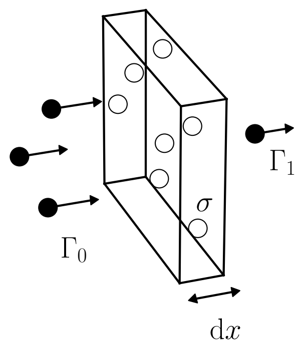

Figure 6.1: Illustration to the model of the change of the particle flux via collisions.

which is the (average) velocity change due to the collisions.

Hence the Euler equation with magnetic field set to zero and with collisional term added reads

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right]=qn\boldsymbol{E}-\nabla p-mn\nu\boldsymbol{u} \tag{6.6}$$

Let's investigate the system where $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}=0$ (in such system we find only a constant motion due to the diffusion). Then

$$\boldsymbol{u}=\frac{q\boldsymbol{E}}{m\nu}-\frac{K_{\mathrm{B}}T\nabla n}{mn\nu}=\frac{\boldsymbol{\Gamma}}{n}, \tag{6.7}$$

where we used the definition of the particle flux. By introducing mobility $\mu=\frac{q\,\mathrm{sgn}\,q}{m\nu}$ and diffusion coefficient $D=\frac{K_{\mathrm{B}}T}{m\nu}$ we have for the particle species j

$$\boldsymbol{\Gamma}_j=\mathrm{sgn}\,q_j\,n\mu_j\boldsymbol{E}-D_j\nabla n. \tag{6.8}$$

For $\boldsymbol{E}=0$ we have

$$\boldsymbol{\Gamma}=-D\nabla n, \tag{6.9}$$

which is the form of *Fick's law* for diffusion. The simple interpretation of this useful relation is that the particles flow from locations with a higher density to lower density regions (as illustrated in Fig. 6.2), which is to be expected. In plasmas, the Fick's law is generalised to (6.8) noting the effect of the electric field. One should also note that the coefficients μ_j and D_j may be generally different for various particle species (most often electrons and ions) and hence the diffusion in plasmas may cause the separation of charges and thus drive the electric field. However, to fulfill the quasineutrality of plasma, $\boldsymbol{\Gamma}_{\mathrm{i}}=\boldsymbol{\Gamma}_{\mathrm{e}}=\boldsymbol{\Gamma}$ on scales $L\gg\lambda_{\mathrm{D}}$, which allows us to derive the necessary electric field which will keep the plasma quasineutral.

By assuming the plasma approximation $n_{\mathrm{e}}=n_{\mathrm{i}}=n$ we have

$$n\mu_{\mathrm{i}}\boldsymbol{E}-D_{\mathrm{i}}\nabla n=-n\mu_{\mathrm{e}}\boldsymbol{E}-D_{\mathrm{e}}\nabla n \tag{6.10}$$

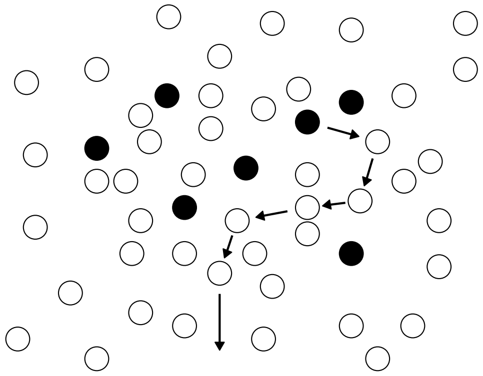

Figure 6.2: Illustrative concept of the diffusion. An overabundance of the black "particles" is decreased in time via collisions with white "particles" and their local density thus decreases.

and hence

$$\boldsymbol{E} = \frac{D_\mathrm{i} - D_\mathrm{e}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \frac{\nabla n}{n} \tag{6.11}$$

is the induced electric field triggered by the separation of charges. This is the repulsive electric field, which stops the charges from further separation due to the diffusion and hence keeps plasma neutral on large scales. This electric field causes the electrons to slow down and accelerate ions simultaneously, so that the bulk of plasma (with charges separated on scales smaller than λ_D) diffuses according to modified Fick's law. This is the basic picture of *ambipolar diffusion*.

Then

$$\begin{aligned} \boldsymbol{\Gamma} = \boldsymbol{\Gamma}_\mathrm{i} &= n\mu_\mathrm{i} \frac{D_\mathrm{i} - D_\mathrm{e}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \frac{\nabla n}{n} - D_\mathrm{i} \nabla n = \\ &= \frac{\mu_\mathrm{i} D_\mathrm{i} - \mu_\mathrm{i} D_\mathrm{e} - \mu_\mathrm{i} D_\mathrm{i} - \mu_\mathrm{e} D_\mathrm{i}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \nabla n = - \frac{\mu_\mathrm{i} D_\mathrm{e} + \mu_\mathrm{e} D_\mathrm{i}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \nabla n = \\ &\equiv -D_\mathrm{a} \nabla n, \end{aligned} \tag{6.12}$$

where D_a is the coefficient of the ambipolar diffusion. For $\mu_\mathrm{e} \gg \mu_\mathrm{i}$ and using the definition of mobility coefficient we have

$$D_\mathrm{a} = \frac{\mu_\mathrm{i} D_\mathrm{e} + \mu_\mathrm{e} D_\mathrm{i}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \approx D_\mathrm{i} + \frac{\mu_\mathrm{i}}{\mu_\mathrm{e}} D_\mathrm{e} = \left(1 + \frac{T_\mathrm{e}}{T_\mathrm{i}} \right) D_\mathrm{i}. \tag{6.13}$$

Thus the coefficient of the ambipolar diffusion is mostly determined by the diffusion coefficients for ions with some correction.

Typical time evolution by diffusion

As an example, let's take the continuum equation

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot n\boldsymbol{u} = 0, \tag{6.14}$$

and hence

$$\frac{\partial n}{\partial t} = -\boldsymbol{\nabla} \cdot n\boldsymbol{u} = -\boldsymbol{\nabla} \cdot \boldsymbol{\Gamma} = D_\mathrm{a} \triangle n. \tag{6.15}$$

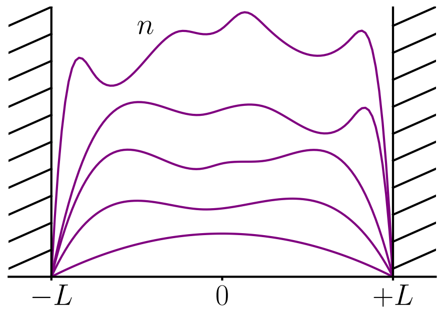

Figure 6.3: A model of plasma diffusion confined between two infinite plane-parallel walls. A series of plasma density snapshots is plotted by the purple lines. At the beginning, the density profile is quite complex. Due to the diffusion, it simplifies with time, large spatial frequencies (small-scale features) decay the fastest.

This equation is similar to the heat conduction equation. It may be solved by assuming the 1-D solution separating the variables $n(x,t) = T(t)S(x)$, where T and S are some functions. Then

$$S\frac{\mathrm{d}T}{\mathrm{d}t} = D_\mathrm{a} T \triangle S, \tag{6.16}$$

and hence

$$\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{D_\mathrm{a}}{S}\triangle S = \mathrm{const} = -\frac{1}{\tau}. \tag{6.17}$$

Both sides of the equations must equal the same constant because the left-hand side depends purely on time; thus, the right-hand side can't depend on time and vice versa for the spatial dependency of the right-hand side. From the dimension reasons, this constant must have the dimension of the reciprocal time. The equation may then be solved by parts

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{T}{\tau} \quad \rightarrow \quad T = T_0 \mathrm{e}^{-\frac{t}{\tau}} \tag{6.18}$$

and

$$\frac{\mathrm{d}^2 S}{\mathrm{d}x^2} = -\frac{1}{D_\mathrm{a}\tau} S, \tag{6.19}$$

which has the solution in the series of harmonic functions,

$$S = S_0 \cos\frac{x}{\sqrt{D_\mathrm{a}\tau}} \quad \text{and} \quad S = S_0 \sin\frac{x}{\sqrt{D_\mathrm{a}\tau}}. \tag{6.20}$$

This solution must be supplied with the boundary conditions. Let's assume a simple model of plasma confined between two infinite plane-parallel walls (see Fig. 6.3), where the density of the plasma vanishes outside interval $x \in (-L, L)$. Then the solution to the problem is

$$n = n_0 \left\{ \sum_{l=0}^{\infty} a_{\mathrm{e},l} \mathrm{e}^{-\frac{t}{\tau_{\mathrm{e},l}}} \cos\left[\frac{(l+\frac{1}{2})\pi x}{L}\right] + \sum_{m=1}^{\infty} a_{\mathrm{o},m} \mathrm{e}^{-\frac{t}{\tau_{\mathrm{o},m}}} \sin\left[\frac{m\pi x}{L}\right] \right\} \tag{6.21}$$

The coefficients $\tau_{\mathrm{e},l}$ and $\tau_{\mathrm{o},m}$ may be determined from inserting the solution above into (6.16) and comparing the terms one by one. For the first term ($l = 0$) in the cosine series we have

$$-a_{\mathrm{e},0}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{e},0}}}\frac{1}{\tau_{\mathrm{e},0}}\cos\frac{\pi x}{2L} = -D_{\mathrm{a}}a_{\mathrm{e},0}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{e},0}}}\left(\frac{\pi}{2L}\right)^2\cos\frac{\pi x}{2L} \tag{6.22}$$

and hence

$$\tau_{\mathrm{e},0} = \frac{4L^2}{\pi^2 D_{\mathrm{a}}}. \tag{6.23}$$

For a general mode l, we would similarly find

$$\tau_{\mathrm{e},l} = \frac{L^2}{(l+\frac{1}{2})^2\pi^2 D_{\mathrm{a}}}. \tag{6.24}$$

For the first term ($m = 1$) in the sine series, we have

$$-a_{\mathrm{o},1}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{o},1}}}\frac{1}{\tau_{\mathrm{o},1}}\sin\frac{\pi x}{L} = -D_{\mathrm{a}}a_{\mathrm{o},1}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{o},1}}}\left(\frac{\pi}{L}\right)^2\sin\frac{\pi x}{L} \tag{6.25}$$

and hence

$$\tau_{\mathrm{o},1} = \frac{L^2}{\pi^2 D_{\mathrm{a}}}. \tag{6.26}$$

For the other modes m we similarly obtain

$$\tau_{\mathrm{o},m} = \frac{L^2}{m^2\pi^2 D_{\mathrm{a}}}. \tag{6.27}$$

The lifetime of various modes $\tau_{\mathrm{e},l}$ and $\tau_{\mathrm{o},m}$ shows that the higher modes are attenuated faster. Therefore, if there is a wild density profile at the beginning (in time $t = 0$), then the profile is being simplified with time. A set of snapshots of the evolution of an arbitrary density profile is given in Fig. 6.3.

Other physics-motivated examples for diffusion

The question is whether it is possible to achieve the stationarity in the presence of diffusion. It is not if the continuity equation holds. It is possible in the presence of the source term,

$$\boldsymbol{\nabla}\cdot n\boldsymbol{u} = Q. \tag{6.28}$$

In case of the collisional ionisation $Q = Zn$, where Z is the ionisation function, we have

$$\boldsymbol{\nabla}\cdot n\boldsymbol{u} = -D_{\mathrm{a}}\,\triangle n = Zn \quad\rightarrow\quad \triangle n = -\frac{Z}{D_{\mathrm{a}}}n, \tag{6.29}$$

solution to which is again the series of harmonic functions.

The recombination, on the other hand, takes a different form because it does change the number of charged particles. The number density of the ionised particles naturally decays with n^2 in recombination, hence the continuity equation may be written in a form

$$\frac{\partial n}{\partial t} - D_{\mathrm{a}}\nabla n = -\alpha n^2. \tag{6.30}$$

Similarly to the previous cases, we consider the solution separated to the temporal and spatial parts $n = TS$. Further, we will only be interested in the temporal part, which is equivalent to modification of (6.30) to

$$\frac{\partial n}{\partial t} = -\alpha n^2, \tag{6.31}$$

where n^2 dependence is the consequence of the recombination being proportional to both the density of electrons and ions and the use of plasma approximation. Then

$$\frac{1}{n^2}\mathrm{d}n = -\alpha \mathrm{d}t \quad \rightarrow \quad -\frac{1}{n} = -\frac{1}{n_0} - \alpha t. \tag{6.32}$$

For $t \to \infty$ we have $n \sim \frac{1}{\alpha t}$, which says that due to the recombination in plasmas, the density of the charged particles drops as $1/t$. The spatial part of the continuity equation is not relevant to the problem; thus, we neglect it from the solution.

This solution is important for understanding which process drives the diffusion of charged particles in a left-alone cloud of plasma in time. Obviously, in the initial moment, the recombination will be the dominant process, as its time dependence is steeper for small t. Contrary, when the density of the plasma gets smaller, the diffusion with exponecial time dependence prevails and further drives the diffusion.

6.2 Diffusion in highly ionised plasmas

We will show that in highly ionised plasmas, the diffusion resembles the process of recombination rather than ambipolar diffusion. Let's take the fluid approximation to deal with the problem.

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p \tag{6.33}$$

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j} \tag{6.34}$$

Note that generally, the specific resistivity of plasma η is not a scalar value but rather a tensor and thus may depend on direction. In plasma, it usually is the case. Therefore, we split the specific resistivity to the parallel and perpendicular components ($\eta = \eta_{\parallel} + \eta_{\perp}$), where in the parallel direction the Ohm's law simply is

$$\boldsymbol{E}_{\parallel} = \eta_{\parallel}\boldsymbol{j}_{\parallel}. \tag{6.35}$$

. To obtain a similar relation in the direction perpendicular to the magnetic field, when assuming the stationarity, by vector-multiplying (6.34) by $\boldsymbol{B}$ and using (6.33) we have

$$\boldsymbol{E} \times \boldsymbol{B} + (\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B} = \boldsymbol{E} \times \boldsymbol{B} - B^2\boldsymbol{u} + B^2\boldsymbol{u}_{\parallel} = \eta_{\perp}\nabla p. \tag{6.36}$$

Since $\boldsymbol{u} = \boldsymbol{u}_{\parallel} + \boldsymbol{u}_{\perp}$, then the two last terms on the left-hand side give together $B^2\boldsymbol{u}_{\perp}$ and thus

$$\boldsymbol{u}_{\perp} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} - \frac{\eta_{\perp}\nabla p}{B^2}. \tag{6.37}$$

The first term on the right-hand side is the same for both (electron and ion) fluids. The second one, on the other hand, may drive the diffusion. The perpendicular particle flux on top of the E-B drift is

$$\boldsymbol{\Gamma}_\perp = n\boldsymbol{u}'_\perp = -\frac{\eta_\perp n(K_\mathrm{B}T_\mathrm{e} + K_\mathrm{B}T_\mathrm{i})}{B^2}\nabla n = -D_\perp \nabla n\,. \tag{6.38}$$

Note that the diffusion coefficient $D_\perp$ is a function of particle density and may be for simpler manipulation rewritten as

$$D_\perp = \frac{\eta_\perp n K_\mathrm{B} \sum\limits_\alpha T_\alpha}{B^2} = 2An. \tag{6.39}$$

The factor of two is introduced so that $\boldsymbol{\nabla}\cdot(2An\nabla n) = 2A\boldsymbol{\nabla}\cdot(n\nabla n) = A\,\triangle n^2$.

