Lecture notes

NAST008: Cosmic electrodynamics

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ToDos

- Liouville theorem
- Eqs. (3.122) to (3.124) to be derived in the appendix.
- Section 4.4
- Relativistic E-B drift in the Appendix fully including the intermediate steps.

Preface

Blah

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Chapter 1

Basics of statistical physics

A physical system, or, more precisely, a mathematical model of a physical system is defined by a certain set of mathematical objects. These, together with their values (in a very general sense) define the state of the system. As an example, let's consider a set of N point masses $(m_1, \ldots, m_N)$ which may be characterised by $2N$ vectors of positions, $\boldsymbol{r}_i$, and momenta, $\boldsymbol{p}_i$, in the framework of classical physics. Such a set of $6N$ numbers may be viewed as a vector, $\boldsymbol{q} \equiv (\boldsymbol{r_1}, \boldsymbol{p_1}, \boldsymbol{r_2}, \boldsymbol{p_2}, ..., \boldsymbol{r_N}, \boldsymbol{p_N})$, in $6N$-dimensional *phase space* and it is usually referred to as *state*.

Proper modelling of the state of a physical system is valuable by itself, nevertheless, more pieces of information are needed if we wish to know how the system evolves in time. This information is carried by equations of motion (again, in a very general sense). In the following, we will usually consider the evolution of the system to be determined by Hamilton's canonical equations, i.e.,

$$\dot{\boldsymbol{r}}_i = \frac{\partial \mathcal{H}(\boldsymbol{q})}{\partial \boldsymbol{p}_i} \qquad \text{and} \qquad \dot{\boldsymbol{p}}_i = -\frac{\partial \mathcal{H}(\boldsymbol{q})}{\partial \boldsymbol{r}_i}, \quad \forall i = 1 \ldots N, \tag{1.1}$$

with $\mathcal{H}(\boldsymbol{q})$ being the *Hamiltonian* of the system.

1.1 Liouville's theorem

Liouville's theorem states that the density of states in the phase space is constant along any evolutionary trajectory provided it is determined by Hamilton's canonical equations of motion. Equivalently, we may consider an arbitrary set of states in the phase space, Ω_0. If each of the states from that set evolve for the same (yet arbitrary) time interval, t, they define another set, Ω_t that specific time, which is schematically sketched in the left panel of Fig. 1.1. Liouville's theorem implies that the volumes of Ω_0 and Ω_t are equal.

As a particular (and probably the simplest possible) example, let's consider a single free point-mass particle which is only allowed to move along the axis x with momentum p_x. The Hamiltonian of this system is $\mathcal{H} = p_x^2/2m$, where m is mass of the particle. The solution of the Hamilton equations of motion is then $x(t) = t\,p_x/m \wedge p_x(t) =$

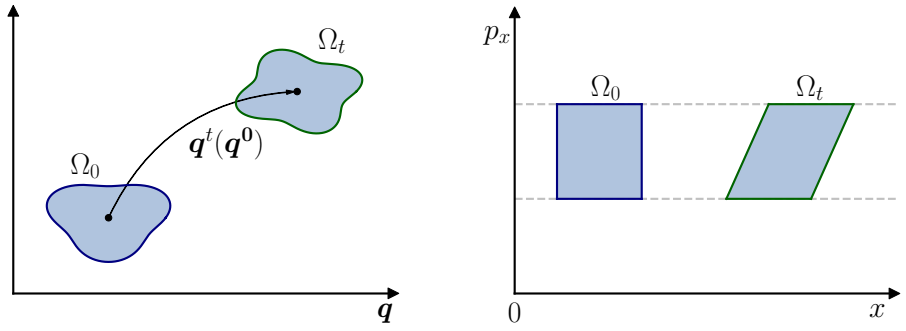

Figure 1.1: Left: Sketch of the time evolution of a set of states in the phase space; $\boldsymbol{q}^t(\boldsymbol{q}_0)$ represents phase space trajectory of one (initial) particular state $\boldsymbol{q}_0$. Right: Evolution of a particular (initially rectangular) set of states of a free particle with two degrees of freedom.

const. This means that the states in the phase space move along lines parallel to the x-axis. Different states with the same value of momentum travel equal distances within an equal time interval t, however, these distances differ between states with different values of momenta. Hence, the rectangular set Ω_0 as depicted in the right panel of Fig. 1.1 transforms to the rhomboid Ω_t with the same base length and height. Either straightforward geometric intuition or simple integration show that their volumes are the same.

In the following, we will provide a proof of Liouville's theorem, i.e., we will show that the volume of the set Ω_t, which equals $\int_{\Omega_t} \mathrm{d}^{6\mathrm{N}}q$, is equal to the volume of Ω_0, i.e.,

$$\int_{\Omega_0} \mathrm{d}^{6\mathrm{N}}q = \int_{\Omega_t} \mathrm{d}^{6\mathrm{N}}q = \int_{\Omega_t} \mathrm{d}^{6\mathrm{N}}q^t \,. \tag{1.2}$$

The second equality in (1.2) can be viewed as just trivial renaming of the integration variables. Next, let us consider the theorem about integration by substitution of variables, which states:

1) Let ϕ be a regular one-to-one mapping from $\mathrm{R_N}$ to $\mathrm{R_N}$ defined on an open set $G \subset \mathrm{R_N}$;
2) let $M \subset \phi(G) \subset \mathrm{R_N}$;
3) let f be a regular mapping $\mathrm{R_N} \to \mathrm{R_1}$ defined almost everywhere on M.

Then,

$$\int_M f(\boldsymbol{y})\mathrm{d}^N\boldsymbol{y} = \int_{\phi^{-1}(M)} f\left(\phi(\boldsymbol{x})\right) \left|\det\frac{\mathrm{D}\phi}{\mathrm{D}x}\right| \mathrm{d}^N\boldsymbol{x} \,. \tag{1.3}$$

For the sake of completeness, we stress that the explicit form of the determinant matrix is

$$\frac{\mathrm{D}\phi}{\mathrm{D}x} = \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_2}{\partial x_1} & \cdots & \frac{\partial\phi_N}{\partial x_1} \\ \frac{\partial\phi_1}{\partial x_2} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \\ \vdots & & & \ddots \end{bmatrix} . \tag{1.4}$$

We will apply (1.3) to the right-hand side of equation (1.2) with $f \equiv 1$, $\boldsymbol{y} \to \boldsymbol{q}^t$, $\boldsymbol{x} \to \boldsymbol{q}^0$, $\phi(\boldsymbol{x}) \to \boldsymbol{q}^t(q)$. The mapping ϕ^t is such that it maps the initial state $\boldsymbol{q}^0$ into the new state $\boldsymbol{q}^t$ for each t according to the Hamiltonian equations of motion. Note that the mathematical meaning of $\boldsymbol{q}^t$ is ambiguous here. When it describes a particular state in the phase space, it represents the coordinates. On the other hand, when it describes the mapping from one state to another according to the equations of motion, it represents a function $\boldsymbol{q}^t(\boldsymbol{q}^0; t)$, where t is then a parameter of this function. That is because the Hamiltonian may also be viewed as a recipe for the transformation of coordinates between the states. Finally, $M = \Omega_t$, $\phi^{-1}(M) = \Omega_0$, and $N \to 6N$, which together yields

$$\int_{\Omega_t} \mathrm{d}^{6\mathrm{N}}\boldsymbol{q} = \int_{\Omega_t} \mathrm{d}^{6\mathrm{N}}\boldsymbol{q}^t = \int_{\Omega_0} \left|\det \frac{\mathrm{D}q^t}{\mathrm{D}q^0}\right| \mathrm{d}^{6N}\boldsymbol{q}^{\boldsymbol{0}} \,. \tag{1.5}$$

Consequently, we want to prove that

$$\int_{\Omega_t} 1\mathrm{d}^{6\mathrm{N}}\boldsymbol{q} = \int_{\Omega_0} 1\mathrm{d}^{6\mathrm{N}}\boldsymbol{q} \,. \tag{1.6}$$

Let us further denote

$$J^t \equiv \left|\det \frac{\mathrm{D}q^t}{\mathrm{D}q^0}\right| \,, \tag{1.7}$$

which is the Jacobian of the transformation $\boldsymbol{q} \to \boldsymbol{q}^t$. It is apparent that Liouville's theorem is valid provided $J^t = 1$. In practise, we will show that $\mathrm{d}J^t/\mathrm{d}t = 0$ which, together with the trivial identity $J^0 \equiv J^{t=0} = 1$, implies the required property of the Jacobian. Let us denote the element of the Jacobi matrix as $J^t_{ij} \equiv \partial q^t_i / \partial q^0_j$. Its time derivative may be expressed as

$$\frac{\mathrm{d}J^t_{ij}}{\mathrm{d}t} = \frac{\partial}{\partial q^0_j}\frac{\mathrm{d}q^t_i}{\mathrm{d}t} = \frac{\partial \dot{q}^t_i}{\partial q^0_j} = \sum_{k=1}^{6N} \frac{\partial \dot{q}^t_i}{\partial q^t_k}\frac{\partial q^t_k}{\partial q^0_j} = \sum_{k=1}^{6N} \frac{\partial \dot{q}^t_i}{\partial q^t_k} J^t_{kj} \,, \tag{1.8}$$

where we denote $\dot{q}^t_i \equiv \mathrm{d}q^t_i/\mathrm{d}t$. Here we used

$$J^t_{kj} = \frac{\partial q^t_k}{\partial q^0_j}. \tag{1.9}$$

In equation (1.8) we have exchanged order of derivatives with respect to t and q_j which is possible provided these two variables are continuous. The time derivative of the Jacobian J^t can be written with the help of eq. (1.8):

$$\frac{\mathrm{d}J^t}{\mathrm{d}t} = \sum_{i,j} \frac{\partial J^t}{\partial J^t_{ij}}\frac{\mathrm{d}J^t_{ij}}{\mathrm{d}t} = \sum_{i,j,k} \frac{\partial J^t}{\partial J^t_{ij}} J^t_{kj}\frac{\partial \dot{q}^t_i}{\partial q^t_k} \,. \tag{1.10}$$

Equation (1.10) can be simplified due to the identity (see Box 1.1)

$$\sum_j \frac{\partial J^t}{\partial J^t_{ij}} J^t_{kj} = \delta_{ik} J^t \tag{1.11}$$

to

$$\frac{\mathrm{d}J^t}{\mathrm{d}t} = J^t \sum_{i=1}^{6N} \frac{\partial \dot{q}_i^t}{\partial q_i^t} \,. \tag{1.12}$$

Finally, let us rewrite (1.12) in terms of the canonical coordinates and momenta and consider Hamilton's equations to evaluate $\dot{r}_i^t$ and $\dot{p}_i^t$:

$$\frac{\mathrm{d}J^t}{\mathrm{d}t} = J^t \sum_{i=1}^{3\mathrm{N}} \left(\frac{\partial \dot{r}_i^t}{\partial r_i^t} + \frac{\partial \dot{p}_i^t}{\partial p_i^t} \right) = J^t \sum_{i=1}^{3\mathrm{N}} \left(\frac{\partial^2 \mathcal{H}}{\partial r_i^t \partial p_i^t} - \frac{\partial^2 \mathcal{H}}{\partial p_i^t \partial r_i^t} \right) \tag{1.13}$$

When we assume that we may switch the order of partial derivatives of $\mathcal{H}$, the value of the expression above vanishes. Hence $J^t = 1$ for each t and Liouville's theorem holds.

(Add example of non-Hamiltonian system (in which case different initial states may converge to common final state).)

1.2 Statistical description of a physical system

The probability density is constant along the evolutionary track of each state through the phase space, i.e. the probability of being realised (found/measured) is conserved in time for each particular state. This, together with the assumptions required for Liouville's theorem to be valid (i.e. Hamiltonian nature of the system) allows us to formulate Liouville's equation for the temporal evolution of the distribution function D_N.

Let's assume an arbitrary *fixed* subset, Ω, of the phase space. The probability of finding the system in some state in Ω is $\mathcal{P}(\Omega) = \int_\Omega D_\mathrm{N} \mathrm{d}\omega$. This property implies the *normalisation* of the distribution function bearing the usual form

$$\int_{-\infty}^{+\infty} D_\mathrm{N} \mathrm{d}\omega = \mathcal{C}, \tag{1.18}$$

where $\mathcal{C}$ is a scalar value. Taking into account the previous paragraph, the natural selection dictates $\mathcal{C} = 1$. The probabilistic interpretation states that if subset Ω covers the whole phase space, the corresponding probability obtained from the distribution function should be 1 (the certainty).

Liouville's theorem states that the fluid of states is incompressible, hence, the only way how the probability $P(\Omega)$ can change is due to the flow of the states through the boundary of Ω, i.e.:

$$\frac{\partial}{\partial t} \int_\Omega D_\mathrm{N} \, \mathrm{d}\omega = - \oint_{\eth\Omega} D_\mathrm{N} \, \dot{\boldsymbol{q}} \cdot \mathrm{d}\boldsymbol{\Sigma} \tag{1.19}$$

where $\mathrm{d}\boldsymbol{\Sigma}$ is an (outward) oriented element of the surface $\eth\Omega$. The minus sign means that the probability decreases when the state flows parallel to $\mathrm{d}\boldsymbol{\Sigma}$, i.e. out from Ω. Provided some physically plausible mathematical conditions are fulfilled (provide an exact formulation of these conditions), we may exchange derivative and integration

Let $B = A^{-1}$ be inverse matrix of A. Then the elements b_{ij} may be computed by using the adjoint (or adjugate[1]) matrix

$$B = \frac{1}{\det A}\mathrm{adj}A, \tag{1.14}$$

defined by

$$(\mathrm{adj}A)^{ij} \equiv (-1)^{i+j}\det(A^{ji}). \tag{1.15}$$

Then the inverse elements are given by

$$b_{ij} = \frac{(-1)^{i+j}}{a}\det\left(A^{ji}\right),$$

where $a \equiv \det A$ indicates the matrix transpose operator. The notation A^{ij} represents the submatrix obtained from matrix A by excluding ith row and jth column.
The determinant a of matrix A may be computed using the expansion as

$$a = \sum_i (-1)^{i+j} a_{ij} \det A^{ij}.$$

Then

$$\frac{\partial a}{\partial a_{ij}} = (-1)^{i+j}\det A^{ij} = a b^{ji}. \tag{1.16}$$

Hence

$$\sum_j \frac{\partial a}{\partial a_{ij}} a_{kj} = \sum_j a b^{ji} a_{kj} = a \sum_j b^{ji} a_{kj} = a(BA)_{ki} = a\delta_{ik}, \tag{1.17}$$

Q.E.D.

Box 1.1: Useful algebraic identity to prove the Liouville's theorem.

operators on the left-hand side of eq. (1.19); on the right-hand side, we apply Gauss theorem, yielding

$$\int_\Omega \frac{\partial D_\mathrm{N}}{\partial t}\,\mathrm{d}\omega = -\int_\Omega \nabla\cdot(D_\mathrm{N}\dot{\boldsymbol{q}})\,\mathrm{d}\omega\ . \tag{1.20}$$

Integral equality (1.20) holds for an arbitrary set Ω which implies the equality of integrands. Therefore, we obtain a continuity equation for the distribution function,

$$\frac{\partial D_\mathrm{N}}{\partial t} = -\nabla\cdot(D_\mathrm{N}\,\dot{\boldsymbol{q}})\ . \tag{1.21}$$

This equation represents the "continuity equation" for the "fluid of states" and implies that the state keeps its D_N while evolving in time according to the Hamiltonian equations. The relation can be further rewritten as

$$\frac{\partial D_\mathrm{N}}{\partial t} = -\sum_{i=1}^{6N}\left[\frac{\partial D_\mathrm{N}}{\partial q_i}\dot{q}_i + D_\mathrm{N}\frac{\partial \dot{q}_i}{\partial q_i}\right] = -\sum_{i=1}^{3N}\left[\frac{\partial D_\mathrm{N}}{\partial r_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial D_\mathrm{N}}{\partial p_i}\frac{\partial \mathcal{H}}{\partial r_i} + D_\mathrm{N}\left(\frac{\partial \mathcal{H}}{\partial r_i\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i\partial r_i}\right)\right]\ . \tag{1.22}$$

The terms in () brackets cancel out provided $\mathcal{H}$ is continuously differentiable with respect to r_i and p_i. Using the Poisson brackets notation, the *Liouville equation* is usually written as:

$$\frac{\partial D_\mathrm{N}}{\partial t} = \{\mathcal{H}, D_\mathrm{N}\}_\mathrm{P}\ . \tag{1.23}$$

(Mention the analogous von Neumann equation from the quantum theory.)