The continuity equation

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\boldsymbol{u}) = 0 \tag{6.40}$$

then transforms to

$$\frac{\partial n}{\partial t} = A\,\triangle n^2. \tag{6.41}$$

Such equation may be solved by separation of variables, hence $n(\boldsymbol{r},t) = T(t)S(\boldsymbol{r})$. Then $S\frac{\mathrm{d}T}{\mathrm{d}t} = AT^2\,\triangle S^2$ and

$$\frac{1}{T^2}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{A}{S}\triangle S^2 = \mathrm{const} = -\frac{1}{\tau^2} \tag{6.42}$$

and in the case we solved previously. Then the temporal part has a solution

$$\frac{1}{T} = \frac{t}{\tau^2} + \frac{1}{T_0} \tag{6.43}$$

and the form of τ depends on the spatial part S. For $t \to \infty$

$$T \propto \frac{\tau^2}{t}, \tag{6.44}$$

which reminds the solution of the change of the particle density caused by recombinations. The difference between the ambipolar diffusion and the result we just obtained is in the diffusion coefficient depending on particle density. Note that the result we obtained is not consistent with the assumptions of the derivation of the Fick's law, as we allow for non-stationarity here.

6.3 Diffusion in the magnetic field

In presence of the magnetic field, the Euler equation retains an additional term. Let us see what happens in the presence of the magnetic field. We must go back to the double-fluid approximation since collisions are essential in the process of diffusion.

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] = \operatorname{sgn} q\, en(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - K_\mathrm{B}T\nabla n - mn\nu\boldsymbol{u} \tag{6.45}$$

Let's assume that the collisions are much faster than the overall evolution of the system in time, or in other words $\frac{\mathrm{d}}{\mathrm{d}t} \ll \omega_\mathrm{c}, \nu$, where ν is the collision frequency with neutrals and $\tau \equiv 1/\nu$ is the corresponding time between collisions. Then the left-hand side may be neglected. Thus in components in the Cartesian system with z axis oriented along the magnetic field ($\boldsymbol{B} = B\boldsymbol{e}_z$) we have

$$mn\nu u_x = \operatorname{sgn} q\, enE_x - K_\mathrm{B}T\frac{\partial n}{\partial x} + \operatorname{sgn} q\, enu_yB, \tag{6.46}$$

$$mn\nu u_y = \operatorname{sgn} q\, enE_y - K_\mathrm{B}T\frac{\partial n}{\partial y} - \operatorname{sgn} q\, enu_xB. \tag{6.47}$$

Along the magnetic field, a classic diffusion as derived above occurs.

By introducing $\mu = \frac{e}{m\nu}$, $D = \frac{K_\mathrm{B}T}{m\nu} = \frac{\omega_\mathrm{c}}{\nu}\frac{K_\mathrm{B}T}{eB}$ we have

$$u_x = \operatorname{sgn} q\, \mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \operatorname{sgn} q\, \frac{\omega_\mathrm{c}}{\nu}u_y, \tag{6.48}$$

$$u_y = \operatorname{sgn} q\, \mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \operatorname{sgn} q\, \frac{\omega_\mathrm{c}}{\nu}u_x, \tag{6.49}$$

and (using $\tau \equiv 1/\nu$)

$$u_x(1+\omega_\mathrm{c}^2\tau^2) = \operatorname{sgn} q\, \mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \omega_\mathrm{c}^2\tau^2\frac{E_y}{B} - \operatorname{sgn} q\, \omega_\mathrm{c}^2\tau^2\frac{K_\mathrm{B}T}{neB}\frac{\partial n}{\partial y}, \tag{6.50}$$

$$u_y(1+\omega_\mathrm{c}^2\tau^2) = \operatorname{sgn} q\, \mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \omega_\mathrm{c}^2\tau^2\frac{E_x}{B} + \operatorname{sgn} q\, \omega_\mathrm{c}^2\tau^2\frac{K_\mathrm{B}T}{neB}\frac{\partial n}{\partial x}. \tag{6.51}$$

The last two terms on the right-hand side obviously indicate the already known E-B and diamagnetic drifts. Hence

$$\boldsymbol{u}_\perp = \frac{\boldsymbol{v}_E + \boldsymbol{v}_\mathrm{D}}{1 + 1/(\omega_\mathrm{c}^2\tau^2)} + \operatorname{sgn} q \frac{\mu}{1+\omega_\mathrm{c}^2\tau^2}\boldsymbol{E} - \frac{D}{1+\omega_\mathrm{c}^2\tau^2}\frac{\nabla n}{n}, \tag{6.52}$$

where $\boldsymbol{v}_E = \boldsymbol{E}\times\boldsymbol{B}/B^2$ and $\boldsymbol{v}_\mathrm{D} = -\nabla p \times \boldsymbol{B}/(qnB^2)$. Let us further identify

$$\mu_\perp \equiv \frac{\mu}{1+\omega_\mathrm{c}^2\tau^2} \quad \text{and} \quad D_\perp \equiv \frac{D}{1+\omega_\mathrm{c}^2\tau^2}. \tag{6.53}$$

Hence the generalised Fick's law transforms to

$$\boldsymbol{u}_\perp = \operatorname{sgn} q\, \mu_\perp \boldsymbol{E} - D_\perp\frac{\nabla n}{n} + \frac{\boldsymbol{v}_E + \boldsymbol{v}_\mathrm{D}}{1 + 1/(\omega_\mathrm{c}^2\tau^2)}. \tag{6.54}$$

The last term represents the already familiar drifts in the plasma fluid, only with the correction term for the diffusion. Obviously, the correcting term is important for $\nu > \omega_\mathrm{c}$. Hence in the weak field regime, drifts are suppressed by collisions.

The correctinon term for the mobility and the diffusion coeficient act in an opposite way. In the strong field, i.e., when $\nu < \omega_\mathrm{c}$, diffusion is suppressed. In particular, we have

$$D_\perp \propto \frac{K_\mathrm{B}T\nu}{m\omega_\mathrm{c}^2} \quad \text{and} \quad D_\parallel = \frac{K_\mathrm{B}T}{m\nu}, \tag{6.55}$$

where $D_\parallel$ is the non-modified diffusion coefficient which occurs in the component of the Euler equation parallel to the magnetic field. Hence the decay caused by diffusion in plasmas has different rates in different directions. In case the collisional frequency is proportional[1] to $m^{-1/2}$ then the diffusion depends also on the species of the particles. Then

$$D_\perp \propto m^{1/2} \quad \text{and} \quad D_\parallel \propto m^{-1/2}. \tag{6.56}$$

In the perpendicular direction, the diffusion is faster for ions than electrons, while in the parallel direction the electrons diffuse faster. It has to be noted that generally speaking, the collisions are important for the particles to be able to penetrate across the magnetic field.

[1] As we discussed at the beginning of this chapter, $\nu = n_\mathrm{n}\langle\sigma v\rangle$. Leaving out a possible dependence of the effective cross-section on velocity, we have $\nu \propto \langle v\rangle \propto \sqrt{\langle v^2\rangle}$. Since $T \propto m\langle v^2\rangle$, for the fixed temperature T we have $\langle v^2\rangle \propto m^{-1}$ and finally $\nu \propto m^{-1/2}$.

Chapter 7

Equilibrium and stability

7.1 Equilibrium

The state of equilibrium indicates the forces acting on plasma are in balance. In the investigation of the possible equilibrium of plasma, we will use the fluid approximation. However, it should be noted that this approximation might be too rough. In plasmas, the drifting motions are usually the sources of instabilities driving the plasma from equilibrium, and the drifting motions are largely not captured by the fluid approximation.

In investigating the stability, we assume that the forces are in balance, hence the solution is stationary. Hence the necessary condition of equilibrium is $\frac{\partial \bullet}{\partial t} \to 0$.

We recognize a *stable equilibrium*, where small perturbations deviating the system from an exact balance are attenuated, and an *unstable equilibrium*, when even a small perturbation of the system grows and brings the system out of equilibrium. The equilibrium generally is a non-linear problem (see Fig. 7.1). We will limit ourselves to discuss the linear problems only by considering the small perturbations. To describe the equilibrium, the MHD description suffices, however it is usually not sufficient to discuss the stability of such equilibrium. Let us remind that the *mechanical stability* equals to the work that is needed to bring the object from the stable equilibrium position to another (investigated) equilibrium position. Should this work be negative, the position at which it is located is not stable.

Let us describe a general concept of investigation of equilibrium on an example of *hydromagnetic equilibrium.* Let's consider a plasma cylinder, where the magnetic field aligns along the cylinder axis (see Fig. 7.2). The simplified equation of motion reads

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p. \tag{7.1}$$

We search for a stationary state, hence we set $\frac{\partial \boldsymbol{u}}{\partial t} = 0$ and the Euler equation degenerates to

$$\nabla p = \boldsymbol{j} \times \boldsymbol{B} \tag{7.2}$$

It is clear now that there is an obvious balance between the Lorentz force which supports the pressure gradient. Such balance may be achieved when an azimuthal current flows

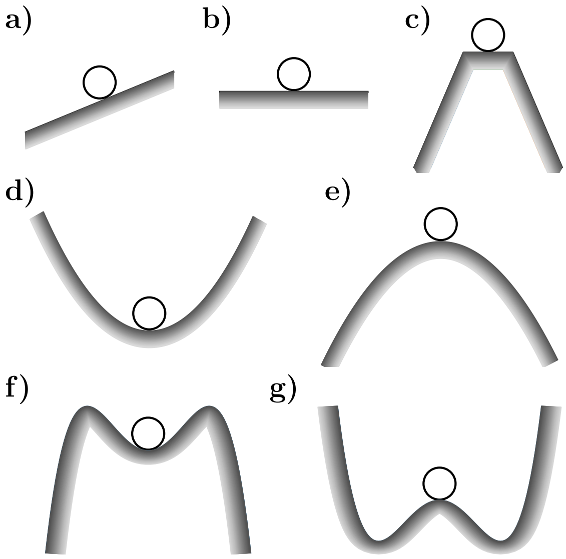

Figure 7.1: Illustrations of various kinds of equilibria. a) inequilibrium, b) neutrally stable, c) metastable, d) stable, e) unstable, f) linearly stable, non-linearly unstable, g) linearly unstable, non-linearly stable.

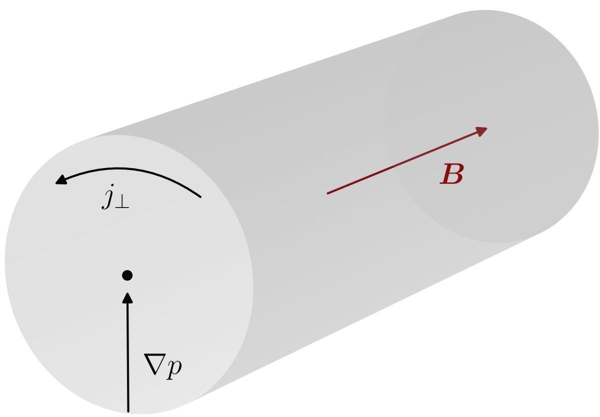

Figure 7.2: Illustration of hydromagnetic equilibrium.

through plasma cylinder. By taking the vector multiplication by $\boldsymbol{B}$ we have

$$\boldsymbol{B} \times \nabla p = \boldsymbol{B} \times (\boldsymbol{j} \times \boldsymbol{B}) = B^2 \boldsymbol{j} - \boldsymbol{B} \cdot \boldsymbol{j} \boldsymbol{B} = B^2 (\boldsymbol{j} - \boldsymbol{j}_{\parallel}). \tag{7.3}$$

By assuming that $\boldsymbol{j} = \boldsymbol{j}_{\parallel} + \boldsymbol{j}_{\perp}$ we obtain the expression for a required azimuthal current

$$\boldsymbol{j}_{\perp} = \frac{\boldsymbol{B} \times \nabla p}{B^2} = (K_{\mathrm{B}} T_{\mathrm{i}} + K_{\mathrm{B}} T_{\mathrm{e}}) \frac{\boldsymbol{B} \times \nabla n}{B^2}, \tag{7.4}$$

which is an expression for a diamagnetic current. Such a situation may be viewed as if a plasma cylinder is to be in equilibrium, there must be a current induced, which takes a form of a diamagnetic current. One should bear in mind that this expression was derived when neglecting all other terms in the Euler equation.

One should also note that both $\boldsymbol{j}$ and $\boldsymbol{B}$ are perpendicular to ∇p, hence both $\boldsymbol{j}$ and $\boldsymbol{B}$ lie in the isosurface of constant pressure. The field lines and the current lines may be curved in this surface.

7.2 Plasma β

By considering Ampére's law

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} \tag{7.5}$$

for the equilibrium situation, we have

$$\begin{aligned} \nabla p = \boldsymbol{j} \times \boldsymbol{B} &= \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = \\ &= \frac{1}{\mu_0} \left[-(\nabla \boldsymbol{B}) \cdot \boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B} \right] = \frac{1}{\mu_0} \left[(\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B} - \frac{1}{2} \nabla B^2 \right] \end{aligned} \tag{7.6}$$

By rearranging we have

$$\nabla \left(p + \frac{1}{\mu_0} B^2 / 2 \right) = \frac{1}{\mu_0} (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B}. \tag{7.7}$$

The right-hand side indicates the action of forces along field lines, the *magnetic tension*. If we assume that there are negligible changes of the magnetic field along the field lines, the right-hand side vanishes. Then

$$p + \frac{B^2}{2\mu_0} = \mathrm{const.} \tag{7.8}$$

The second term has a physical meaning of the *magnetic field pressure*. By comparing the two terms we define the *plasma beta*,

$$\beta \equiv \frac{p}{p_{\mathrm{mag}}} = \frac{n \sum_{\alpha} K_{\mathrm{B}} T_{\alpha}}{\frac{B^2}{2\mu_0}}, \tag{7.9}$$

where α identificates the particles by type.

For $\beta > 1$ the gas pressure prevails. This high-beta regime is common to the interior of stars and most of the laboratory experiments. For $\beta < 1$ the magnetic field pressure prevails. This low-beta regime is common to solar/stellar atmosphere(s) and also to interstellar matter.

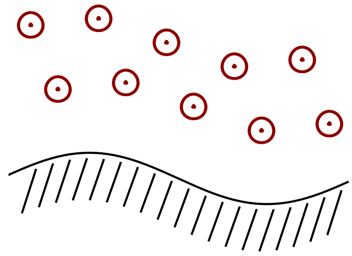

Figure 7.3: Plasma and magnetic field boundary and definition of plasma β,

7.2.1 Diffusion of the magnetic field into plasmas

Let us discuss the system of two limited half-spaces, one of them filled with plasmas only, and the other one containing the magnetic field in a vacuum (Fig. 7.3). In case there is no resistivity in plasmas, the two regions will remain disconnected. The electromotive force induced by any possible movement of the field would drive an infinite electric current on the boundary separating these two regions, but the field would never penetrate the plasma. However, the shape of the boundary may change significantly. If a pressure balance (7.8) is to be held, then it must be that if we somehow *force* the magnetic field to penetrate the plasma region, the gas must leave such a region.

Since (7.8) holds, automatically $\beta = 1$ on the boundary. For the "plasma half-space" the second term vanishes and vice versa for the "magnetic field half-space". However, by considering the finite resistivity of the plasma we practically allow the two distinct "fluids" to mix. In the following, we want to estimate the time scale of this process. Let us start from a Faraday's law of induction, Ampére's law, and a generalised Ohm's law:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{7.10}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} \tag{7.11}$$

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j} \tag{7.12}$$

Let us assume that the plasma is stationary, hence $\boldsymbol{u} = 0$ and thus the second term on the left-hand side of the Ohm's law vanishes. Note that even when we assume no bulk motion of the plasma, there still might flow electric currents through plasma, as we have two distinct charged species in the plasma. Remind that the velocity of the bulk motion is defined as

$$\boldsymbol{u} = \frac{1}{\rho}(n_{\mathrm{i}} M \boldsymbol{u}_{\mathrm{i}} + n_{\mathrm{e}} m \boldsymbol{u}_{\mathrm{e}}) \sim \frac{M\boldsymbol{u}_{\mathrm{i}} + m\boldsymbol{u}_{\mathrm{e}}}{M + m}, \tag{7.13}$$

hence as a weighted mean of the velocities of the two species, while the current is defined as a *difference* of the velocities of the two species:

$$\boldsymbol{j} = e(n_{\mathrm{i}} \boldsymbol{u}_{\mathrm{i}} - n_{\mathrm{e}} \boldsymbol{u}_{\mathrm{e}}) \sim ne(\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}). \tag{7.14}$$

By combining equations (7.10), (7.11), and (7.12) we have

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \eta \boldsymbol{j} = -\frac{\eta}{\mu_0} \left[\nabla(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \triangle \boldsymbol{B}\right] = \frac{\eta}{\mu_0} \triangle \boldsymbol{B}, \tag{7.15}$$

where we used $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. The above equation is a type of "heat-conduction equation" and may be understood using the following model: the equation describes the temporal evolution of the magnetic field when it is distorted from the equilibrium. We seek the solution and study whether it is possible to recover the equilibrium and at what costs.

Such an equation may be solved using the separation of variables. For simplicity, we approximate the ∇ operator by division over the *length-scale* L, which may be understood as a penetration depth of the magnetic field. Then we solve the equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \frac{\eta}{\mu_0 L^2} \boldsymbol{B}, \tag{7.16}$$

which has a solution in a form of exponential

$$\boldsymbol{B} = \boldsymbol{B}_0 \exp\left[\pm \frac{t}{\tau}\right], \tag{7.17}$$

from which we take only the negative solution as physical since we solve for a dissipation of the magnetic field into plasma.

$$\tau = \frac{\mu_0 L^2}{\eta} \tag{7.18}$$

is a characteristic time-scale of the penetration of the magnetic field into plasma.

The solution might be also understood as the magnetic field dissipating due to the induction of the electric currents in plasma, which dissipate via the Ohmic heating. Let us estimate how much of the energy of the magnetic field dissipates due to the Ohmic heating processes.

The loss output of the Ohmic heating is given by

$$P = \eta j^2. \tag{7.19}$$

Hence the energy drained from the magnetic field over the time τ is

$$\Delta W = \eta j^2 \tau. \tag{7.20}$$

The electric current is given by the Ampére's law (7.11). Hence the energy loss (again, when approximating the ∇ operator) we have

$$\Delta W = \eta j^2 \tau = \eta \left(\frac{B}{\mu_0 L}\right)^2 \frac{\mu_0 L^2}{\eta} = \frac{B^2}{\mu_0}. \tag{7.21}$$

It turned out that all the total energy of the magnetic field dissipates and heats up the plasma. Actually it is by factor two larger, which is due to our rough estimates.

7.3 Instabilities

The systems out of the thermodynamical equilibrium contain some of the free energy, which may drive the rise of random fluctuations. In this case, for instance, the self-excitation of plasma waves is possible. There is no other mean to self-excite plasma waves. Of course, they might be driven from the outside. In this case, we speak about the unstable equilibrium.

We recognise basically four types of plasma instabilities and in the further sections, we will study some of the examples.

Stream instabilities A stream of plasma or an electric current propagates through plasma. A spectrum of waves is excited that leads to an instability of the system.

Rayleigh-Taylor instability Should there be a sharp density gradient in plasmas or the plasma be confined *and* there is an external force (for instance, gravity), the external force drives the growth of instabilities. A classical example of non-plasmatic Rayleigh-Taylor instability is a glass of water turned upside down when the atmospheric pressure supports the water in the glass. The system is allowed to be in an unstable equilibrium, however once perturbed, the instability grows.

Universal instability Is called universal because it is always present in the confined plasmas (and the plasmas are always confined somehow). The confined plasma is not in the total thermodynamical balance even when there are no important external forces. The pressure (expanding the plasma) drives the instabilities.

Kinetic instability The kinetic instability is driven by the non-equilibrium distribution function (usually non-Maxwellian). An anisotropy in the distribution function may drive the instabilities.

7.3.1 Two-stream instability

Let us study the first one example of stream instabilities, the two-stream instability. Let us assume a homogeneous plasma in the coordinate system, where ions are stationary. Let us allow electrons to have a systematical speed $\boldsymbol{u}_0$ with respect to the ion background. We assume $\nabla \cdot \boldsymbol{u}_0 = 0$. The plasma is cold ($T_\mathrm{i} = T_\mathrm{e} = 0$) and there is no magnetic field (hence $\boldsymbol{B}_0 = 0$, $\boldsymbol{B}_1 = 0$).

Then the linearised Euler equations read:

$$Mn_0 \frac{\partial \boldsymbol{u}_{\mathrm{i}1}}{\partial t} = en_0 \boldsymbol{E}_1, \tag{7.22}$$

$$mn_0 \left[\frac{\partial \boldsymbol{u}_{\mathrm{e}1}}{\partial t} + (\boldsymbol{u}_0 \cdot \boldsymbol{\nabla}) \boldsymbol{u}_{\mathrm{e}1} \right] = -en_0 \boldsymbol{E}_1. \tag{7.23}$$

We search for electrostatic waves propagating in the x direction, where $\boldsymbol{E}_1 \parallel \boldsymbol{e}_x \parallel \boldsymbol{k} \parallel \boldsymbol{u}_0$. Let us use the approach of solving the equations in the Fourier space (using the

Fourier transform (5.10)) similarly to the problem of deriving the dispersion relation of plasma waves. Hence

$$-\mathrm{i}\omega M u_{\mathrm{i1}} = eE_1, \tag{7.24}$$

$$m(-\mathrm{i}\omega + \mathrm{i}ku_0)u_{\mathrm{e1}} = -eE_1. \tag{7.25}$$

The continuity equation for ions reads (with assumptions applied: $\nabla n_0 = 0$ and $\boldsymbol{u}_{\mathrm{i0}} \equiv 0$)

$$\frac{\partial n_{\mathrm{i1}}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i1}} = 0, \tag{7.26}$$

where we again consider only the x-component of the differential operator. We express n_{i1} and further use (7.24) to get

$$n_{\mathrm{i1}} = \frac{k}{\omega} n_0 u_{\mathrm{i1}} = \frac{\mathrm{i}en_0 k}{M\omega^2} E_1. \tag{7.27}$$

Similarly, the continuity equation for electrons reads

$$\frac{\partial n_{\mathrm{e1}}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e1}} + (\boldsymbol{u}_0 \cdot \boldsymbol{\nabla}) n_{\mathrm{e1}} = 0, \tag{7.28}$$

hence the parallel component in the Fourier space is

$$(-\mathrm{i}\omega + \mathrm{i}ku_0)n_{\mathrm{e1}} + \mathrm{i}kn_0 u_{\mathrm{e1}} = 0, \tag{7.29}$$

from which we express n_{e1} by using (7.25):

$$n_{\mathrm{e1}} = \frac{kn_0 u_{\mathrm{e1}}}{\omega - ku_0} = -\frac{\mathrm{i}ekn_0}{m(\omega - ku_0)^2} E_1. \tag{7.30}$$

We do not assume the plasma approximation to be valid in this case, thus we further use Poisson equation to finally derive the dispersion relation

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}_1 = \frac{e}{\varepsilon_0}(n_{\mathrm{i1}} - n_{\mathrm{e1}}), \tag{7.31}$$

where the density perturbations of the two species are given by (7.27) and (7.30). Then

$$\mathrm{i}kE_1 = \frac{e}{\varepsilon_0} \mathrm{i}ekn_0 \left[\frac{1}{M\omega^2} + \frac{1}{m(\omega - ku_0)^2} \right] E_1 \tag{7.32}$$

and thus

$$1 = \frac{e^2 n_0}{\varepsilon_0 m} \left[\frac{m/M}{\omega^2} + \frac{1}{(\omega - ku_0)^2} \right] = \omega_{\mathrm{p}}^2 \left[\frac{m/M}{\omega^2} + \frac{1}{(\omega - ku_0)^2} \right]. \tag{7.33}$$

This is the dispersion relation of the two-stream instability. Please note that for $u_0 = 0$ and $m/M \to 0$ this relation degenerates to $\omega^2 = \omega_{\mathrm{p}}^2$, hence to normal plasma oscillations.

This is the equation of the fourth-order in ω. The equation may either have four real roots, each of them will correspond to the individual oscillatory mode, hence it will not describe an instability. The other option is that the equation will have two real roots and the remaining two will be complex, one conjugated to the other. Let us discuss these two. They might be symbolically written as

$$\omega_j = \alpha_j + \mathrm{i}\gamma_j, \quad \text{where} \quad \alpha_j = \Re[\omega_j] \quad \text{and} \quad \gamma_j = \Im[\omega_j]. \tag{7.34}$$

In the case of the complex roots, the solution for e.g. the electric field then reads

$$E_1 = E\boldsymbol{e}_x \exp\left[\mathrm{i}(kx - \omega t)\right] \mathrm{e}^{\gamma_j t}. \tag{7.35}$$

Since the two complex roots are complex conjugated, one of them will have $\gamma_j > 0$ and at least one of the modes will depict an exponential growth of the perturbation.

To study the dispersion relation at least qualitatively, we introduce new variables $\xi = \omega/\omega_\mathrm{p}$ and $\xi_0 = ku_0/\omega_\mathrm{p}$. Then the dispersion relation reads

$$1 = \frac{m/M}{\xi^2} + \frac{1}{(\xi - \xi_0)^2} = F(\xi; \xi_0). \tag{7.36}$$

We draw $F(\xi;\xi_0)$ as a function of ξ for the given ξ_0. Two possible plots are displayed in Fig. 7.4.

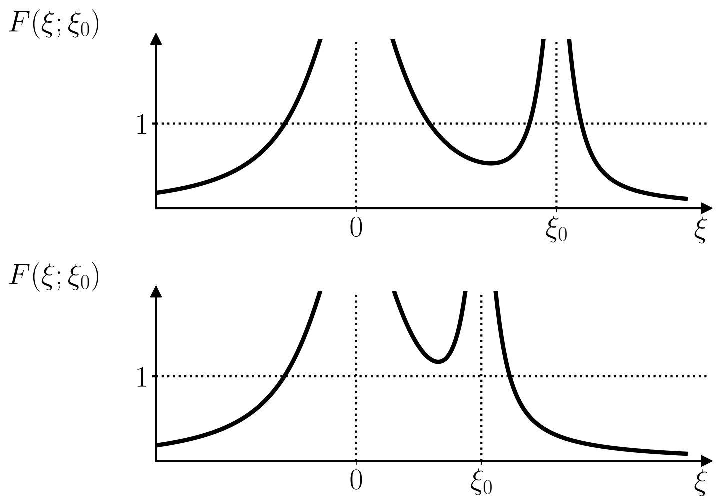

Figure 7.4: A graphical solution of the dispersion relation of the two-stream instability. In the case displayed above, the solution has four real roots. No instability takes place. In the case below, only two real roots solve the dispersion relation. Necessarily, the remaining root must have an imaginary component, which is responsible for the instability.

It turns out that for small ξ_0 the plasma is unstable (function $F(\xi;\xi_0)$ has two conjugated complex roots). In other words, for a given u_0, the plasma is unstable with respect to perturbations with small k (and hence large wavelength). In this case, an internal inconsistency appears. The linearisation approach to solve the system of initial equations will no longer be valid, the assumptions of small perturbations will break. Our solution is therefore no longer correct. We just derived that such a solution qualitatively is unstable (that is a correct result), however quantitatively (deriving the exponential growth) is not correct.[1]

7.3.2 Gravitational instability

Let us consider another example which belongs into the family of Rayleigh-Taylor instabilities: the space is divided into two half-spaces, one empty and one occupied by plasma. The Cartesian coordinate system is set so that the $\boldsymbol{e}_x$ is normal to the boundary between the two half-spaces and $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ lie within the boundary, perpendicular to each other. The background magnetic field is parallel to axis z. The plasma has the density gradient parallel to axis x. The plasma is cold and the magnetic field is homogeneous. We allow for a plasma bulk velocity parallel to axis y. Then in the state of equilibrium, the Euler equation reads

$$Mn_{\mathrm{i}0}[(\boldsymbol{u}_{\mathrm{i}0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{i}0}] = en_{\mathrm{i}0}\boldsymbol{u}_{\mathrm{i}0}\times\boldsymbol{B}_0 + Mn_{\mathrm{i}0}\boldsymbol{g}. \tag{7.37}$$

Note that we added an *ad hoc* term of the gravity force. For $\boldsymbol{g}$ constant also $\boldsymbol{u}_{\mathrm{i}0}$ is constant and hence $(\boldsymbol{u}_{\mathrm{i}0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{i}0}\to 0$. Then

$$0 = en_{\mathrm{i}0}\boldsymbol{u}_{\mathrm{i}0}\times\boldsymbol{B}_0 + Mn_{\mathrm{i}0}\boldsymbol{g} \tag{7.38}$$

We multiply (7.38) by $\boldsymbol{B}_0$ to obtain

$$0 = e\boldsymbol{B}_0\times(\boldsymbol{u}_{\mathrm{i}0}\times\boldsymbol{B}_0) + M\boldsymbol{B}_0\times\boldsymbol{g} = eB_0^2\boldsymbol{u}_{\mathrm{i}0} - e\boldsymbol{B}_0\cdot\boldsymbol{u}_{\mathrm{i}0}\boldsymbol{B}_0 + M\boldsymbol{B}_0\times\boldsymbol{g}, \tag{7.39}$$

where $\boldsymbol{B}_0\cdot\boldsymbol{u}_{\mathrm{i}0}$ due to the orthogonality of these vectors and hence

$$\boldsymbol{u}_{\mathrm{i}0} = -\frac{M}{e}\frac{\boldsymbol{B}_0\times\boldsymbol{g}}{B_0^2} = -\frac{g}{\Omega_{\mathrm{c}}}\boldsymbol{e}_y, \tag{7.40}$$

where $\boldsymbol{u}_{\mathrm{i}0}$ is the velocity of the ion drift in the field of gravity. The drift of electrons is negligible since they have m/M-times smaller effect. Regarding the other drifts, there is no E-B drift, since $\boldsymbol{E}_0 = 0$, there is also no diamagnetic drift because we assumed the cold plasma. However, the conclusions might be different, when the boundary is perturbed and corrugated (see Fig. 7.5). Then the following effects occur:

1. The ions drift and will cause the accumulation of the charge. The electrons react with some delay.
2. The accumulated charges cause the additional electric field $\boldsymbol{E}_1$.
3. Now the E-B drift appears with drift velocity pointing parallel to the normal direction to the boundary.

[1] The solution has a form of the growth in time. The growth factor gives an approximate idea on time scales of the growth of the instability. It is also important to compare this timescale with the timescale of the study we are performing. Should, for instance, the time span of the experiment be much shorter than the instability timescale, the instability will simply not have enough time to fully develop and may thus be neglected.