1.3 Reduced distribution functions

Distribution function D_N holds the maximum information of a particular (usually very complex) physical system. This maximum information may often be more than what we need. In some circumstances, it is convenient to define reduced distribution functions (for details see Box 1.2):

$$f(\boldsymbol{r},\boldsymbol{p}) \equiv \int D_\mathrm{N}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{r}_2,\boldsymbol{p}_2,\ldots,\boldsymbol{r}_\mathrm{N},\boldsymbol{p}_\mathrm{N})\,\mathrm{d}\tau_{\mathrm{N}-1}\ , \tag{1.24}$$

where

$$\mathrm{d}\tau_{\mathrm{N}-1} \equiv \frac{\mathrm{d}^3\boldsymbol{r}_2\mathrm{d}^3\boldsymbol{p}_2\ldots\mathrm{d}^3\boldsymbol{r}_\mathrm{N}\mathrm{d}^3\boldsymbol{p}_\mathrm{N}}{(\mathrm{N}-1)!} \tag{1.25}$$

and the integration is taken over the whole definition space of variables $\boldsymbol{r}_2\ldots\boldsymbol{p}_\mathrm{N}$. Definition (1.24) generally gives different results when different pairs $(\boldsymbol{r}_i,\boldsymbol{p}_i)$ are excluded from integration (which means that the result should keep some mark according to the excluded particle). Definition (1.24), however, is usually considered for specific systems of *indistinguishable* particles which imply that D_N as well as $\mathcal{H}$ are symmetric with respect to permutations of individual particles (exchange of pairs of $(\boldsymbol{r}_i,\boldsymbol{p}_i)$ and $(\boldsymbol{r}_j,\boldsymbol{p}_j)$). For these systems, integration (1.24) gives identical functions regardless of over which $\mathrm{N}-1$ pairs of coordinates and momenta it is performed. Function $f(\boldsymbol{r},\boldsymbol{p})$ is usually

Let us comment on different normalisations of the reduced distribution functions. The distribution function D_{N} holds the maximum information of a particular physical system, which is often not necessary. For instance, we may only be interested in the probability density of finding the particle carrying tag 1 at $\boldsymbol{r}_1$ with the momentum $\boldsymbol{p}_1$, where we do not need any information about the remaining particles. Then we may integrate D_{N} over the definition space of all the other particles yielding

$$f^1(\boldsymbol{r}_1,\boldsymbol{p}_1)=\int D_{\mathrm{N}}(\boldsymbol{r}_1,\boldsymbol{p}_1,\ldots,\boldsymbol{r}_N,\boldsymbol{p}_N)\mathrm{d}^3\boldsymbol{r}_2\mathrm{d}^3\boldsymbol{p}_2\ldots\mathrm{d}^3\boldsymbol{r}_N\mathrm{d}^3\boldsymbol{p}_N.$$

Function f^1 is in a good sense a one-particle distribution function. A similar reduction may be written for the particle carrying tag 2:

$$f^2(\boldsymbol{r}_2,\boldsymbol{p}_2)=\int D_{\mathrm{N}}(\boldsymbol{r}_1,\boldsymbol{p}_1,\ldots,\boldsymbol{r}_N,\boldsymbol{p}_N)\mathrm{d}^3\boldsymbol{r}_1\mathrm{d}^3\boldsymbol{p}_1\mathrm{d}^3\boldsymbol{r}_3\mathrm{d}^3\boldsymbol{p}_3\ldots\mathrm{d}^3\boldsymbol{r}_N\mathrm{d}^3\boldsymbol{p}_N$$

and so forth. As long as the particles are distinguishable, functions f^1, f^2, ... may in principle be different. Note that $f^1\ldots$ functions are each normalised to unity.
In the above given examples, we may always formally rename the considered particle so that its position and momentum is formally written without the subscript, hence $(\boldsymbol{r},\boldsymbol{p})$. Still, generally, $f^1(\boldsymbol{r},\boldsymbol{p})\neq f^2(\boldsymbol{r},\boldsymbol{p})$. They are identical only in the case of the indistinguishable particles.
Now, in the case we are interested in a probability density $f(\boldsymbol{r},\boldsymbol{p})$ of finding *any* indistinguishable particle at $(\boldsymbol{r},\boldsymbol{p})$ we must sum the probabilities of finding particle 1, particle 2,... at $(\boldsymbol{r},\boldsymbol{p})$. Hence, for example,

$$f(\boldsymbol{r},\boldsymbol{p})=\sum_{i=1}^{N}f^i(\boldsymbol{r},\boldsymbol{p})=Nf^1(\boldsymbol{r},\boldsymbol{p})=Nf^2(\boldsymbol{r},\boldsymbol{p})=\ldots.$$

Note that f is now normalised to N. This is not convenient for the interpretation. Hence we may naturally re-normalise D_{N} to obtain f which is normalised to unity. When integrating from D_{N} we get

$$\begin{aligned}f(\boldsymbol{r},\boldsymbol{p}) &= N\int D_{\mathrm{N}}(\boldsymbol{r},\boldsymbol{p},\ldots,\boldsymbol{r}_N,\boldsymbol{p}_N)\mathrm{d}^3\boldsymbol{r}_2\mathrm{d}^3\boldsymbol{p}_2\ldots\mathrm{d}^3\boldsymbol{r}_N\mathrm{d}^3\boldsymbol{p}_N=\\ &= N\int\frac{D'_{\mathrm{N}}}{N!}\mathrm{d}^3\boldsymbol{r}_2\mathrm{d}^3\boldsymbol{p}_2\ldots\mathrm{d}^3\boldsymbol{r}_N\mathrm{d}^3\boldsymbol{p}_N=\int D'_{\mathrm{N}}\frac{\mathrm{d}^3\boldsymbol{r}_2\mathrm{d}^3\boldsymbol{p}_2\ldots\mathrm{d}^3\boldsymbol{r}_N\mathrm{d}^3\boldsymbol{p}_N}{(N-1)!},\end{aligned}$$

where D'_{N} has a different normalisation (to $N!$) and we naturally see the definition of the new volume element $\mathrm{d}\tau_{N-1}$. In the following, we drop the $'$ for simplicity.

Box 1.2: On the normalisation of the distribution functions.

called the single-particle distribution function and it gives the probability density of finding *any* particle at position $\boldsymbol{r}$ with momentum $\boldsymbol{p}$. Note that the single-particle distribution function really holds considerably reduced information with respect to D_{N}. We could add that to the meaning of $f(\boldsymbol{r},\boldsymbol{p})$ that "... and we have no information about the positions and momenta of the other particles." The single-particle distribution function holds a sufficient amount of information to calculate the mean values of physical quantities which may be written in the form:

$$A(\boldsymbol{r}_1,\boldsymbol{p}_1,\ldots,\boldsymbol{r}_{\mathrm{N}},\boldsymbol{p}_{\mathrm{N}}) = \sum_{i=1}^{\mathrm{N}} a(\boldsymbol{r}_i,\boldsymbol{p}_i)\ , \tag{1.26}$$

i.e. it is a sum of N identical functions. The mean value is then

$$\langle A\rangle \equiv \int A\, D_{\mathrm{N}}\,\mathrm{d}\tau_{\mathrm{N}} = \int a(\boldsymbol{r},\boldsymbol{p})\, f(\boldsymbol{r},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}\ . \tag{1.27}$$

Similarly to (1.24) we may define the "two-particle distribution function,"

$$f_2(\boldsymbol{r}_1,\boldsymbol{p}_1,\boldsymbol{r}_2,\boldsymbol{p}_2) \equiv \int D_{\mathrm{N}}\,\mathrm{d}\tau_{\mathrm{N}-2}\ , \tag{1.28}$$

which gives the probability density of finding *some* particle at position $\boldsymbol{r}_1$ with momentum $\boldsymbol{p}_1$ and *any other* particle at position $\boldsymbol{r}_2$ with momentum $\boldsymbol{p}_2$. Mean values of physical quantities that depend on combinations of positions and momenta of two particles,

$$B(\boldsymbol{r}_1,\boldsymbol{p}_1,\ldots,\boldsymbol{r}_{\mathrm{N}},\boldsymbol{p}_{\mathrm{N}}) = \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} b(\boldsymbol{r}_i,\boldsymbol{p}_i,\boldsymbol{r}_j,\boldsymbol{p}_j)\ , \tag{1.29}$$

can be calculated through the integration of the two-particle distribution function:

$$\langle B\rangle = \int b(\boldsymbol{r}_1,\boldsymbol{p}_1,\boldsymbol{r}_2,\boldsymbol{p}_2)\, f_2(\boldsymbol{r}_1,\boldsymbol{p}_1,\boldsymbol{r}_2,\boldsymbol{p}_2)\,\mathrm{d}^3\boldsymbol{r}_1\,\mathrm{d}^3\boldsymbol{p}_1\mathrm{d}^3\boldsymbol{r}_2\,\mathrm{d}^3\boldsymbol{p}_2\ . \tag{1.30}$$

An important example of a physical quantity which often takes the form of (1.26) or (1.29) or combination of both is the total energy of the system which is defined by the Hamiltonian. Let's introduce its specific (i.e. not general!) form:

$$\mathcal{H} = \sum_{i=1}^{\mathrm{N}} \frac{\boldsymbol{p}_i^2}{2m} + \sum_{i=1}^{\mathrm{N}} V(\boldsymbol{r}_i) + \sum_{i=1}^{\mathrm{N}}\sum_{\substack{j=1\\ j\neq i}}^{\mathrm{N}} w(\boldsymbol{r}_i,\boldsymbol{r}_j)\ . \tag{1.31}$$

Here, the first sum corresponds to a system of free particles; the first plus the second sum defines a system of non-interacting particles (of equal mass, m, which is an inevitable consequence of their indistinguishability) in an external potential $V(\boldsymbol{r})$. The third term is a possible form of the interaction term. The mean value of the energy of the statistically described system determined by the Hamiltonian (1.31) is

$$\langle\mathcal{H}\rangle = \int h(\boldsymbol{r},\boldsymbol{p})\, f(\boldsymbol{r},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p} + \int w(\boldsymbol{r}_1,\boldsymbol{r}_2)\, f_2(\boldsymbol{r}_1,\boldsymbol{p}_1,\boldsymbol{r}_2,\boldsymbol{p}_2)\,\mathrm{d}^3\boldsymbol{r}_1\,\mathrm{d}^3\boldsymbol{p}_1\,\mathrm{d}^3\boldsymbol{r}_2\,\mathrm{d}^3\boldsymbol{p}_2\ , \tag{1.32}$$

where we denote

$$h(\boldsymbol{r},\boldsymbol{p}) \equiv \frac{p^2}{2m} + V(\boldsymbol{r}) \,. \tag{1.33}$$

In the following, we will use a particular example of the interaction term which describes mutual gravitational interaction among particles:

$$w(\boldsymbol{r}_i,\boldsymbol{r}_j) = -\frac{1}{2}\frac{Gm^2}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \tag{1.34}$$

with G representing the gravitational constant and the factor of $1/2$ is to respect the symmetry of the gravitational force.

1.4 Boltzmann equation

Similarly to Liouville's equation (1.23) which defines the evolution of the N-particle distribution function, D_N, we may derive a partial differential equation for the evolution of the single-particle distribution function, $f(\boldsymbol{r},\boldsymbol{p})$. The procedure lies in the integration of (1.23) over the whole definition space of positions and momenta of all but one particles. More precisely, let's define $\boldsymbol{r} \equiv \boldsymbol{r}_1$ and $\boldsymbol{p} \equiv \boldsymbol{p}_1$ and perform the integration over all coordinates and momenta except for $\boldsymbol{r}$ and $\boldsymbol{p}$:

$$\int \frac{\partial D_\mathrm{N}}{\partial t}\,\mathrm{d}\tau_{\mathrm{N}-1} = \int \sum_{i=1}^{\mathrm{N}} \left(\frac{\partial D_\mathrm{N}}{\partial \boldsymbol{p}_i}\cdot\frac{\partial h}{\partial \boldsymbol{r}_i} - \frac{\partial D_\mathrm{N}}{\partial \boldsymbol{r}_i}\cdot\frac{\partial h}{\partial \boldsymbol{p}_i} \right) \mathrm{d}\tau_{\mathrm{N}-1} + \int \sum_{i,j} \frac{\partial w(\boldsymbol{r}_i,\boldsymbol{r}_j)}{\partial \boldsymbol{r}_i}\cdot\frac{\partial D_\mathrm{N}}{\partial \boldsymbol{p}_i}\,\mathrm{d}\tau_{\mathrm{N}-1} \,. \tag{1.35}$$

Note that when inserting (1.31) into (1.22), the first term of (1.35) should contain a double sum and the second term even a triple sum. However, according to (1.22) a derivative of the Hamiltonian is taken with respect to the generalised coordinates, which only keeps certain terms corresponding to the considered r_i or p_i from the sum in (1.31), thereby effectively removing one summation from both terms in (1.35).

Let's first consider the left-hand side of equation (1.35). Provided the conditions of the theorem of the derivative of integral with respect to the parameter are fulfilled (put them down explicitly here?), we may write

$$\int \frac{\partial D_\mathrm{N}}{\partial t}\,\mathrm{d}\tau_{\mathrm{N}-1} = \frac{\partial}{\partial t}\int D_\mathrm{N}\,\mathrm{d}\tau_{\mathrm{N}-1} = \frac{\partial f(\boldsymbol{r},\boldsymbol{p})}{\partial t} \,. \tag{1.36}$$

Considering the second term in (1.35), we first split it into three parts ($i = 1$, $i = 2$ and

$i > 2$):

$$\int \sum_{i=1}^{\mathrm{N}} \left(\frac{\partial D_\mathrm{N}}{\partial \boldsymbol{p}_i} \cdot \frac{\partial h}{\partial \boldsymbol{r}_i} - \frac{\partial D_\mathrm{N}}{\partial \boldsymbol{r}_i} \cdot \frac{\partial h}{\partial \boldsymbol{p}_i} \right) \mathrm{d}\tau_{\mathrm{N}-1} = \int \frac{\partial D_\mathrm{N}}{\partial \boldsymbol{p}} \cdot \frac{\partial h}{\partial \boldsymbol{r}} \, \mathrm{d}\tau_{\mathrm{N}-1} - \int \frac{\partial D_\mathrm{N}}{\partial \boldsymbol{r}} \cdot \frac{\partial h}{\partial \boldsymbol{p}} \, \mathrm{d}\tau_{\mathrm{N}-1} + \int \left(\frac{\partial D_\mathrm{N}}{\partial \boldsymbol{p}_2} \cdot \frac{\partial h}{\partial \boldsymbol{r}_2} - \frac{\partial D_\mathrm{N}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial h}{\partial \boldsymbol{p}_2} \right) \mathrm{d}\tau_{\mathrm{N}-1} + \int \sum_{i=3}^{\mathrm{N}} \left(\frac{\partial D_\mathrm{N}}{\partial \boldsymbol{p}_i} \cdot \frac{\partial h}{\partial \boldsymbol{r}_i} - \frac{\partial D_\mathrm{N}}{\partial \boldsymbol{r}_i} \cdot \frac{\partial h}{\partial \boldsymbol{p}_i} \right) \mathrm{d}\tau_{\mathrm{N}-1} \, . \tag{1.37}$$

Two specific and very important properties of the first two terms on the right-hand side of eq. (1.37) are that first, h is a function of just $\boldsymbol{r}$ and $\boldsymbol{p}$ and second, the partial derivation is with respect to these quantities, over which we do not integrate. Hence, we may put the terms $\partial h/\partial \boldsymbol{r}$ and $\partial h/\partial \boldsymbol{p}$ in front of the integrals and, under the assumptions of the derivative of an integral with respect to a parameter, we may write:

$$\int \frac{\partial D_\mathrm{N}}{\partial \boldsymbol{p}} \cdot \frac{\partial h}{\partial \boldsymbol{r}} \, \mathrm{d}\tau_{\mathrm{N}-1} - \int \frac{\partial D_\mathrm{N}}{\partial \boldsymbol{r}} \cdot \frac{\partial h}{\partial \boldsymbol{p}} \, \mathrm{d}\tau_{\mathrm{N}-1} = \frac{\partial h}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} - \frac{\partial h}{\partial \boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} \, . \tag{1.38}$$