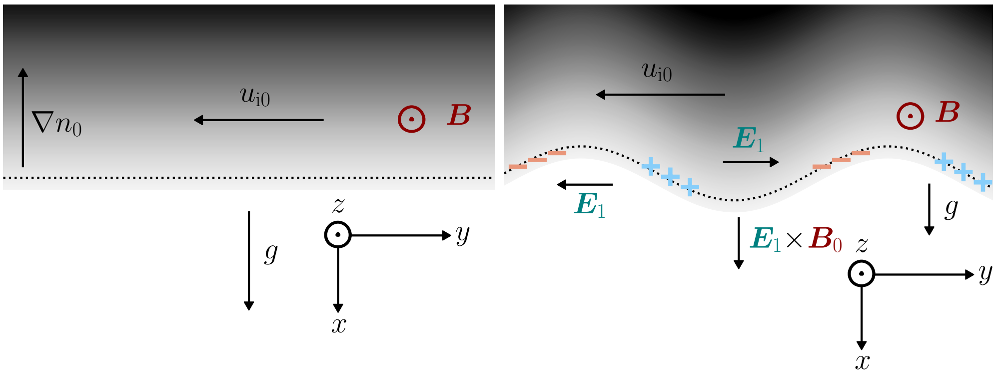

Figure 7.5: Left: The coordinate setup for the gravitational instability. Note that the bulk velocity $\boldsymbol{u}$ is a mass-weighted combination of background electron $\boldsymbol{u}_{e0}$ and ion $\boldsymbol{u}_{i0}$ velocities. However, in the drift of the gravity field, $\boldsymbol{u}_{e0} \ll \boldsymbol{u}_{i0}$. Right: The E-B drift in the evolved gravitational instability.

Let us search for the increment of the growth. Let us assume that there is a wave propagating parallel to $\boldsymbol{e}_y$. For ions, we introduce the perturbations to the density and velocity, similarly as we did in the case of the plasma waves. The Euler equation reads:

$$M(n_{i0}+n_{i1})\left\{\frac{\partial}{\partial t}(\boldsymbol{u}_{i0}+\boldsymbol{u}_{i1}) + [(\boldsymbol{u}_{i0}+\boldsymbol{u}_{i1})\cdot\boldsymbol{\nabla}](\boldsymbol{u}_{i0}+\boldsymbol{u}_{i1})\right\} = \\ e(n_{i0}+n_{i1})\left[\boldsymbol{E}_1 + (\boldsymbol{u}_{i0}+\boldsymbol{u}_{i1})\times\boldsymbol{B}_0\right] + M(n_{i0}+n_{i1})\boldsymbol{g} \tag{7.41}$$

We further proceed slightly differently from the procedure we took when studying the waves. We append (7.41) with (7.37) multiplied by $1 + n_{i1}/n_{i0}$ to obtain the system of two equations:

$$M(n_{i0}+n_{i1})\left\{\frac{\partial}{\partial t}(\boldsymbol{u}_{i0}+\boldsymbol{u}_{i1}) + (\boldsymbol{u}_{i0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i0} + (\boldsymbol{u}_{i0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i1} + (\boldsymbol{u}_{i1}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i0}\right\} = \\ e(n_{i0}+n_{i1})\left[\boldsymbol{E}_1 + (\boldsymbol{u}_{i0}+\boldsymbol{u}_{i1})\times\boldsymbol{B}_0\right] + M(n_{i0}+n_{i1})\boldsymbol{g} \tag{7.42}$$

$$M(n_{i0}+n_{i1})(\boldsymbol{u}_{i0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i0} = (n_{i0}+n_{i1})e\boldsymbol{u}_{i0}\times\boldsymbol{B}_0 + M(n_{i0}+n_{i1})\boldsymbol{g}. \tag{7.43}$$

By subtracting the two some of the terms cancel, and when we further use $\frac{\partial \boldsymbol{u}_{i0}}{\partial t} = 0$ and $(\boldsymbol{u}_{i1}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i0} = 0$ we have to the first order

$$Mn_{i0}\left[\frac{\partial \boldsymbol{u}_{i1}}{\partial t} + (\boldsymbol{u}_{i0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i1}\right] = en_{i0}(\boldsymbol{E}_1 + \boldsymbol{u}_{i1}\times\boldsymbol{B}_0). \tag{7.44}$$

It seems that apparently the term with the gravity acceleration disappeared. An explicit term containing $\boldsymbol{g}$ indeed disappeared, however we have to keep in mind that the gravity acceleration is still hidden in $\boldsymbol{u}_{i0}$!

Equation (7.44) in the Fourier space transforms to

$$M(\omega - \boldsymbol{k}\cdot\boldsymbol{u}_{i0})\boldsymbol{u}_{i1} = \mathrm{i}e(\boldsymbol{E}_1 + \boldsymbol{u}_{i1}\times\boldsymbol{B}_0). \tag{7.45}$$

Let us write this equation in components by assuming $E_{1x} = 0$, since we search for wave propagating in the y-direction. We further use $\boldsymbol{k} = k\boldsymbol{e}_y$ and $\boldsymbol{u}_{i0} = u_{i0}\boldsymbol{e}_y$. For ions we have

$$u_{i1x} = \frac{\mathrm{i}e}{M(\omega - ku_{i0})} u_{i1y} B_0, \tag{7.46}$$

$$u_{i1y} = \frac{-\mathrm{i}e}{M(\omega - ku_{i0})} (E_{1y} - u_{i1x} B_0). \tag{7.47}$$

By combining the two we obtain for u_{iy}

$$u_{i1y} = \frac{-\mathrm{i}e}{M(\omega - ku_{i0})} \left(E_{1y} + \frac{-\mathrm{i}eB_0^2}{M(\omega - ku_{i0})} u_{i1y} \right) \tag{7.48}$$

and hence

$$u_{i1y} = \frac{-\mathrm{i}eE_{1y}}{M(\omega - ku_{i0})} \left[1 - \frac{e^2 B_0^2 / M^2}{(\omega - ku_{i0})^2} \right]^{-1} = \frac{-\mathrm{i}eE_{1y}}{M(\omega - ku_{i0})} \left[1 - \frac{\Omega_\mathrm{c}^2}{(\omega - ku_{i0})^2} \right]^{-1}. \tag{7.49}$$

To obtain a relation for u_{i1x} we simply insert (7.49) into (7.46) and have

$$u_{i1x} = -\frac{e\Omega_\mathrm{c} E_{1y}}{M(\omega - ku_{i0})^2} \left[1 - \frac{\Omega_\mathrm{c}^2}{(\omega - ku_{i0})^2} \right]^{-1}. \tag{7.50}$$

By further assuming low-frequency waves, hence using approximation $\Omega_\mathrm{c}^2 \gg (\omega - ku_{i0})^2$, we have

$$u_{i1x} = \frac{E_{1y}}{B_0}, \quad u_{i1y} = -\mathrm{i}\frac{\omega - ku_{i0}}{\Omega_\mathrm{c}} \frac{E_{1y}}{B_0} \tag{7.51}$$

and analogously for electrons

$$u_{e1x} = \frac{E_{1y}}{B_0}, \quad u_{e1y} = -\mathrm{i}\frac{\omega - ku_{e0}}{\omega_\mathrm{c}} \frac{E_{1y}}{B_0}, \tag{7.52}$$

where $\omega_\mathrm{c} \gg \Omega_\mathrm{c}$ and thus $u_{e1y} \to 0$.

The continuity equation for ions is

$$\frac{\partial n_{i1}}{\partial t} + \boldsymbol{\nabla} \cdot (n_{i0}\boldsymbol{u}_{i0}) + (\boldsymbol{u}_{i0} \cdot \boldsymbol{\nabla}) n_{i1} + n_{i1} \boldsymbol{\nabla} \cdot \boldsymbol{u}_{i0} + (\boldsymbol{u}_{i1} \cdot \boldsymbol{\nabla}) n_{i0} + n_{i0} \boldsymbol{\nabla} \cdot \boldsymbol{u}_{i1} + \boldsymbol{\nabla} \cdot (n_{i1}\boldsymbol{u}_{i1}) = 0. \tag{7.53}$$

The second term vanishes, because by assumptions $\nabla n_{i0} \perp \boldsymbol{u}_{i0}$, the fourth term vanishes because we assumed that $\boldsymbol{\nabla} \cdot \boldsymbol{u}_{i0} = 0$ and the last term on the left-hand side also vanishes because it is the second-order term. Hence in the Fourier space, we have

$$-\mathrm{i}\omega n_{i1} + \mathrm{i}ku_{i0} n_{i1} + u_{i1x} \frac{\partial n_{i0}}{\partial x} + \mathrm{i}kn_{i0} u_{i1y} = 0. \tag{7.54}$$

Similarly, we construct the continuity equation for electrons, however it is simpler, because we assumed that $u_{e0} \ll u_{i0}$ and we showed that $u_{e1y} \to 0$. We will also expect the low-frequency waves, hence the use of plasma approximation $n_1 = n_{e1} = n_{i1}$ and

$n_0 = n_{e0} = n_{i0}$ is justified. In the end, we should show the consistency of the results with the assumptions. Thus for electrons, we have

$$-i\omega n_1 + u_{e1x}\frac{\partial n_0}{\partial x} = 0. \tag{7.55}$$

By combining (7.54) and (7.51) we have

$$-i\omega n_1 + iku_{i0}n_1 + \frac{E_{1y}}{B_0}\frac{\partial n_0}{\partial x} - i^2 kn_0\frac{\omega - ku_{i0}}{\Omega_c}\frac{E_{1y}}{B_0} = 0, \tag{7.56}$$

which trivially turns into

$$(\omega - ku_{i0})n_1 + i\frac{E_{1y}}{B_0}\frac{\partial n_0}{\partial x} + ikn_0\frac{\omega - ku_{i0}}{\Omega_c}\frac{E_{1y}}{B_0} = 0. \tag{7.57}$$

Similarly we combine (7.55) and (7.52) to get the equivalent of (7.57):

$$-i\omega n_1 + \frac{E_{1y}}{B_0}\frac{\partial n_0}{\partial x} = 0, \tag{7.58}$$

which allows us to express the term E_{1y}/B_0 and insert it into (7.57). Then

$$(\omega - ku_{i0})n_1 - \frac{\omega n_1}{\frac{\partial n_0}{\partial x}}\frac{\partial n_0}{\partial x} - kn_0\frac{\omega - ku_{i0}}{\Omega_c}\frac{\omega n_1}{\frac{\partial n_0}{\partial x}} = 0. \tag{7.59}$$

After some algebra we obtain

$$\omega(\omega - ku_{i0}) = -\frac{u_{i0}\Omega_c}{n_0}\frac{\partial n_0}{\partial x}. \tag{7.60}$$

Now we use (7.40) and have

$$\omega(\omega - ku_{i0}) = \frac{g}{n_0}\frac{\partial n_0}{\partial x}, \tag{7.61}$$

or

$$\omega^2 - ku_{i0}\omega - g\left(\frac{\frac{\partial n_0}{\partial x}}{n_0}\right) = 0. \tag{7.62}$$

This is the dispersion relation of the wave we searched for. The solution is

$$\omega = \frac{1}{2}ku_{i0} \pm \left[\frac{1}{4}k^2u_{i0}^2 + g\left(\frac{\frac{\partial n_0}{\partial x}}{n_0}\right)\right]^{\frac{1}{2}}. \tag{7.63}$$

We will have instability (and exponential growth) in case ω is complex. That will be fulfilled for

$$-g\left(\frac{\frac{\partial n_0}{\partial x}}{n_0}\right) > \frac{1}{4}k^2u_{i0}^2, \tag{7.64}$$

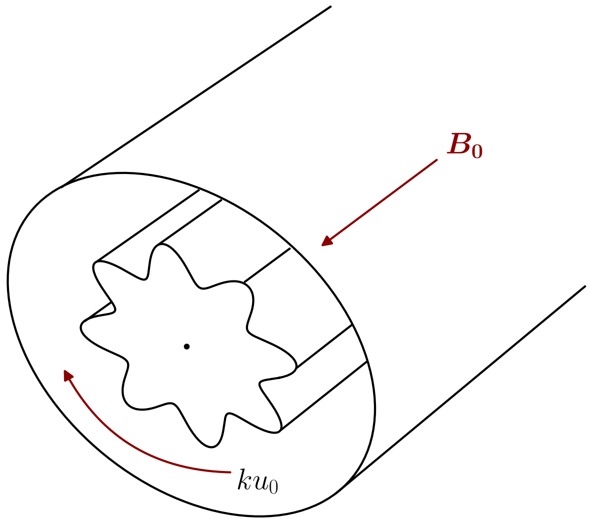

Figure 7.6: A flute instability.

thus $\boldsymbol{g}$ and ∇n_0 must have opposite signs. By waving the hands, "the lighter fluid supports the heavier one". Similar situation we had in case of the glass of water turned upside down.

The interpretation is seen in Fig. 7.5 on the right. The corrugated boundary together with the flow along the boundary causes the concentration of charges on the opposite sides of the "wave". The perturbed E_1 electric field is induced by this charge separation and the resulting $\boldsymbol{v}_\mathrm{E}$ drift causes the initial perturbation to grow.

Similar instability appears in the column of plasma when the gravity force is replaced by the action of the centrifugal force (Fig. 7.6). The conclusions are the same. The boundary of the plasma column is then corrugated and the instability is termed *flute instability*. The onset of this instability is observed in the experiments of the laser-induced fusion. The idea of the laser-induced fusion is that the fusion begins faster than the flute instability sets on and disintegrates the plasma confinement (Fig. 7.7).

7.3.3 Universal instability

We will proceed similarly to the previous section, however, the $\boldsymbol{g} \times \boldsymbol{B}$ drift will be replaced by the diamagnetic drift. Since in this drift there is no scaling with mass of the particles, we have to solve equations for both ions and electrons.

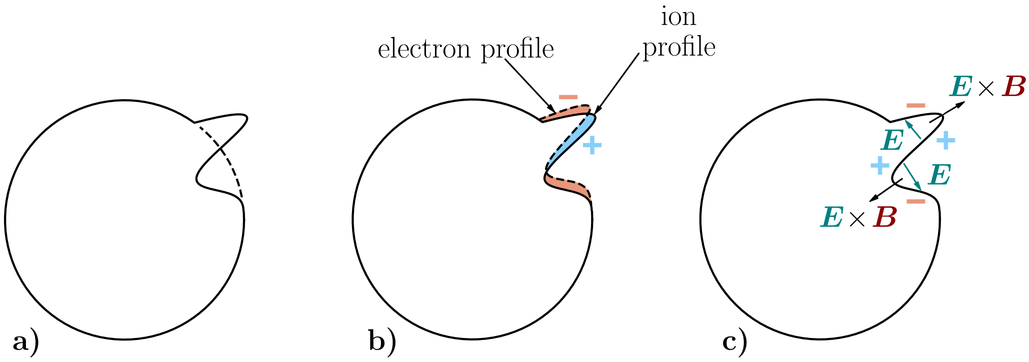

Figure 7.7: Development of flute instability. a) Initial disturbance. b) Effect of ion and electron azimuthal drifts. c) Resulting $\boldsymbol{E} \times \boldsymbol{B}$ drifts increase the amplitude.

For the background velocities we have relations:

$$\boldsymbol{u}_{\mathrm{i}0} = \boldsymbol{u}_{\mathrm{Di}} = \frac{K_{\mathrm{B}}T_{\mathrm{i}}}{eB_0}\frac{1}{n_0}\frac{\partial n_0}{\partial x}\boldsymbol{e}_y, \tag{7.65}$$

$$\boldsymbol{u}_{\mathrm{e}0} = \boldsymbol{u}_{\mathrm{De}} = -\frac{K_{\mathrm{B}}T_{\mathrm{e}}}{eB_0}\frac{1}{n_0}\frac{\partial n_0}{\partial x}\boldsymbol{e}_y, . \tag{7.66}$$

We expect the non-zero component k_z of the wavevector, the motion of electrons along the background magnetic field is important to shield the electrostatic field induced by ions. We will further assume that the Boltzmann relation (4.92) fully describes the density perturbation of the electrons,

$$\frac{n_{\mathrm{e}1}}{n_0} = \frac{e\phi_1}{K_{\mathrm{B}}T_{\mathrm{e}}}. \tag{7.67}$$

The geometry of the problem is sketched in Fig. 7.8.