Finally, we will show that the second (and analogically any subsequent) term of the sum in the third term on the right-hand side of the equation is zero. Let's reorganise the differentials in $\mathrm{d}\tau_{\mathrm{N}-1}$ and evaluate a sub-integral

$$I_1 \equiv \int \int \left(\frac{\partial D_\mathrm{N}}{\partial \boldsymbol{p}_2} \cdot \frac{\partial h}{\partial \boldsymbol{r}_2} - \frac{\partial D_\mathrm{N}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial h}{\partial \boldsymbol{p}_2} \right) \mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 \, . \tag{1.39}$$

Performing integration per partes with respect to variables $\boldsymbol{p}_2$ and $\boldsymbol{r}_2$ in the first and second term, respectively, we will get

$$I_1 = \int \left[D_\mathrm{N} \frac{\partial h}{\partial \boldsymbol{r}_2} \right]_{\boldsymbol{p}_2 \to \infty} \mathrm{d}^3\boldsymbol{r}_2 - \int \int D_\mathrm{N} \frac{\partial h}{\partial \boldsymbol{r}_2 \, \partial \boldsymbol{p}_2} \, \mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 - \int \left[D_\mathrm{N} \frac{\partial h}{\partial \boldsymbol{p}_2} \right]_{\boldsymbol{r}_2 \to \infty} \mathrm{d}^3\boldsymbol{p}_2 + \int \int D_\mathrm{N} \frac{\partial h}{\partial \boldsymbol{p}_2 \, \partial \boldsymbol{r}_2} \, \mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 \tag{1.40}$$

The notation of individual terms in (1.40) is somewhat fuzzy. More precisely, one should perform the integration over the components of vectors $\boldsymbol{r}_2$ and $\boldsymbol{p}_2$. In the first and third term on the right-hand side of (1.40), the argument of squared brackets is to be evaluated in an appropriate infinity. Hence, these terms are both zero as D_N goes to zero at infinity, which is a necessary condition for the normalisation (1.18) to be fulfilled. The second and fourth term in (1.40) cancel out provided the function $h(\boldsymbol{r}, \boldsymbol{p})$ is smoothly differentiable with respect to both arguments (which is a condition for the possible exchange of the order of differentiation).

Each individual part of the sum in the third term on the right-hand side of Eq. (1.37) gives zero after we manipulate it analogically as above. Equation (1.35) then reduces

to

$$\frac{\partial f}{\partial t}+\frac{\partial h}{\partial \boldsymbol{p}}\cdot\frac{\partial f}{\partial \boldsymbol{r}}-\frac{\partial h}{\partial \boldsymbol{r}}\cdot\frac{\partial f}{\partial \boldsymbol{p}}=\int\sum_{i,j}\frac{\partial w(\boldsymbol{r}_i,\boldsymbol{r}_j)}{\partial \boldsymbol{r}_i}\cdot\frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}_i}\,\mathrm{d}\tau_{\mathrm{N}-1}\,. \tag{1.41}$$

The right-hand side of Eq. (1.41) could be simplified by using a similar approach when dealing with (1.37). It can be easily shown that only $\mathrm{N}-1$ terms with $i=1$ are non-zero and they are all identical due to the symmetries of D_{N}. With an alternate notation $\boldsymbol{r}'\equiv\boldsymbol{r}_2$ and $\boldsymbol{p}'\equiv\boldsymbol{p}_2$ we write

$$\begin{aligned}\frac{\partial f}{\partial t}+\frac{\partial h}{\partial \boldsymbol{p}}\cdot\frac{\partial f}{\partial \boldsymbol{r}}-\frac{\partial h}{\partial \boldsymbol{r}}\cdot\frac{\partial f}{\partial \boldsymbol{p}} &= (\mathrm{N}-1)\int\frac{\partial w(\boldsymbol{r},\boldsymbol{r}')}{\partial \boldsymbol{r}}\cdot\frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}}\,\frac{\mathrm{d}^3\boldsymbol{r}'\mathrm{d}^3\boldsymbol{p}'\mathrm{d}^3\boldsymbol{r}_3\mathrm{d}^3\boldsymbol{p}_3\ldots\mathrm{d}^3\boldsymbol{r}_{\mathrm{N}}\mathrm{d}^3\boldsymbol{p}_{\mathrm{N}}}{(\mathrm{N}-1)!}\\ &= \int\mathrm{d}^3\boldsymbol{r}'\mathrm{d}^3\boldsymbol{p}'\,\frac{\partial w(\boldsymbol{r},\boldsymbol{r}')}{\partial \boldsymbol{r}}\cdot\int\frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}}\,\mathrm{d}\tau_{\mathrm{N}-2}\\ &= \int\frac{\partial w(\boldsymbol{r},\boldsymbol{r}')}{\partial \boldsymbol{r}}\cdot\frac{\partial f_2(\boldsymbol{r},\boldsymbol{p},\boldsymbol{r}',\boldsymbol{p}')}{\partial \boldsymbol{p}}\,\mathrm{d}^3\boldsymbol{r}'\mathrm{d}^3\boldsymbol{p}'\,.\end{aligned} \tag{1.42}$$

In the last step we, again, assumed the it was possible to exchange the order of integration and differentiation with respect to $\boldsymbol{p}$. The right-hand side of eq. (1.42) covers the mutual interaction of the particles and is usually called the collision term, being denoted as $(\partial f/\partial t)_{\mathrm{coll}}$. Using the Hamilton canonical equations, we may write one of several "standard" forms of the *Boltzmann equation*:

$$\frac{\partial f}{\partial t}+\frac{\partial h}{\partial \boldsymbol{p}}\cdot\frac{\partial f}{\partial \boldsymbol{r}}-\frac{\partial h}{\partial \boldsymbol{r}}\cdot\frac{\partial f}{\partial \boldsymbol{p}}=\left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}}\,. \tag{1.43}$$

Keep in mind that this is just a notation and the collision term still involves the integral of the two-body distribution function. This is a principial problem which makes it impossible to solve the equation. There are several ways how to overcome this obstacle.

1.4.1 BBGKY hierarchy

Deriving the partial differential equation the for two-body distribution function, f_2, in a similar way, i.e. integrating the Liouville equation over $6(\mathrm{N}-2)$ variables may seem to be a straightforward way how to make the system of equations closed. Unfortunately, the problem would reappear – in the equation for f_2 we would obtain an integral term including the three-body distribution function. The procedure may be repeated again and again, leading to a closed set of $N-1$ equations. This set is called the BBGKY hierarchy (after scientists who independently studied this problem: Bogolyubov, Born, Green, Kirkwood and Yvon) but in its completeness, it is no more suitable for solving than the Liouville equation.

A general class of approximations with an intent to make a smaller set of equations closed lies in the assumption that the higher-order distribution function can be written as a combination of lower-order distribution functions, e.g. $f_2(\boldsymbol{r},\boldsymbol{p},\boldsymbol{r}',\boldsymbol{p}')=f(\boldsymbol{r},\boldsymbol{p})\,f(\boldsymbol{r}',\boldsymbol{p}')$.

1.4.2 Vlasov approximation

The above presented approximation is often used and is the basis of the Vlasov equation. Let's start without the approximation and rewrite f_2 in the following way:

$$f_2(\boldsymbol{r},\boldsymbol{p},\boldsymbol{r}',\boldsymbol{p}') = f(\boldsymbol{r},\boldsymbol{p})\, f(\boldsymbol{r}',\boldsymbol{p}') + g_2(\boldsymbol{r},\boldsymbol{p},\boldsymbol{r}',\boldsymbol{p}') \,. \tag{1.44}$$

The first term on the right-hand side of formula (1.44), a product of two single particle distribution functions, represents the probablility density of two uncorrelated events (finding one particle at position $\boldsymbol{r}$ with momentum $\boldsymbol{p}$ and another one at position $\boldsymbol{r}'$ with momentum $\boldsymbol{p}'$. Function g_2 is then a correlation function which gives the probablity density of finding one particle at position $\boldsymbol{r}$ with momentum $\boldsymbol{p}$ *due to* finding another particle at position $\boldsymbol{r}'$ and momentum $\boldsymbol{p}'$ and vice versa.