At point A, the density is larger than in equilibrium, which is denoted by the solid line, hence $n_{\mathrm{e}1} > 0$ and thus $\phi_1 > 0$. At point B, the density is smaller than in equilibrium, hence $n_{\mathrm{e}1} < 0$ and $\phi_1 < 0$. Thus between points A and B there must be an electric field $\boldsymbol{E}_1$, which drives the E-B drift with the drift velocity

$$\boldsymbol{u}_1 = \frac{\boldsymbol{E}_1 \times \boldsymbol{B}_0}{B_0^2}. \tag{7.68}$$

There will be a wave in direction $\boldsymbol{e}_y$, hence both $n_{\mathrm{e}1}$ and ϕ_1 oscillate in time. Therefore also $\boldsymbol{u}_{\mathrm{e}1}$ oscillates in time and in fact the oscillations in the drift velocity are the cause for the oscillations of density. As a result, there will be a motion of the plasma fluid in x direction. Let us support this physical scenario by some equations. The Fourier image of (7.68) is

$$u_{1x} = \frac{E_y}{B_0} = -\frac{\mathrm{i}k_y\phi_1}{B_0}, \tag{7.69}$$

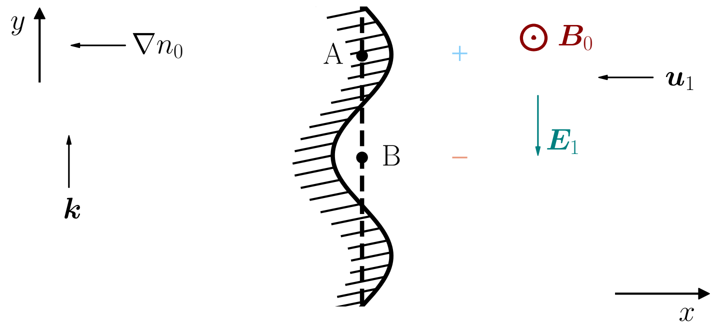

Figure 7.8: An illustration of the universal instability. The corrugated boundary induces the charge separation, which drives the perturbation of the electric field. This together with the background magnetic field yiels the $E - B$ drift, which causes the instability to grow.

where we used the Poisson equation $\boldsymbol{E}_1 = -\nabla\phi_1 = -\mathrm{i}\boldsymbol{k}\phi_1$. The speed of the drift is the same for both ions and electrons. Let us further assume for simplicity, that the fluid of plasmas is incompressible and oscillates only in the x direction. Mathematically speaking: $u_{1x} \neq u_{1x}(x)$ and $k_z \ll k_y$. Then the continuity equation for divergence centres reads:

$$\frac{\partial n_1}{\partial t} = -u_{1x}\frac{\partial n_0}{\partial x}, \tag{7.70}$$

other components are being neglected. We express (7.70) in the Fourier space and use Boltzmann relation to handle n_1 for electrons and (7.69) to handle u_{1x}. We have:

$$-\mathrm{i}\omega n_{\mathrm{e}1} = -u_{\mathrm{e}1x}\frac{\partial n_0}{\partial x} = \frac{\mathrm{i}k_y\phi_1}{B_0}\frac{\partial n_0}{\partial x} = -\mathrm{i}\omega n_0\frac{e\phi_1}{K_\mathrm{B}T_\mathrm{e}}, \tag{7.71}$$

which solves as

$$\frac{\omega}{k} = -\frac{K_\mathrm{B}T_\mathrm{e}}{eB_0}\frac{1}{n_0}\frac{\partial n_0}{\partial x}, \tag{7.72}$$

which is exactly the expression (7.66). We confirmed the physical picture sketched in words a few paragraphs above. The perturbation waves propagate with the speed of the diamagnetic drift, hence termed *drift waves.*

However, the dispersion relation shows that no instability occurs, it does not have any term assuring the growth. We did not prove that this configuration is unstable. Let us only comment that the approach we took is not correct enough and that we neglected two important ingredients: the polarisation drift and the drift in the inhomogeneous electric field. The polarisation drift shows up in the case we have a variable electric field. Then an additional drift with a speed of $v_\mathrm{p} = \operatorname{sgn} q\frac{1}{\omega_\mathrm{c}B}\frac{\mathrm{d}E}{\mathrm{d}t}$ needs to be taken into account. We will further only draw a physical picture in words: the corrections by the additional two drifts causes the phase shift between ϕ_1, which depicts a delay between u_1 and n_1. Hence, in places, where the plasma is already skewed ($n_1 > 0$), u_1 directs out of the plasma and vice versa. The perturbation on the boundary grows.

7.3.4 Landau damping

The highlight of these lecture notes consists of an example of the kinetic instability. We started this text with an introduction to the statistical physics to describe important properties of the ensemble particles. We will end the text by investigating the oscillatory motions in plasmas, which occur in the case of the particle distribution function not being in equilibrium. We need to leave the fluid approximation and need to go one step deeper, to the single-particle distribution function. Let us remind that for the single-particle distribution function f the Boltzmann equation may be derived,

$$\frac{\partial f}{\partial t} + \frac{\partial h}{\partial p_i}\frac{\partial f}{\partial r_i} - \frac{\partial h}{\partial r_i}\frac{\partial f}{\partial p_i} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}, \tag{7.73}$$

where $h = \frac{p^2}{2m} + V(r)$ is a Hamiltonian of one particle. We showed that this equation may be rewritten as

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f + \frac{F}{m}\cdot\frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}, \tag{7.74}$$

where F are the external forces coming from the external potential $V(r)$. By playing explicitly with the term on the right-hand side to extract the term of the self-potential, which has a similar shape as the action of the external potential, we got

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f + \frac{\langle F\rangle}{m}\cdot\frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)'_{\text{coll}}, \tag{7.75}$$

where $\langle F\rangle$ is the averaged action of both external forces and the self potential combined. Then the Vlasov approximation consists of neglecting the right-hand side, which is equivalent of neglecting the mutual correlation of distribution function of different particles. For the further case, we substitute $\langle F\rangle$ by the Lorentz force (as an effect of the sum of both external and internal electric and magnetic fields). Hence finally we have a Vlasov equation to solve:

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0, \tag{7.76}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ represent the joint action of both external and internal electric and magnetic fields. In the following, we will neglect the external fields and in the case of the magnetic field, we will neglect also the internal one. Altogether, we use the following approximations:

1. $\boldsymbol{B} = 0$,
2. $\boldsymbol{E}_0 = 0$, but $\boldsymbol{E}_1 \neq 0$,
3. $f = f_0 + f_1$, $f_0 = f_0(v)$, $f_1 = f_1(t, \boldsymbol{r}, \boldsymbol{v})$,
4. all perturbations are expressed by means of the Fourier series as $A_1 = \bar{A}_1 \exp\left[\mathrm{i}(kx - \omega t)\right]$,

5. and $A_1 \ll A_0$ and $\frac{\partial A_1}{\partial a} \ll \frac{\partial A_0}{\partial a}$. The perturbations are much smaller than the background values and also the derivatives of the perturbations are much smaller than the derivatives of the background.

6. We will further assume that the electric field has only a component along axis $\hat{x}$.

We solve the problem for electrons. Thus the Vlasov equation (7.76) under the assumptions listed above reduces to

$$-\mathrm{i}\omega f_1 + \mathrm{i}k f_1 v_x = \frac{e}{m} E_{1x} \frac{\partial f_0}{\partial v_x}, \tag{7.77}$$

hence

$$f_1 = \frac{\mathrm{i}eE_{1x}}{m} \frac{\partial f_0/\partial v_x}{\omega - kv_x}. \tag{7.78}$$

we add the Poisson equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}_1 = \mathrm{i}kE_{1x} = -\frac{en_1}{\varepsilon_0} = -\frac{e}{\varepsilon_0} \int\limits_{-\infty}^{+\infty} f_1 \mathrm{d}^3\boldsymbol{v}. \tag{7.79}$$

Let us rescale the distribution function so that $f_0 = n_0 \hat{f}_0(\boldsymbol{v})$, where the normalisation of the newly introduced $\hat{f}_0$ is $\int \hat{f}_0 \mathrm{d}^3\boldsymbol{v} = 1$. This definition has an advantage of n_0, which may be put outside of the integrals, because it is not a function of $\boldsymbol{v}$. Let us integrate (7.78) over the whole velocity space

$$\int\limits_{-\infty}^{+\infty} f_1 \mathrm{d}^3\boldsymbol{v} = \frac{\mathrm{i}eE_{1x}}{m} n_0 \int\limits_{-\infty}^{+\infty} \frac{\partial \hat{f}_0/\partial v_x}{\omega - kv_x} \mathrm{d}^3\boldsymbol{v} = -\frac{\mathrm{i}kE_{1x}\varepsilon_0}{e}, \tag{7.80}$$

where we also used (7.79). From that we obtain

$$1 = -\frac{e^2 n_0}{\varepsilon_0 mk} \int\limits_{-\infty}^{+\infty} \frac{\partial \hat{f}_0/\partial v_x}{\omega - kv_x} \mathrm{d}^3\boldsymbol{v} = +\frac{\omega_\mathrm{p}^2}{k^2} \int\limits_{-\infty}^{+\infty} \frac{\partial \hat{f}_0/\partial v_x}{v_x - \frac{\omega}{k}} \mathrm{d}^3\boldsymbol{v}. \tag{7.81}$$

Let us assume that $\hat{f}_0$ is a background equilibrium distribution, essentially having the shape of Maxwell-Boltzmann distribution. Then we may decompose $\hat{f}_0$ as a product of

$$\hat{f}_0(\boldsymbol{v}) = \tilde{f}_0(v_x)\tilde{f}_0(v_y)\tilde{f}_0(v_z), \tag{7.82}$$

with a normalisation $\int \tilde{f}_0(v)\mathrm{d}v = 1$. Then (7.81) simplifies to

$$1 = \frac{\omega_\mathrm{p}^2}{k^2} \int\limits_{-\infty}^{+\infty} \mathrm{d}v_x \frac{\partial \tilde{f}_0(v_x)/\partial v_x}{v_x - \frac{\omega}{k}} \int\limits_{-\infty}^{+\infty} \mathrm{d}v_y \tilde{f}_0(v_y) \int\limits_{-\infty}^{+\infty} \mathrm{d}v_z \tilde{f}_0(v_z) = \frac{\omega_\mathrm{p}^2}{k^2} \int\limits_{-\infty}^{+\infty} \frac{\partial \tilde{f}_0/\partial v_x}{v_x - \frac{\omega}{k}} \mathrm{d}v_x. \tag{7.83}$$

We will let ω be complex. The function (7.83) has a pole in ω/k. By using the residuum

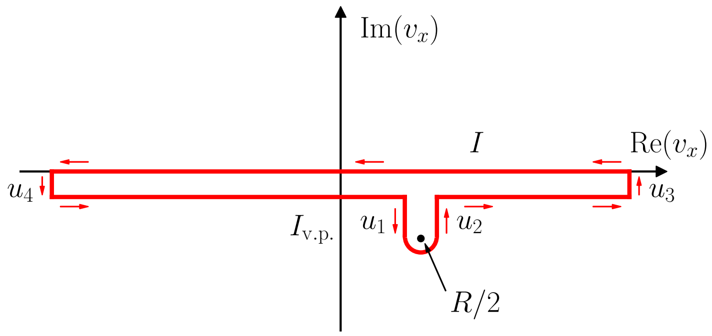

Figure 7.9: A sketch of the integration path including the pole in ω/k.

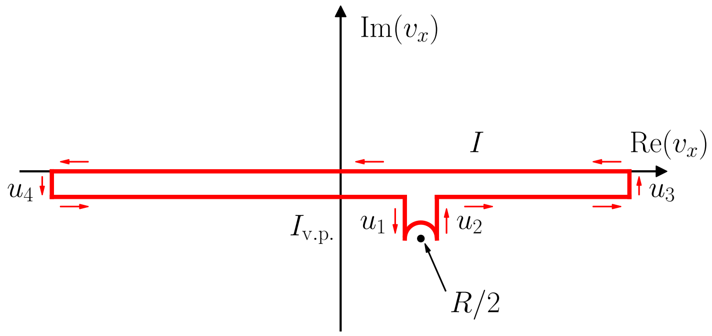

Figure 7.10: A sketch of the integration path bypassing the pole in ω/k.

theorem for the integration of the complex functions, we must integrate along the closed curve, but bypass a pole, which results in an additional contribution of the residuum of this pole. According to the integration path displayed in Fig. 7.9 we have

$$-I + I_{\text{v.p.}} + u_1 + u_2 + \frac{R}{2} + u_3 + u_4 = 0, \tag{7.84}$$

where I is the integral we want to evaluate (note the negative sign as we integrate against the direction of the x axis), $I_{\text{v.p.}}$ is the principal value of this integral, $R = \left[2\pi \mathrm{i} \frac{\partial \tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}$ is a residuum in the pole and the segments u_3 and u_4 may be made infinitesimally small. The segments u_1 and u_2 are in the close vicinity of the pole and we may assume that their length will be similar. Their orientation is the opposite on the integration path, hence we may safely assume that their total contribution, effectively the *difference* of their lengths, will be negligible.

UNVERIFIED: Why did we take only one half of the residuum in the pole? We assume that the damping is small and therefore the pole is very close to the real axis, effectively lying on it. The principal value integral may then be computed as an average of the integrals below and above the real axis. There is no pole (and no residuum) for *the integral above the axis, for the integral below the axis one needs to account for the whole residuum, therefore the total contribution contains only one half of the residuum. The case of the non-negligible damping is discussed elsewhere*[2].

ALTERNATIVELY: The integration path may also be drawn that bypasses the pole, as in Fig. 7.10. In that case, we use the integration path that avoids the pole with a semicircular detour towards the upper half of the complex plane. The theorem from the complex analysis from mathematics says that if we bypass the pole of the complex differentiable (meromorphic)function using a trajectory in a form of an arc, we need to consider only proportional fraction of the corresponding residuum in that pole. The fraction is then given as a ratio of the length of the arc and the length of the corresponding full circle. In our case, since we bypass the residuum using a half-circle, we consider only one half of the residuum value.

(7.83) may thus be written as

$$1 = \frac{\omega_\mathrm{p}^2}{k^2}\left\{\mathrm{v.p.}\int\limits_{-\infty}^{+\infty}\frac{\partial\tilde{f}_0/\partial v_x}{v_x-\frac{\omega}{k}}\mathrm{d}v_x + \left[\mathrm{i}\pi\frac{\partial\tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.85}$$

The integral may be solved by parts

$$\int\limits_{-\infty}^{+\infty}\frac{\partial\tilde{f}_0}{\partial v_x}\frac{\mathrm{d}v_x}{v_x-\omega/k} = \left[\frac{\tilde{f}_0}{v_x-\omega/k}\right]_{+\infty}^{-\infty} + \int\limits_{-\infty}^{+\infty}\frac{\tilde{f}_0\mathrm{d}v_x}{(v_x-\omega/k)^2}. \tag{7.86}$$

The first term vanishes due to the fast decay of the distribution function. Note that the phase speed $v_\varphi = \omega/k$ and let v_φ be positive, i.e., we will deal with the positive velocities only. Let's approximate using the Taylor expansion

$$(v_x - v_\varphi)^{-2} \sim v_\varphi^{-2}\left(1+\frac{2v_x}{v_\varphi}+\frac{3v_x^2}{v_\varphi^2}\right). \tag{7.87}$$

Then (7.86) turns into

$$\int\limits_{-\infty}^{+\infty}\frac{\tilde{f}_0\mathrm{d}v_x}{(v_x-v_\varphi)^2} \sim v_\varphi^{-2}\left[\int\limits_{-\infty}^{+\infty}\tilde{f}_0\mathrm{d}v_x + \int\limits_{-\infty}^{+\infty}\frac{2v_x}{v_\varphi}\tilde{f}_0\mathrm{d}v_x + \int\limits_{-\infty}^{+\infty}\frac{3v_x^2}{v_\varphi^2}\tilde{f}_0\mathrm{d}v_x\right] =$$

$$= v_\varphi^{-2}\left[1+\frac{2}{v_\varphi}\langle v_x\rangle + \frac{3\langle v_x^2\rangle}{v_\varphi^2}\right] = v_\varphi^{-2}\left(1+\frac{3\langle v_x^2\rangle}{v_\varphi^2}\right), \tag{7.88}$$

where we used $\langle v_x\rangle = 0$, which certainly is true for the symmetrical velocity distribution, such as the Maxwell-Boltzmann one.