The Vlasov approximation lies in an assumption that the correlation function, g_2 may be neglected with respect to the uncorrelated probability density, i.e., $f_2(\boldsymbol{r},\boldsymbol{p},\boldsymbol{r}',\boldsymbol{p}') = f(\boldsymbol{r},\boldsymbol{p})\, f(\boldsymbol{r}',\boldsymbol{p}')$. Then, the collision term in the Boltzmann's equation (1.43) reads,

$$\begin{aligned}\left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}} &= \int \frac{\partial w(\boldsymbol{r},\boldsymbol{r}')}{\partial \boldsymbol{r}} \cdot \frac{\partial f(\boldsymbol{r},\boldsymbol{p})}{\partial \boldsymbol{p}}\, f(\boldsymbol{r}',\boldsymbol{p}')\, \mathrm{d}^3\boldsymbol{r}'\mathrm{d}^3\boldsymbol{p}' \;= \\ &= \frac{\partial f(\boldsymbol{r},\boldsymbol{p})}{\partial \boldsymbol{p}} \cdot \frac{\partial}{\partial \boldsymbol{r}} \int w(\boldsymbol{r},\boldsymbol{r}')\, f(\boldsymbol{r}',\boldsymbol{p}')\, \mathrm{d}^3\boldsymbol{r}'\mathrm{d}^3\boldsymbol{p}' \,. \end{aligned} \tag{1.45}$$

Here we put the term that does not depend on the integration variable in front of the integral and, similarly to several cases above, we assumed that the order of integration and differentiation with respect to parameter $\boldsymbol{r}$ may be mutually exchanged. The integral in (1.45) is, by definition, the mean value of quantity $w(\boldsymbol{r},\boldsymbol{r}')$. In the case of example (1.34), this represents the mean value of gravitational potential energy of the system of N particles distributed according to the distribution function $f(\boldsymbol{r}',\boldsymbol{p}')$ at (arbitrary) position $\boldsymbol{r}$. Let us denote

$$\Phi(\boldsymbol{r}) \equiv \langle W\rangle(\boldsymbol{r}) = \int w(\boldsymbol{r},\boldsymbol{r}')\, f(\boldsymbol{r}',\boldsymbol{p}')\, \mathrm{d}^3\boldsymbol{r}'\mathrm{d}^3\boldsymbol{p}' = -\int \frac{Gm^2}{|\boldsymbol{r}-\boldsymbol{r}'|} f(\boldsymbol{r}',\boldsymbol{p}')\, \mathrm{d}^3\boldsymbol{r}'\mathrm{d}^3\boldsymbol{p}' \tag{1.46}$$

and, consequently, we may rewrite Boltzmann's equation (1.43) in the Vlasov approximation:

$$\frac{\partial f}{\partial t} + \frac{\partial h}{\partial \boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \boldsymbol{\nabla}\left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0 \,, \tag{1.47}$$

which will be called Vlasov's equation hereafter.

For the case of the system described by Hamiltonian (1.31) with interaction term (1.34), we have $\boldsymbol{v} \equiv \dot{\boldsymbol{r}} = \boldsymbol{p}/m$, which allows us to rewrite Vlasov's equation in the form

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \tfrac{1}{m}\boldsymbol{\nabla}\left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0 \,. \tag{1.48}$$

Above, we considered the distribution function f to be an explicit function of $\boldsymbol{r}$ and $\boldsymbol{p}$ with $\boldsymbol{p}$ being the *canonical* impulse and with normalisation $\int f(\boldsymbol{r},\, \boldsymbol{p})\, \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{p} = 1$.

Quite often, Vlasov's equation is formulated for another distribution function, $f(\boldsymbol{r}, \boldsymbol{v})$,

$$\frac{\partial f(\boldsymbol{r},\boldsymbol{v})}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f(\boldsymbol{r},\boldsymbol{v})}{\partial \boldsymbol{r}} + \frac{\boldsymbol{F}(\boldsymbol{r})}{m} \cdot \frac{\partial f(\boldsymbol{r},\boldsymbol{v})}{\partial \boldsymbol{v}} = 0 \tag{1.49}$$

with $\boldsymbol{F}(\boldsymbol{r})$ representing all external and mean internal forces and with normalisation $\int f(\boldsymbol{r},\, \boldsymbol{v})\, \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{v} = 1$. Note that while being trivial for $\boldsymbol{v} = \boldsymbol{p}/m$, the transition from eq. (1.48) to (1.49) needs to be carried out carefully when dealing with systems of charged particles, in which case the linear relation between velocity and momentum does not hold. This is (currently) beyond the scope of this text and we will assume that an even more generalised version of Vlasov's equation (1.49) holds, in which $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}, \boldsymbol{v})$ may also cover the Lorentz force.

1.5 Fluid equations

Some variant of the Boltzmann or Vlasov equation appears to be suitable for solving various physical systems. Nevertheless, in other cases, it is either too difficult, or unnecessary to consider all information held by the distribution function. Fluid equations, which subsequently allow various heuristic modifications, are quite commonly used. Usually, they are derived from the Vlasov equation in the form (1.48).

The basic quantities which stand in the fluid equations are defined as various, linearly independent moments of the single-particle distribution function:

$$\begin{pmatrix} \rho(\boldsymbol{r};t) \\ \rho(\boldsymbol{r};t)\, \boldsymbol{u}(\boldsymbol{r};t) \\ \rho(\boldsymbol{r};t)\, \mathcal{E}(\boldsymbol{r};t) \end{pmatrix} \equiv \int \begin{pmatrix} m \\ m\, \boldsymbol{v} \\ \frac{1}{2} m\, |\boldsymbol{v} - \boldsymbol{u}|^2 \end{pmatrix} f(\boldsymbol{r};\boldsymbol{t},\boldsymbol{v})\, \mathrm{d}^3\boldsymbol{v}\ . \tag{1.50}$$

The quantities ρ, $\boldsymbol{u}$ and $\mathcal{E}$ are mean values of density, velocity and specific (per unit mass and unit volume) internal energy respectively of the statistically described system. They are functions of position and time, but we have dropped the dependency on velocity $\boldsymbol{v}$. In the following, we will (again) drop the implicit dependence on time.

The fluid equations are obtained by the multiplication of the Boltzmann or Vlasov equation by m, $\boldsymbol{v}$ and $\frac{1}{2}mv^2$ and subsequent integration over the velocity space. They are sometimes called moments of the Boltzmann/Vlasov equation.

1.5.1 Continuity equation

The simplest equation to derive is the continuity equation which starts from

$$\int m \frac{\partial f}{\partial t}\, \mathrm{d}^3\boldsymbol{v} + \int m\boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}}\, \mathrm{d}^3\boldsymbol{v} - \int \boldsymbol{\nabla}\left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] \cdot \frac{\partial f}{\partial \boldsymbol{v}}\, \mathrm{d}^3\boldsymbol{v} = 0\ . \tag{1.51}$$

Under certain assumptions for the form of $f(\boldsymbol{r},\, \boldsymbol{v})$, we may exchange the order of integration and differentiation with respect to t and $\boldsymbol{r}$ in the first and second term, respectively. In the last term, we may put all but the $\partial f/\partial \boldsymbol{v}$ term in front of the integral:

$$\frac{\partial}{\partial t} \int m\, f\, \mathrm{d}^3\boldsymbol{v} + \frac{\partial}{\partial \boldsymbol{r}} \cdot \int m\, \boldsymbol{v} f\, \mathrm{d}^3\boldsymbol{v} - \boldsymbol{\nabla}\left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] \cdot \int \frac{\partial f}{\partial \boldsymbol{v}}\, \mathrm{d}^3\boldsymbol{v} = 0\ . \tag{1.52}$$

The first two integrals are by definion ρ and $\rho\,\boldsymbol{u}$, the third integral evaluates to $f(\boldsymbol{r},\,\boldsymbol{v})$ evaluated in infinity which is zero due to the normalisation condition (see Sec. 1.4 for discussion of properties of the distribution function implied by the normalisation constraint). Hence, the zeroth moment of the Boltzmann/Vlasov equation reads:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{u}) = 0\,. \tag{1.53}$$