Let's continue to deal with (7.85):

$$1 = \frac{\omega_\mathrm{p}^2}{k^2}\frac{k^2}{\omega^2}\left\{1+\frac{3\langle v_x^2\rangle}{v_\varphi^2} + \mathrm{i}\pi\left[\frac{\omega^2}{k^2}\frac{\partial\tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.89}$$

[2]T. H. Stix: Waves in Plasmas, 1992, AIP-Press, ISBN 978-0-88318-859-0

Let us take a step to the side. If we were to neglect the residuum in the pole, we would neglect the last term. By further using $\frac{1}{2}m\langle v_x^2\rangle = \frac{1}{2}K_\mathrm{B}T_\mathrm{e}$ we have

$$1 = \frac{\omega_\mathrm{p}^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}\frac{K_\mathrm{B}T_\mathrm{e}}{m}\right). \tag{7.90}$$

If we further assume that $\omega \to \omega_\mathrm{p}$ in the second-order (in other words $\omega^4 \sim \omega^2\omega_\mathrm{p}^2$) we have

$$\omega^2 = \omega_\mathrm{p}^2 + 3k^2\frac{K_\mathrm{B}T_\mathrm{e}}{m}, \tag{7.91}$$

which is the dispersion relation for electron electrostatic waves. By neglecting the contribution in the pole we obtained the classical result.

The new physics shows up when we properly treat the residuum in the pole. For simplicity, let's omit the term with the thermal motion, which (as we just showed) results in the electrostatic electron waves. The dispersion relation then reads

$$\omega^2 = \omega_\mathrm{p}^2\left\{1 + \mathrm{i}\pi\left[\frac{\omega^2}{k^2}\frac{\partial\tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.92}$$

Using the expansion to the Taylor series to the first order and assuming $\omega_\mathrm{p} > 0$ we have

$$\omega \sim \omega_\mathrm{p}\left\{1 + \frac{1}{2}\mathrm{i}\pi\left[\frac{\omega^2}{k^2}\frac{\partial\tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.93}$$

The Taylor expansion is justified because we expect ω to be very close to ω_p. In that case (7.92) may be written as

$$\frac{\omega^2}{\omega_\mathrm{p}^2} = \left\{1 + \mathrm{i}\pi\left[\frac{\omega^2}{k^2}\frac{\partial\tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\} \sim 1 \tag{7.94}$$

and thus the second term on the right-hand side must be small compared to unity. For $\omega \to \omega_\mathrm{p}$ then the equation (7.93) gives

$$\omega \sim \omega_\mathrm{p}\left\{1 + \frac{1}{2}\mathrm{i}\pi\left[\frac{\omega_\mathrm{p}^2}{k^2}\frac{\partial\tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.95}$$

This relation has a suitable form of the complex number with separated purely real and purely imaginary parts.

By going back to the definition of the Fourier series $A_1 = \bar{A}_1 \exp\left[\mathrm{i}(kx - \omega t)\right]$, the new results will come obvious. For $\frac{\partial\tilde{f}_0}{\partial v_x} > 0$ $\Im(\omega) > 0$ and thus we have an instability. For $\frac{\partial\tilde{f}_0}{\partial v_x} < 0$ $\Im(\omega) < 0$ the external perturbations (such as waves) are damped.

So if we plot the distribution function and the perturbation appears at the given phase speed, the derivative of the distribution function decides whether this perturbation

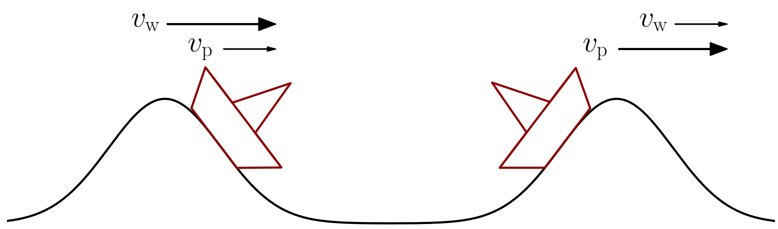

Figure 7.11: The system in the frame moving with phase speed v_φ. At point A the speed of the particle is a little smaller than v_φ, hence the particle falls behind into the point x', where the electric potential is higher and thus pushes the particle forward. Particles having $v < v_\varphi$ are being accelerated. Contrary, particles with $v > v_\varphi$ are decelerated.

will be damped or will grow. Note that for the Maxwell-Boltzmann distribution, $\frac{\partial \tilde{f}_0}{\partial v_x} < 0$ for all speeds, thus in for plasma in a perfect equilibrium the perturbations are always damped. This effect is termed the *Landau damping*.

For a Maxwell-Boltzmann distribution, we may obtain an explicit term of the imaginary part, which is

$$\Im\left(\frac{\omega}{\omega_\mathrm{p}}\right) \sim 0.22\sqrt{\pi}\left(\frac{\omega_\mathrm{p}}{kv_\mathrm{T}}\right)^3 \exp\left[-\frac{1}{2k^2\lambda_\mathrm{D}}\right], \tag{7.96}$$

which shows that the waves with wavenumbers comparable to the Debye shielding length are the most effectively damped.

A different situation appears when the stream of particles having a different typical speed perturbs the distribution function and creates a bump. Part of this bump will have $\frac{\partial \tilde{f}_0}{\partial v_x} > 0$. If the system is perturbed e.g. by the waves with the phase speeds in the susceptible interval, the instability will occur. The energy exchange between the wave and the particles will occur (see Fig. 7.11). A mixture of two Doppler-shifted Maxwellian distributions is unstable. An alternative interpretation can be seen in Fig. 7.12.

Landau damping effect plays an important role in the heating of the solar corona (and also coronae of Sun-like stars), where the outer layers of the atmosphere (coronae) are much hotter (millions of degrees) than the below located photospheres (thousands of degrees). The heating is therefore due to the non-thermodynamical effect, such as the dissipation of the coronal currents flowing along with the loops of the magnetic fields, thermal energy from the all-scale reconnection processes and the heating due to the MHD waves propagating to the coronae. The waves exchange the energy with the coronal plasma by means of the Landau damping.

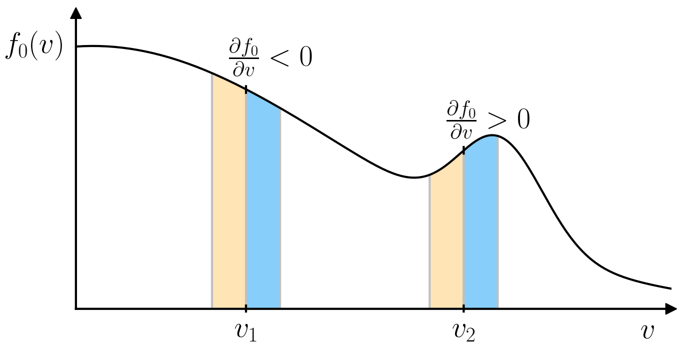

Figure 7.12: The interpretation of Landau damping: If in total there are less faster-than-the-wave particles than slower-than-the-wave particles, the energy streams from the wave to the particles and the perturbation is damped. This is the case of Landau damping, when $\frac{\partial \tilde{f}_0}{\partial v_x} < 0$. The other case is obvious.

Appendices

Appendix A

Basics of statistical physics

A.1 Equation of the energy balance as a second moment of the Vlasov equation

Energy balance equation will be obtained multiplying eq. (1.48) with $\frac{1}{2}m|\boldsymbol{v}|^2$ and integrating over the velocity space.

It is again convenient to use the Einstein summation rule and perform some of the algebra in the components.

We remind that

$$|\boldsymbol{v}|^2 = (u_i + w_i)(u_i + w_i) \tag{A.1}$$

for $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$, where $\boldsymbol{u}$ is the bulk velocity and $\boldsymbol{w}$ the random (thermal) component. We also remind that

$$\rho\langle\mathcal{X}\rangle \equiv \int m\mathcal{X} f \, \mathrm{d}^3\boldsymbol{v}, \tag{A.2}$$

defines a mean value of quantity $\mathcal{X}$.

Multiplying the Boltzmann equation (only repeating Eq. 1.48)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{1}{m}\boldsymbol{\nabla}\left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0 \, . \tag{A.3}$$

by $\frac{1}{2}m|\boldsymbol{v}|^2$ and integrating over the velocity space we get

$$\int \frac{\partial f}{\partial t}\frac{1}{2}m|\boldsymbol{v}|^2 \, \mathrm{d}^3\boldsymbol{v} + \int v_k \frac{\partial f}{\partial r_k}\frac{1}{2}m|\boldsymbol{v}|^2 \, \mathrm{d}^3\boldsymbol{v} - \int \frac{1}{m}\frac{\partial}{\partial r_k}\left[V + \Phi\right]\frac{\partial f}{\partial v_k}\frac{1}{2}m|\boldsymbol{v}|^2 \, \mathrm{d}^3\boldsymbol{v} = 0 \, . \tag{A.4}$$

We will further deal with the terms of this equation one by one. In the first term, we swap the order of the time derivative and integration over velocity space, assuming that it is mathematically possible. We further utilise the definition of the mean value (A.2) and decomposition of the velocity to bulk and random components.

$$\frac{\partial}{\partial t}\int \frac{1}{2}m|\boldsymbol{v}|^2 f\,\mathrm{d}^3\boldsymbol{v} = \frac{\partial}{\partial t}\langle\frac{\rho}{2}|\boldsymbol{v}|^2\rangle = \frac{\partial}{\partial t}\frac{\rho}{2}\langle u^2+w^2\rangle = \frac{\partial}{\partial t}\left[\frac{\rho}{2}u^2+\frac{\rho}{2}\langle w^2\rangle\right] = \\ = \frac{\partial}{\partial t}\left[\frac{\rho}{2}u^2+\rho\mathcal{E}\right], \tag{A.5}$$

where we used the definition of the internal energy as $\rho\mathcal{E} \equiv \rho\langle 1/2w^2\rangle$.

When dealing with the second term of (A.4), we will utilise the following relation:

$$\langle(v_iv_i)v_k\rangle = \langle(u_i+w_i)^2(u_k+w_k)\rangle = \langle(u_i^2+2u_iw_i+w_i^2)(u_k+w_k)\rangle = \\ = \langle u_i^2u_k+2u_iw_iu_k+w_i^2u_k+u_i^2w_k+2u_iw_iw_k+w_i^2w_k\rangle = \\ = u^2u_k+u_k\langle w^2\rangle+2u_i\langle w_iw_k\rangle+\langle w_i^2w_k\rangle, \tag{A.6}$$

where we used the fact that the mean value is a linear operator and $\langle w_i\rangle = 0$. We will further define the following quantities:

$$\text{heat conduction} \qquad F_k \equiv \rho\langle w_k\frac{1}{2}w^2\rangle, \tag{A.7}$$

$$\text{pressure} \qquad P \equiv \frac{1}{3}\rho\langle w^2\rangle, \tag{A.8}$$

$$\text{stress tensor} \qquad \pi_{ij} \equiv \rho\langle\frac{1}{3}|w^2|\delta_{ij}-w_iw_j\rangle = P\delta_{ij}-\rho\langle w_iw_j\rangle. \tag{A.9}$$

Note that from the last line it follows that $\langle w_iw_k\rangle = 1/\rho(P\delta_{ik}-\pi_{ik})$.

Using the identity (A.6) we may evaluate the second term of (A.4):

$$\int v_k\frac{\partial f}{\partial r_k}\frac{1}{2}m|\boldsymbol{v}|^2\,\mathrm{d}^3\boldsymbol{v} = \frac{\partial}{\partial r_k}\int v_kf\frac{1}{2}m|\boldsymbol{v}|^2\,\mathrm{d}^3\boldsymbol{v} = \\ = \frac{\partial}{\partial r_k}\frac{\rho}{2}\langle v^2v_k\rangle = \frac{\partial}{\partial r_k}\frac{\rho}{2}\langle(v_iv_i)v_k\rangle = \\ = \frac{\partial}{\partial r_k}\left[\frac{\rho}{2}u^2u_k+F_k+\rho\mathcal{E}u_k+u_i(P\delta_{ik}-\pi_{ik})\right]. \tag{A.10}$$

The third term of (A.4) gives:

$$\int\frac{1}{m}\frac{\partial\,[V+\Phi]}{\partial r_k}\frac{\partial f}{\partial v_k}\frac{1}{2}m|\boldsymbol{v}|^2\,\mathrm{d}^3\boldsymbol{v} = \frac{\partial\,[V+\Phi]}{\partial r_k}\int\frac{1}{2}\frac{m}{m}v_iv_i\frac{\partial f}{\partial v_k}\,\mathrm{d}^3\boldsymbol{v} = -\frac{\rho}{m}u_k\frac{\partial[V+\Phi]}{\partial r_k}, \tag{A.11}$$

where we used the integration by parts and also the normalisation condition of the distribution function

$$\int v_iv_i\frac{\partial f}{\partial r_k}\,\mathrm{d}^3\boldsymbol{v} = [v_iv_if]_{-\infty}^{+\infty} - \int\frac{\partial(v_iv_i)}{\partial v_k}f\,\mathrm{d}^3\boldsymbol{v} = -2\int v_i\frac{\partial v_i}{\partial v_k}f\,\mathrm{d}^3\boldsymbol{v} = -2\int v_i\delta_{ik}f\,\mathrm{d}^3\boldsymbol{v} = \\ = -2\int v_kf\,\mathrm{d}^3\boldsymbol{v} = -2\frac{\rho}{m}u_k. \tag{A.12}$$

So in total, the (A.4) may be written as

$$\frac{\partial}{\partial t}\left[\frac{\rho}{2}u^2+\rho\mathcal{E}\right]+\frac{\partial}{\partial r_k}\left[\frac{\rho}{2}u^2u_k+F_k+\rho\mathcal{E}u_k+u_i(P\delta_{ik}-\pi_{ik})\right]=-\frac{\rho}{m}u_k\frac{\partial[V+\Phi]}{\partial r_k}. \tag{A.13}$$

Now we take the Euler equation (1.60)

$$\rho\frac{\partial u_k}{\partial t}+\rho u_j\frac{\partial u_k}{\partial r_j}=-\frac{\rho}{m}\frac{\partial[V+\Phi]}{\partial r_k}-\frac{\partial P}{\partial r_k}+\frac{\partial\pi_{kj}}{\partial r_j} \tag{A.14}$$

and multiply it by u_k:

$$u_k\rho\frac{\partial u_k}{\partial t}+\rho u_ku_j\frac{\partial u_k}{\partial r_j}=-\frac{\rho}{m}u_k\frac{\partial[V+\Phi]}{\partial r_k}-u_k\frac{\partial P}{\partial r_k}+u_k\frac{\partial\pi_{kj}}{\partial r_j}. \tag{A.15}$$

The left-hand side becomes:

$$\begin{aligned}u_k\rho\frac{\partial u_k}{\partial t}+\rho u_ku_j\frac{\partial u_k}{\partial r_j}&=\frac{\rho}{2}\frac{\partial u_k^2}{\partial t}+\frac{\rho}{2}u_j\frac{\partial u_k^2}{\partial r_j}+\frac{u_k^2}{2}\frac{\partial\rho}{\partial t}+\frac{u_k^2}{2}\frac{\partial\rho u_j}{\partial r_j}=\\&=\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_k^2\right)+\frac{\partial}{\partial r_j}\left(\frac{1}{2}\rho u_ju_k^2\right)=\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}\right)+\frac{\partial}{\partial r_j}\left(\frac{\rho u^2}{2}u_j\right),\end{aligned} \tag{A.16}$$

where as a trick we added a zero term in a form of the $u^2/2$-multiple of the continuity equation. Hence the u_k-multiple of the Euler equation reads:

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}\right)+\frac{\partial}{\partial r_k}\left(\frac{\rho u^2}{2}u_k\right)=-\frac{\rho}{m}u_k\frac{\partial[V+\Phi]}{\partial r_k}-u_k\frac{\partial P}{\partial r_k}+u_k\frac{\partial\pi_{kj}}{\partial r_j}. \tag{A.17}$$

Now we subtract the previous from (A.13) and get

$$\frac{\partial\rho\mathcal{E}}{\partial t}=-\frac{\partial F_k}{\partial r_k}-\frac{\partial\rho\mathcal{E}u_k}{\partial r_k}-\frac{\partial}{\partial r_k}u_iP\delta_{ik}+u_k\frac{\partial P}{\partial r_k}+\frac{\partial u_i\pi_{ik}}{\partial r_k}-u_k\frac{\partial\pi_{kj}}{\partial r_j}. \tag{A.18}$$

The term on the left-hand side together with the second term on the right-hand side give

$$\frac{\partial\rho\mathcal{E}}{\partial t}+\frac{\partial\rho\mathcal{E}u_k}{\partial r_k}=\mathcal{E}\frac{\partial\rho}{\partial t}+\rho\frac{\partial\mathcal{E}}{\partial t}+\mathcal{E}\frac{\partial u_k\rho}{\partial r_k}+\rho u_k\frac{\partial\mathcal{E}}{\partial r_k}=\rho\frac{\partial\mathcal{E}}{\partial t}+\rho u_k\frac{\partial\mathcal{E}}{\partial r_k}=\rho\frac{\partial\mathcal{E}}{\partial t}+\rho\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\mathcal{E}=\rho\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}, \tag{A.19}$$

where in the first step we identified the $\mathcal{E}$-multiple of the continuity equation, hence two terms cancelled out.

The third and forth terms on the right-hand side of (A.18) give

$$-\frac{\partial}{\partial r_k}u_iP\delta_{ik}+u_k\frac{\partial P}{\partial r_k}=-\frac{\partial u_kP}{\partial r_k}+u_k\frac{\partial P}{\partial r_k}=-P\frac{\partial u_k}{\partial r_k}-u_k\frac{\partial P}{\partial r_k}+u_k\frac{\partial P}{\partial r_k}=-P\frac{\partial u_k}{\partial r_k}. \tag{A.20}$$

And finally, the fifth and sixth terms on the right-hand side of (A.18) give

$$\frac{\partial u_i\pi_{ik}}{\partial r_k}-u_k\frac{\partial\pi_{kj}}{\partial r_j}=\frac{\partial u_i\pi_{ik}}{\partial r_k}-u_i\frac{\partial\pi_{ik}}{\partial r_k}=\pi_{ik}\frac{\partial u_i}{\partial r_k}+u_i\frac{\partial\pi_{ik}}{\partial r_k}-u_i\frac{\partial\pi_{ik}}{\partial r_k}=\pi_{ik}\frac{\partial u_i}{\partial r_k}. \tag{A.21}$$

The resulting expression is usually termed the local heat dissipation rate, $\Psi \equiv \pi_{ik}\frac{\partial u_i}{\partial r_k}$.

Putting the intermediate results together we finally have

$$\rho\frac{\partial \mathcal{E}}{\partial t} + \rho u_k \frac{\partial \mathcal{E}}{\partial r_k} = -\frac{\partial F_k}{\partial r_k} - P\frac{\partial u_k}{\partial r_k} + \pi_{ik}\frac{\partial u_i}{\partial r_k} \tag{A.22}$$

or in a vector form

$$\rho\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -\boldsymbol{\nabla}\cdot\boldsymbol{F} - P\boldsymbol{\nabla}\cdot\boldsymbol{u} + \boldsymbol{\pi} : (\boldsymbol{\nabla}\boldsymbol{u}), \tag{A.23}$$

where we need to stress out that the object $(\boldsymbol{\nabla}\boldsymbol{u})$ is a dyadic (a second-order tensor) and the colon (:) operator indicates the double-scalar product between the tensors.

A.2 Properties of the stress tensor and heat flow with the isotropic distribution function

Let us properly derive the properties of the stress tensor π_{ij} and (conductive) heat flow F_i in the special case of the distribution function, which is isotropic in the velocity space, hence $f(\boldsymbol{r},\boldsymbol{v}) = f(\boldsymbol{r},|\boldsymbol{v}|)$.

Note that the isotropic distribution function is the special case of the symmetric distribution function, for which we have $f(\boldsymbol{r},\boldsymbol{v}) = f(\boldsymbol{r},-\boldsymbol{v})$. In such a case, when splitting the particle velocity component into its bult and fluctuating components, $v_i = u_i + w_i$ we have for the mean values

$$\langle w_i\rangle = 0,\ \langle v_i\rangle = u_i = 0,\ \forall i, \tag{A.24}$$

where we followed the description for the mean value by (A.2). This is due to the symmetry of the distribution function. Such useful properties will also be valid for the isotropic distribution function (which again is a special case of the symmetric one) and further more, the velocity dispersions in all directions will be the same, symbolically

$$\langle v_i^2\rangle = \langle w_i^2\rangle \equiv \langle w^2\rangle,\ \forall i. \tag{A.25}$$

Note that $\langle w^2\rangle$ indicates the velocity dispersion in any direction, which is a different quantity than the square of the length of the fluctuating velocity vector $\boldsymbol{w}$. We would indicate the latter as $|\boldsymbol{w}|^2$. For the isotropic distribution function, these two quantities differ by a factor of 3, which we will use immediately in what is to follow. For the stress tensor we can thus write:

$$\begin{aligned}\pi_{ij} &= \rho\left\langle \frac{1}{3}|\boldsymbol{w}|^2\delta_{ij} - w_iw_j\right\rangle = \rho\left[\frac{1}{3}\langle|\boldsymbol{w}|^2\rangle\delta_{ij} - \langle w_iw_j\rangle\right] = \\ &= \rho\left[\frac{1}{3}\left\langle\sum_{i=1}^{3} w_i^2\right\rangle\delta_{ij} - \langle w_iw_j\rangle\right] = \rho\left[\frac{1}{3}3\langle w^2\rangle\delta_{ij} - \langle w^2\rangle\delta_{ij}\right] = 0,\end{aligned} \tag{A.26}$$

where we applied the properties of the velocity dispersions for the system isotropic in the velocity space.

Similarly, for conductive heat flow we have

$$F_k = \frac{\rho}{2}\left\langle w_k|\boldsymbol{w}|^2\right\rangle = \frac{\rho}{2}\left\langle w\sum_{i=1}^{3}w^2\right\rangle = \frac{1}{2}m\int w\,3w^2 f\mathrm{d}^3\boldsymbol{v} = \frac{3}{2}m\int v^3 f\mathrm{d}^3\boldsymbol{v} = 0,\ \forall k \tag{A.27}$$

again due to the symmetries (the odd powers of velocity component average to zero).

In the case of the symmetric distribution function ($f(\boldsymbol{r},\boldsymbol{v}) = f(\boldsymbol{r},-\boldsymbol{v})$), the system symmetry also simplifies the two considered quantites significantly. For the heat flow we have

$$\begin{aligned} F_k &= \frac{\rho}{2}\left\langle w_k|\boldsymbol{w}|^2\right\rangle = \frac{\rho}{2}\left\langle w_k\sum_{i=1}^{3}w_i^2\right\rangle = \frac{m}{2}\int v_k\sum_{i=1}^{3}v_i^2 f\mathrm{d}^3\boldsymbol{v} = \\ &= \frac{m}{2}\sum_{i=1}^{3}\int v_k v_i^2 f\mathrm{d}^3\boldsymbol{v} = \frac{m}{2}\left[\int v_k^3 f\mathrm{d}^3\boldsymbol{v} + \sum_{i\neq k}\int v_i^2\left(\int v_k f\mathrm{d}v_k\right)\mathrm{d}^2\boldsymbol{v}\right] = 0, \end{aligned} \tag{A.28}$$

where we symbolically split the integration element to $\mathrm{d}^3\boldsymbol{v} = \mathrm{d}v_k\mathrm{d}^2\boldsymbol{v}$. In the last step and the second term, note that the integral in the round parenthesis (integration of v_k) could in principle still depend on the two remaining velocity components. The symmetry of f in the velocity space implies the symmetry in the individual components, hence the averaging of the velocity component vanishes and the whole term becomes zero.

When assessing the properties of the stress tensor for the symmetric distribution function, let us go back to the velocity correlations, from which the stress-tensor components are defined. The elements of the correlation tensor $\langle v_i v_j\rangle = \int v_i v_j f\mathrm{d}^3\boldsymbol{v}$ substantially differ for the diagonal and off-diagonal elements. For the diagonal elements we trivially have

$$\langle v_i v_i\rangle = \langle v_i^2\rangle = \langle w_i^2\rangle, \tag{A.29}$$

hence on the diagonal, we have velocity dispersions in each direction. Note that the values of the diagonal terms need not to be the same. They are for the isotropic distribution function. For the off-diagonal terms we have

$$\langle v_i v_j\rangle = \frac{m}{\rho}\int v_i v_j f\mathrm{d}^3\boldsymbol{v} = \frac{m}{\rho}\int\left(\int v_i f\mathrm{d}v_i\right)v_j\mathrm{d}v_j\mathrm{d}v = 0, \tag{A.30}$$

where similarly to the previous case we split the integration element to $\mathrm{d}^3\boldsymbol{v} = \mathrm{d}v_i\mathrm{d}v_j\mathrm{d}v$ and do the averaging of the components with caution.

Since the velocity-correlation tensor has a diagonal form, the stress tensor π_{ij} must also have a diagonal form. Only two of them are however independent, as the sum of the diagonal elements is bound to the value of pressure P,

$$\pi_{ii} = \rho\left\langle\frac{1}{3}|\boldsymbol{w}|^2 - w_i^2\right\rangle \text{ and } \pi_{ij} = 0 \text{ for } i\neq j. \tag{A.31}$$

Appendix B

Guiding centre motion

B.1 Relativistic E-B drift

In Section 3.2 we derived an expression for a classical E-B drift. It is interesting to show that the E-B drift naturally shows up as a consequence of the Lorentz transform of the electromagnetic field tensor. That is we transform $F_{\mu\nu}$,

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \tag{B.1}$$

using the tensor of Lorentz transform Λ^μ_ν

$$\Lambda^\mu_\nu = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}, \tag{B.2}$$

where $\beta = v/c$ and $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. Note that the velocity v is not known yet and we will obtain it as a solution to the problem. The transformed $F'_{\xi\zeta}$ is

$$F'_{\xi\zeta} = \Lambda^\mu_\xi F_{\mu\nu} \Lambda^\nu_\zeta =$$
$$= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}. \tag{B.3}$$

Now we need to properly multiply these three matrices to obtain

$$F'_{\xi\zeta} = \begin{bmatrix} 0 & \gamma(E_x/c - \beta B_y) & \gamma(E_y/c + \beta B_x) & E_z/c \\ \gamma(\beta B_y - E_x/c) & 0 & B_z & \gamma(\beta E_x/c - B_y) \\ -\gamma(E_y/c + \beta B_x) & -B_z & 0 & \gamma(\beta E_y/c + B_x) \\ -E_z/c & \gamma(B_y - \beta E_x/c) & -\gamma(\beta E_y/c + B_x) & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & E'_x/c & E'_y/c & E'_z/c \\ -E'_x/c & 0 & B'_z & -B'_y \\ -E'_y/c & -B'_z & 0 & B'_x \\ -E'_z/c & B'_y & -B'_x & 0 \end{bmatrix}. \tag{B.4}$$

Hence

$$E'_x/c = \gamma(E_x/c - \beta B_y), \tag{B.5}$$
$$E'_y/c = \gamma(E_y/c + \beta B_x), \tag{B.6}$$
$$E'_z = E_z, \tag{B.7}$$
$$B'_x = \gamma(\beta E_y/c + B_x), \tag{B.8}$$
$$B'_y = \gamma(B_y - \beta E_x/c), \tag{B.9}$$
$$B'_z = B_z \tag{B.10}$$

are the transformation equations for all the components of the electromagnetic tensor. The target mutual velocity of both coordinate systems is the one when the effect of the magnetic and electric fields will be separable, thus when $\boldsymbol{E} \parallel \boldsymbol{B}$, which is an equivalent of $\boldsymbol{E} \times \boldsymbol{B} = 0$. For reasons which will show up later, let's compute the square of the cross product amplitude, which will certainly be zero for parallel vectors. Then by using the rules for mixed product we have

$$|\boldsymbol{E} \times \boldsymbol{B}|^2 = (\boldsymbol{E} \times \boldsymbol{B}) \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \boldsymbol{E} \cdot [\boldsymbol{B} \times (\boldsymbol{E} \times \boldsymbol{B})] = E^2 B^2 - (\boldsymbol{E} \cdot \boldsymbol{B})^2 = 0 \tag{B.11}$$

Let's compute this relation for transformed vectors $\frac{\boldsymbol{E}'}{c}$ and $\boldsymbol{B}'$ and let's deal only with perpendicular components, because the parallel componets are already separated. Doing this algebra we have

$$\begin{aligned} \left|\frac{\boldsymbol{E}'_\perp}{c}\right|^2 |\boldsymbol{B}'_\perp|^2 &= \left[\gamma^2(E_x/c - \beta B_y)^2 + \gamma^2(E_y/c + \beta B_x)^2\right] \\ &\quad \left[\gamma^2(\beta E_y/c + B_x)^2 + \gamma^2(B_y - \beta E_x/c)^2\right] = \\ &= \gamma^4(E_x/c - \beta B_y)^2(\beta E_y/c + B_x)^2 + \gamma^4(E_x/c - \beta B_y)^2(B_y - \beta E_x/c)^2 + \\ &+ \gamma^4(E_y/c + \beta B_x)^2(\beta E_y/c + B_x)^2 + \gamma^4(E_y/c + \beta B_x)^2(B_y - \beta E_x/c)^2 \end{aligned} \tag{B.12}$$

and

$$\begin{aligned} \left|\frac{\boldsymbol{E}'_\perp}{c} \cdot \boldsymbol{B}'_\perp\right|^2 &= \gamma^4(E_x/c - \beta B_y)^2(\beta E_y/c + B_x)^2 + \gamma^4(E_y/c + \beta B_x)^2(B_y - \beta E_x/c)^2 + \\ &+ 2\gamma^4(E_x/c - \beta B_y)(E_y/c + \beta B_x)(\beta E_y/c + B_x)(B_y - \beta E_x/c). \end{aligned} \tag{B.13}$$