1.5.2 Euler equation

The first moment of the Vlasov equation is sometimes called the Euler equation or the fluid equation of motion. It is a vector equation and for the sake of better clarity, we will write it in an index notation, using the Einstein summation convention:

$$\int m\,v_i\frac{\partial f}{\partial t}\,\mathrm{d}^3\boldsymbol{v} + \int m\,v_i\,v_j\frac{\partial f}{\partial r_j}\,\mathrm{d}^3\boldsymbol{v} - \int v_i\frac{\partial}{\partial r_j}\left[V(\boldsymbol{r})+\Phi(\boldsymbol{r})\right]\frac{\partial f}{\partial v_j}\,\mathrm{d}^3\boldsymbol{v} = 0\,. \tag{1.54}$$

Let's assume that derivative with respect to t and r_j may be exchanged with integration over the velocity space. We will get

$$\frac{\partial\rho\,u_i}{\partial t} + \frac{\partial}{\partial r_j}(\rho\langle v_i\,v_j\rangle) - \frac{\partial}{\partial r_j}\left[V(\boldsymbol{r})+\Phi(\boldsymbol{r})\right]\int v_i\,\frac{\partial f}{\partial v_j}\,\mathrm{d}^3\boldsymbol{v} = 0\,, \tag{1.55}$$

where for arbitrary function $\mathcal{X}$ we denote $\rho\langle\mathcal{X}\rangle \equiv \int m\,\mathcal{X}\,f\,\mathrm{d}^3\boldsymbol{v}$. It can easily be shown (e.g. by means of integration by parts) that

$$\int v_i\,\frac{\partial f}{\partial v_j}\,\mathrm{d}^3\boldsymbol{v} = -\frac{\rho}{m}\,\delta_{ij} \tag{1.56}$$

and

$$\langle v_i\,v_j\rangle = u_i\,u_j + \langle w_i\,w_j\rangle\,,^2 \tag{1.57}$$

where $w_i \equiv v_i - u_i$ represents "random" deviations of the particles' motion from the mean bulk velocity. The symmetric tensor $\rho\langle w_i\,w_j\rangle$ can be split into its trace, $P \equiv \frac{1}{3}\rho\langle|\boldsymbol{w}|^2\rangle$, and traceless tensor $\pi_{ij} \equiv \rho\langle\frac{1}{3}|\boldsymbol{w}|^2\delta_{ij} - w_i\,w_j\rangle$ taken with a negative sign. The newly introduced quantities P and π_{ij} are called pressure and stress tensor, respectively. Using them, we can rewrite the Euler equation into the form

$$\frac{\partial}{\partial t}(\rho\,u_i) + \frac{\partial}{\partial r_j}(\rho\,u_i\,u_j + P\delta_{ij} - \pi_{ij}) = -\frac{\rho}{m}\frac{\partial}{\partial r_i}\left[V(\boldsymbol{r})+\Phi(\boldsymbol{r})\right] \tag{1.58}$$

and with further small alterations, we obtain

$$u_i\frac{\partial\rho}{\partial t} + u_i\frac{\partial}{\partial r_j}(\rho\,u_j) + \rho\frac{\partial u_i}{\partial t} + \rho\,u_j\frac{\partial u_i}{\partial r_j} = -\frac{\rho}{m}\frac{\partial}{\partial r_i}\left[V(\boldsymbol{r})+\Phi(\boldsymbol{r})\right] - \frac{\partial P}{\partial r_i} + \frac{\partial\pi_{ij}}{\partial r_j}\,. \tag{1.59}$$

[2] $\langle v_i v_j\rangle = \frac{m}{\rho}\int\mathrm{d}^3\boldsymbol{v}\,f v_i v_j = \frac{m}{\rho}\int\mathrm{d}^3\boldsymbol{v}\,f(u_i+w_i)(u_j+w_j) = \frac{m}{\rho}\int\mathrm{d}^3\boldsymbol{v}\,[f u_i u_j + f u_i w_j + f u_j w_i + w_i w_j]$. Considering that u_i may be taken out of the integrals and $\int w_i f\mathrm{d}^3\boldsymbol{v} = \int(v_i-u_i)f\mathrm{d}^3\boldsymbol{v} = \int v_i f\mathrm{d}^3\boldsymbol{v} - u_i\int f\mathrm{d}^3\boldsymbol{v} = \frac{\rho}{m}u_i - u_i\frac{\rho}{m} = 0$ and using the definition of the mean value, we have $\langle v_i\,v_j\rangle = u_i\,u_j + \langle w_i\,w_j\rangle$.

The sum of the first two terms on the left-hand side of eq. (1.59) is zero as it is equal to the left-hand side of the continuity equation (1.53) multiplied by u_i. Hence, the standard form of the Euler equation reads:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial r_j} = -\frac{1}{m}\frac{\partial}{\partial r_i}[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})] - \frac{1}{\rho}\frac{\partial P}{\partial r_i} + \frac{1}{\rho}\frac{\partial \pi_{ij}}{\partial r_j} \ . \tag{1.60}$$

Let us also put down a common vector notation of the Euler equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{f} - \frac{1}{\rho}\boldsymbol{\nabla}P + \frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\pi} \ , \tag{1.61}$$

where we have replaced the gradient of potential energy by the specific force density $\boldsymbol{f}$. Let's put stress on meaning of the term $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$, which can be inferred from the comparison of eqs. (1.60) and (1.61). The so-called "convective derivative" is an imporant term in expressing the full time derivative,

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}, \tag{1.62}$$

hence the convective derivative represents the change of the vector field due to the motion of the particle, where the partial derivative represents the temporal evolution of the field at a given position.

1.5.3 Energy equation

The energy balance equation will be obtained by multiplying eq. (1.48) with $\frac{1}{2}m|\boldsymbol{v}|^2$ and integrating over the velocity space. We will be briefer than in the previous two cases as the steps taken are qualitatively the same. Full derivation may be found in Appendix A.1. Hence, after some algebra, we will obtain

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho(u^2 + \langle w^2\rangle)\right] + \frac{\partial}{\partial r_i}\left[\tfrac{1}{2}\rho\langle(u_i + w_i)|\boldsymbol{u}+\boldsymbol{w}|^2\rangle\right] = -\frac{\rho}{m}\,u_i\frac{\partial}{\partial r_i}(V+\Phi) \ , \tag{1.63}$$

where we denote $u^2 \equiv |\boldsymbol{u}|^2$ for the sake of brevity. Introducing the (conductive) heat flow, $\boldsymbol{F} \equiv \frac{1}{2}\rho\langle \boldsymbol{w}\, w^2\rangle$ and keeping in mind that $\rho\mathcal{E} \equiv \frac{1}{2}\rho\langle w^2\rangle \equiv \frac{3}{2}P$, we rewrite (1.63) into the form

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + \rho\mathcal{E}\right) + \frac{\partial}{\partial r_i}\left[\tfrac{1}{2}\rho u^2\, u_i + u_j(P\delta_{ij} - \pi_{ij}) + \rho\mathcal{E}\, u_i + F_i\right] = -\frac{\rho}{m}\,u_i\frac{\partial}{\partial r_i}(V+\Phi) \ , \tag{1.64}$$

which can further be simplified by subtracting the scalar product of the Euler equation (1.58) and u_i,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho\, u^2) + \frac{\partial}{\partial r_i}(\tfrac{1}{2}\rho\, u^2\, u_i) = -\frac{\rho}{m}\,u_i\frac{\partial}{\partial r_i}(V+\Phi) - u_i\frac{\partial P}{\partial r_i} + u_i\frac{\partial \pi_{ij}}{\partial r_j} \ , \tag{1.65}$$

yielding

$$\frac{\partial}{\partial t}(\rho\,\mathcal{E}) + \frac{\partial}{\partial r_i}\,(\rho\,\mathcal{E}\,u_i) = -P\frac{\partial u_i}{\partial r_i} - \frac{\partial F_i}{\partial r_i} + \Psi \ , \tag{1.66}$$

where $\Psi \equiv \pi_{ij}\,\partial u_i/\partial r_j$ is the local dissipation measure. Equation (1.66) can be rewritten with help of the continuity equation into vector form

$$\rho\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -P\,\boldsymbol{\nabla}\cdot\boldsymbol{u} - \boldsymbol{\nabla}\cdot\boldsymbol{F} + \Psi\ . \tag{1.67}$$

Sometimes, there are extra terms added rather heuristically to the right-hand side of the energy balance equation, e.g. the divergence of radiative heat transfer, in order to match the considered physical system. Note, however, that the rigorous addition of such terms from the first principles is usually not trivial.