Hence [we immediately see that when subtracting (B.13) from (B.12), first terms cancel with each other, as do also the fourth term from (B.12) and second term from (B.13)]:

$$\begin{aligned}\left|\frac{\boldsymbol{E}'_\perp}{c}\cdot\boldsymbol{B}'\right|^2-\left|\frac{\boldsymbol{E}'_\perp}{c}\right|^2|\boldsymbol{B}'|^2 &= \gamma^4(E_x/c-\beta B_y)^2(B_y-\beta E_x/c)^2+\\ &+\gamma^4(E_y/c+\beta B_x)^2(\beta E_y/c+B_x)^2-\\ &-2\gamma^4(E_x/c-\beta B_y)(E_y/c+\beta B_y)(\beta E_y/c+B_x)(B_y-\beta E_x/c)=\\ &=\gamma^4\left[\left(\frac{E_x}{c}-\beta B_y\right)\left(B_y-\beta\frac{E_x}{c}\right)-\left(\frac{E_y}{c}+\beta B_x\right)\left(\beta\frac{E_y}{c}+B_x\right)\right]^2=\\ &=\gamma^4\left[-\beta\left(\frac{E_x^2}{c^2}+\frac{E_y^2}{c^2}+B_x^2+B_y^2\right)+(1+\beta^2)\left(\frac{E_y}{c}B_x-\frac{E_x}{c}B_y\right)\right]^2=\\ &=\gamma^4\left[-\beta\left(\frac{E^2}{c^2}+B^2\right)+(1+\beta^2)\left(\boldsymbol{B}\times\frac{\boldsymbol{E}}{c}\right)_z\right]=0, \quad \text{(B.14)}\end{aligned}$$

where we used

$$(1+\beta^2)\left(\boldsymbol{B}\times\frac{\boldsymbol{E}}{c}\right)_z=(1+\beta^2)\left(B_x\frac{E_y}{c}-B_y\frac{E_x}{c}\right). \quad \text{(B.15)}$$

Equation (B.14) may be true if and only if we have

$$\frac{\beta}{1+\beta^2}=\frac{\frac{\boldsymbol{E}}{c}\times\boldsymbol{B}}{\frac{E^2}{c^2}+B^2}, \quad \text{(B.16)}$$

which implicitly defines the target speed v_E of the coordinate system, in which the effects of magnetic and electric fields are separable. This speed is hidden in $\beta = v_E/c$. We will not solve this equation properly but rather take an approximation of $\beta < 1$, when we may use the β^2 as a correction term and obtain

$$\boldsymbol{v}_E=(1+\beta^2)\frac{\boldsymbol{E}\times\boldsymbol{B}}{\frac{E^2}{c^2}+B^2}. \quad \text{(B.17)}$$

This is the relativistic version of (3.29). When assuming a non-relativistic case with $\beta \ll 1$ and further $|\boldsymbol{E}| \ll c$, the two expressions equal exactly. The latter condition applies to most of the real-world systems.

B.2 Alternative derivation of (3.118)

$$\begin{aligned}
\langle(\boldsymbol{e}_\perp \times \boldsymbol{b})(\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla})\boldsymbol{B}\rangle &= \left\langle \left[\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}\right] \left[\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \nabla_1 B \\ \nabla_2 B \\ \nabla_3 B \end{pmatrix}\right]\right\rangle = \\
&= \left\langle \begin{pmatrix} b\sin\phi \\ -b\cos\phi \\ 0 \end{pmatrix} (\cos\phi\nabla_1 B + \sin\phi\nabla_2 B) \right\rangle = \\
&= \left\langle \begin{pmatrix} b\sin\phi\cos\phi\nabla_1 B + b\sin^2\phi\nabla_2 B \\ -b\cos^2\phi\nabla_1 B - b\sin\phi\cos\phi\nabla_2 B \\ 0 \end{pmatrix} \right\rangle = \\
&= -\frac{1}{2}\begin{pmatrix} -b\nabla_2 B \\ b\nabla_1 B \\ 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} \times \begin{pmatrix} \nabla_1 B \\ \nabla_2 B \\ \nabla_3 B \end{pmatrix} = \\
&= -\frac{1}{2}\boldsymbol{b} \times \nabla B \qquad \text{(B.18)}
\end{aligned}$$

$$\begin{aligned}
\langle(\boldsymbol{e}_\perp \times [\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla}]\,\boldsymbol{b}\rangle &= \left\langle \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \times \left[\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \nabla_1 \boldsymbol{b} \\ \nabla_2 \boldsymbol{b} \\ \nabla_3 \boldsymbol{b} \end{pmatrix}\right]\right\rangle = \\
&= \left\langle \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \times \left[(\cos\phi, \sin\phi, 0)\begin{pmatrix} \nabla_1 b_1 & \nabla_1 b_2 & \nabla_1 b_3 \\ \nabla_2 b_1 & \nabla_2 b_2 & \nabla_2 b_3 \\ \nabla_3 b_1 & \nabla_3 b_2 & \nabla_3 b_3 \end{pmatrix}\right]\right\rangle = \\
&= \left\langle \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\phi\nabla_1 b_1 + \sin\phi\nabla_2 b_1 \\ \cos\phi\nabla_1 b_2 + \sin\phi\nabla_2 b_2 \\ \cos\phi\nabla_1 b_3 + \sin\phi\nabla_2 b_3 \end{pmatrix}\right\rangle = \\
&= \left\langle \begin{pmatrix} \sin\phi\cos\phi\nabla_1 b_3 + \sin^2\phi\nabla_2 b_3 \\ -\cos^2\phi\nabla_1 b_3 - \cos\phi\sin\phi\nabla_2 b_3 \\ -\sin\phi\cos\phi\nabla_1 b_1 - \sin^2\phi\nabla_2 b_1 + \cos^2\phi\nabla_1 b_2 + \cos\phi\sin\phi\nabla_2 b_2 \end{pmatrix}\right\rangle = \\
&= \begin{pmatrix} \frac{1}{2}\nabla_2 b_3 \\ -\frac{1}{2}\nabla_1 b_3 \\ -\frac{1}{2}\nabla_2 b_1 + \frac{1}{2}\nabla_1 b_2 \end{pmatrix} = \frac{1}{2}(\boldsymbol{e}_1\nabla_2 - \boldsymbol{e}_2\nabla_1)b_3 + \frac{1}{2}\boldsymbol{b}(\nabla_1 b_2 - \nabla_2 b_1) = \\
&= \frac{1}{2}\boldsymbol{b}(\boldsymbol{\nabla} \times \boldsymbol{b})_3 + \mathcal{O}\left[(\nabla b_{[1,2]})^2\right] = \frac{1}{2}\boldsymbol{b}[\boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b})]. \qquad \text{(B.19)}
\end{aligned}$$

Note that in the second step we expressed the dot product by means of matrix multiplication respecting the rule $\boldsymbol{a} \cdot \boldsymbol{b} = a^{\mathrm{T}} b = a b^{\mathrm{T}}$, where a is the matrix representation of the vector $\boldsymbol{a}$.

B.3 Derivation of equations (3.122)–(3.124)

Derive equations for variables in the perpendicular plane

$$\left\langle \frac{\mathrm{d}v_\parallel}{\mathrm{d}t} \right\rangle = \frac{E_\parallel}{\epsilon} - \frac{u_\perp^2}{2B}\nabla_\parallel B + \boldsymbol{v}_\mathrm{E} \cdot D_t \boldsymbol{b} \tag{B.20}$$

$$\left\langle \frac{\mathrm{d}u_\perp}{\mathrm{d}t} \right\rangle = \frac{v_\parallel u_\perp}{2B}\nabla_\parallel B - \frac{u_\perp}{2}(\boldsymbol{\nabla} \cdot \boldsymbol{v}_\mathrm{E} - \boldsymbol{b} \cdot \nabla_\parallel \boldsymbol{v}_\mathrm{E}) \tag{B.21}$$

$$\left\langle \frac{\mathrm{d}\phi}{\mathrm{d}t} \right\rangle = -\frac{B}{\epsilon} - \boldsymbol{e}_2 \cdot D_t \boldsymbol{e}_1 - \frac{v_\parallel}{2}\boldsymbol{b} \cdot \boldsymbol{\nabla} \times (v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E}) \tag{B.22}$$

Appendix C

Magnetic dipole

C.1 Magnetic dipole expressed in various coordinate systems

The magnetic induction $\boldsymbol{B}$ of a dipole with a magnetic moment $\mathcal{M}$ is given[1] by

$$\boldsymbol{B} = (B_r, B_\vartheta, B_\varphi) = \frac{\mathcal{M}}{r^3}\left[2\cos\vartheta\boldsymbol{e}_r + \sin\vartheta\boldsymbol{e}_\vartheta\right] \tag{C.1}$$

in the spherical coordinates (r, ϑ, φ). For the sake of an example studied in Section 3.9, it is convenient to express the same in the cylindrical coordinates (R, φ, z). To complete this, we consider the following transform equations

$$R = r\sin\vartheta, \tag{C.2}$$
$$\varphi = \varphi, \tag{C.3}$$
$$z = r\cos\vartheta \tag{C.4}$$

that can easily be derived from the definition of each using the usual Cartesian system. Consequently

$$r = \sqrt{R^2 + z^2}, \tag{C.5}$$
$$\sin\vartheta = \frac{R}{r} = \frac{R}{\sqrt{R^2 + z^2}}, \tag{C.6}$$
$$\cos\vartheta = \frac{z}{r} = \frac{z}{\sqrt{R^2 + z^2}}. \tag{C.7}$$

To transform (C.1) to cylindrical coordinates, we first express the unit vectors from (C.1) in the cylindrical coordinates. For the position vector of a point we have

$$\boldsymbol{R} + \boldsymbol{z} = R\boldsymbol{e}_r + z\boldsymbol{e}_z = \boldsymbol{r} = r\boldsymbol{e}_r, \tag{C.8}$$

[1] e.g. `https://en.wikipedia.org/wiki/Magnetic_dipole`

where vectors $\boldsymbol{e}$ are unit vectors in the appropriate coordinate direction indicated by their subscript. Hence

$$\boldsymbol{e}_r = \frac{R}{r}\boldsymbol{e}_R + \frac{z}{r}\boldsymbol{e}_z = \sin\vartheta\boldsymbol{e}_R + \cos\vartheta\boldsymbol{e}_z. \tag{C.9}$$

To express $\boldsymbol{e}_\vartheta$ we use the orthonormality of both the spherical and cylindrical coordinate systems:

$$\boldsymbol{e}_\vartheta = \boldsymbol{e}_\varphi \times \boldsymbol{e}_r \quad \text{and} \quad \boldsymbol{e}_\varphi = \boldsymbol{e}_z \times \boldsymbol{e}_R. \tag{C.10}$$

Then

$$\boldsymbol{e}_\vartheta = (\boldsymbol{e}_z \times \boldsymbol{e}_R) \times \boldsymbol{e}_{\boldsymbol{r}} = -\boldsymbol{e}_z\boldsymbol{e}_R \cdot \boldsymbol{e}_r + \boldsymbol{e}_R\boldsymbol{e}_r \cdot \boldsymbol{e}_z = -\sin\vartheta\boldsymbol{e}_z + \cos\vartheta\boldsymbol{e}_R. \tag{C.11}$$

We use these expression in (C.1) to have

$$\boldsymbol{B} = \frac{\mathcal{M}}{r^3}\left[3\cos\vartheta\sin\vartheta\boldsymbol{e}_R + (2 - 3\sin^2\vartheta)\boldsymbol{e}_z\right]. \tag{C.12}$$

By expressing r in cylindrical coordinates we obtain components of the magnetic induction:

$$B_R = \frac{3\mathcal{M}}{r^3}\cos\vartheta\sin\vartheta = \frac{3\mathcal{M}Rz}{(R^2+z^2)^{5/2}} \tag{C.13}$$

and

$$B_z = \frac{\mathcal{M}}{r^3}\left(2 - 3\sin^2\vartheta\right) = \frac{\mathcal{M}}{(R^2+z^2)^{3/2}}\left[2 - \frac{3R^2}{R^2+z^2}\right] = \frac{\mathcal{M}}{(R^2+z^2)^{5/2}}\left[2z^2 - R^2\right]. \tag{C.14}$$

In Section 3.9 we claimed that for the axisymmetric magnetic field, we may utilise the flux function

$$F = \mathcal{M}\frac{R^2}{(R^2+z^2)^{3/2}} \tag{C.15}$$

to derive the magnetic induction components as

$$B_z = \frac{1}{R}\frac{\partial F}{\partial R} \quad \text{and} \quad B_R = -\frac{1}{R}\frac{\partial F}{\partial z}, \tag{C.16}$$

which is consistent with the approach described in Section 4.4.2. Let us verify that we indeed get the dipole components (C.13) and (C.14).

For the polar component, we have

$$B_R = -\frac{\mathcal{M}R^2}{R}\frac{\partial}{\partial z}\frac{1}{(R^2+z^2)^{3/2}} = -\mathcal{M}R\left[-\frac{3}{2}\frac{2z}{(R^2+z^2)^{5/2}}\right] = \frac{3\mathcal{M}Rz}{(R^2+z^2)^{5/2}}. \tag{C.17}$$

For the vertical component, we have

$$B_z = \frac{\mathcal{M}}{R}\frac{\partial}{\partial R}\frac{R^2}{(R^2+z^2)^{3/2}} = \frac{\mathcal{M}}{R}\left[\frac{2R}{(R^2+z^2)^{3/2}} - \frac{3}{2}\frac{R^2 2R}{(R^2+z^2)^{5/2}}\right] = \tag{C.18}$$

$$= \frac{\mathcal{M}}{(R^2+z^2)^{5/2}}\left[2(R^2+z^2) - 3R^2\right] = \frac{\mathcal{M}}{(R^2+z^2)^{5/2}}\left[2z^2 - R^2\right]. \tag{C.19}$$

Both components were therefore verified.

C.2 Flux function in spherical coordinates

In Fig. 3.12 we plotted the flux function F conveniently in the spherical coordinates, whereas its expression used was derived in the cylindrical coordinates. Following the transforms (C.2) and (C.4) we have

$$F = \mathcal{M}\frac{R^2}{(R^2+z^2)^{3/2}} = \frac{\mathcal{M}r^2\sin^2\vartheta}{(r^2\sin^2\vartheta + r^2\cos^2\vartheta)^{3/2}} = \tag{C.20}$$

$$\frac{\mathcal{M}r^2\sin^2\vartheta}{(r^2)^{3/2}} = \frac{\mathcal{M}}{r}\sin^2\vartheta, \tag{C.21}$$

which constitutes the explicit expression for F in the spherical coordinates.

I may be instructive to show that the approach of the use of the flux function to derive the components of the magnetic induction is applicable regardless the coordinate system. The requirement is that the Gauss law $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ holds. That leads to different expressions to derive the components, namely:

$$B_r = \frac{1}{r^2\sin\vartheta}\frac{\partial F}{\partial\vartheta} = \frac{1}{r^2\sin\vartheta}\frac{\mathcal{M}}{r}\frac{\partial\sin^2\vartheta}{\partial\vartheta} = \frac{2\mathcal{M}}{r^3}\cos\vartheta \tag{C.22}$$

and

$$B_\vartheta = -\frac{1}{r\sin\vartheta}\frac{\partial F}{\partial r} = -\frac{\mathcal{M}\sin^2\vartheta}{r\sin\vartheta}\frac{\partial}{\partial r}\frac{1}{r} = \frac{\mathcal{M}\sin\vartheta}{r}\frac{1}{r^2} = \frac{\mathcal{M}}{r^3}\sin\vartheta. \tag{C.23}$$

These components are consistent with (C.1).