1.5.4 Closing the equations set

We have obtained a set of five linearly independent momentum equations,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{u}) \;=\; 0\ , \tag{1.68}$$

$$\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} \;=\; \boldsymbol{f} - \frac{1}{\rho}\boldsymbol{\nabla}P + \frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\pi}\ , \tag{1.69}$$

$$\rho\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \;=\; -P\,\boldsymbol{\nabla}\cdot\boldsymbol{u} - \boldsymbol{\nabla}\cdot\boldsymbol{F} + \Psi \tag{1.70}$$

for 14 independent variables: ρ, $\boldsymbol{u}$ (3 components), P, $\boldsymbol{\pi}$ (5 independent components), $\boldsymbol{F}$ (3 components) and Φ (which is hidden in $\boldsymbol{f}$). Internal energy $\mathcal{E}$ is not independent, as it is proportional to pressure p, similarly the local dissipation measure Ψ is defined using the stress tensor $\boldsymbol{\pi}$ and $\boldsymbol{u}$. Such a description of a physical problem is clearly not solvable. Hence, various approximations are used to overcome this problem.

A large simplification, just enough to close the problem from a mathematical point of view, is achieved for isotropic systems with $f(\boldsymbol{r},\,\boldsymbol{v}) = f(\boldsymbol{r},\,|\boldsymbol{v}|)$, which implies that $\pi_{ij} = 0$ and $F_i = 0$ (see Appendix A.2).

Quite often, the so-called Navière-Stokes approximation is used, assuming the stress tensor to be proportional to the tensor of deformation D_{ij},

$$\pi_{ij} = \mu D_{ij} = \mu\left[\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} - \frac{3}{2}\frac{\partial u_k}{\partial r_k}\delta_{ij}\right]\ , \tag{1.71}$$

which reduces the number of independent variables by 5. The constant of proportionality, μ, is called dynamical viscosity.

Another step towards a closed set of equations may lie in the heuristic introduction of temperature T, the ideal gas equation $P = \rho/m\,K_{\mathrm{B}}T$, and the assumption

$$\boldsymbol{F} = -\kappa\,\boldsymbol{\nabla}T\ . \tag{1.72}$$

With κ being a constant determining the heat conduction.[3]

[3]Quite often the conductive heat flow, $\boldsymbol{F}$, is actually assumed to be zero or negligible, but on the other hand, radiative heat flow is considered. In such a case, the equation of radiative transfer, formally similar to (1.72), is introduced with κ being the opacity.

Finally, if the internal potential Φ is not neglected, some variant of the Poisson equation, e.g.

$$\triangle\Phi = 4\pi\, G\, \rho\ , \tag{1.73}$$

completes the set. Note that a similar Poisson equation holds for the external potential

$$\triangle V = 4\pi\, G\, \rho_{\mathrm{ext}}\ , \tag{1.74}$$

where ρ_{ext} indicates the external density generating the external potential.

1.6 Thermodynamical equilibria

So far, we have attempted to downscale the problem of evolution of the statistically described system to a space with (much) fewer dimensions than the phase space. There is, however, a good reason to describe the system in its full complexity in a specific case – when it is in a thermodynamic equilibrium. We postulate that the distribution function corresponding to the equilibrium state is the one for which the statistical entropy,

$$S(D) \equiv -K_{\mathrm{B}}\mathrm{Tr}\left(D \ln D\right)\ , \tag{1.75}$$

reaches a maximum value. (Where to add a section on the statistical entropy? Here, or to Sec. 1.2?) For example, let us consider a system with a finite number of possible states, N. The distribution function (in this case the probability) for each state i is then $D(i) = 1/N$. Nevertheless, the definition of the equilibrium system is more general and can be applied also to statistical systems with some additional information. In particular, we usually consider a given mean value, $\langle A\rangle \equiv \mathrm{Tr}(AD)$, of some physical quantity which can be assigned to the state of the system (i.e. measured in the real world) to play the role of the constraint on the distribution function.

Let's have a set of n physical quantities A_i, mean values of which are given. We are looking for the form of the distribution function which maximises the value of $S(D)$ and fulfills $n+1$ constraints $\langle A_i\rangle = \mathrm{Tr}(A_i D)$ and $\mathrm{Tr}D = 1$. A possible way to solve this task is via variation:

$$\eth\left\{\frac{1}{K_{\mathrm{B}}}S(D) - \sum_{i=0}^{n}\lambda_i\left[\mathrm{Tr}(A_i D) - \langle A_i\rangle\right]\right\} = 0\ , \tag{1.76}$$

where λ_i are Lagrange multipliers and $A_0 \equiv \mathbb{1}$. Using the explicit form of A_0 and $S(D)$, equation (1.76) leads to

$$-\mathrm{Tr}\left[(\ln D + \mathbb{1})\eth D\right] - \lambda_0\mathrm{Tr}(\eth D) - \sum_{i=1}^{n}\lambda_i\mathrm{Tr}(A_i\eth D) = 0\,, \tag{1.77}$$

which may further be rewritten as

$$-\mathrm{Tr}\left[\ln(D)\eth D\right] - \lambda_0'\mathrm{Tr}(\eth D) - \sum_{i=1}^{n}\lambda_i\mathrm{Tr}(A_i\eth D) = 0\,, \tag{1.78}$$

where $\lambda_0' \equiv \lambda_0 + 1$. Finally, dropping the prime from λ_0', we obtain:

$$\mathrm{Tr}\left[\left(-\ln D - \sum_{i=0}^{n} \lambda_i A_i\right) \eth D\right] = 0 \,. \tag{1.79}$$

It can be shown (do it!) that in order to fulfill relation (1.79) for the arbitrary variation $\eth D$, the term in the round brackets has to be zero, i.e.

$$D = \exp\left[-\sum_{i=0}^{n} \lambda_i A_i\right] . \tag{1.80}$$

Hence, we have an explicit form of the distribution function (functions/operators A_i are given) with $n+1$ parameters λ_i which are related to the given values of $\langle A_i \rangle$. Let us express λ_0 for convenience. First, let us rewrite (1.80) to

$$D = \mathrm{e}^{-\lambda_0} \exp\left[-\sum_{i=1}^{n} \lambda_i A_i\right] \,. \tag{1.81}$$

Then, consider that $\langle A_0 \rangle = \mathrm{Tr} D = 1$ and apply the operator Tr to equation (1.81) which gives

$$\mathrm{e}^{-\lambda_0} \mathrm{Tr}\left\{\exp\left[-\sum_{i=1}^{n} \lambda_i A_i\right]\right\} = 1 \,. \tag{1.82}$$

Instead of λ_0 itself, let us introduce the so-called partition function (also known as partition sum):

$$Z \equiv \mathrm{e}^{\lambda_0} = \mathrm{Tr}\left\{\exp\left[-\sum_{i=1}^{n} \lambda_i A_i\right]\right\} \,. \tag{1.83}$$

The equilibrium distribution function 1.80 can then be rewritten as

$$D = \frac{1}{Z} \exp\left[-\sum_{i=1}^{n} \lambda_i A_i\right] \,. \tag{1.84}$$

The partition function Z keeps a lot of information about the system. It can easily be shown that, e.g.

$$\frac{\partial \ln Z}{\partial \lambda_i} = -\langle A_i \rangle \quad \text{and} \quad \frac{\partial^2 \ln Z}{\partial \lambda_i \partial \lambda_j} = \langle A_i A_j \rangle - \langle A_i \rangle \langle A_j \rangle \,. \tag{1.85}$$

Similarly, we may write the relation for entropy S and the Lagrange multipliers:

$$\frac{\partial S}{\partial \langle A_i \rangle} = K_\mathrm{B} \lambda_i \,. \tag{1.86}$$

(Can the equilibrium state (of isolated system) be reached starting from non-equilibrium one?)

1.6.1 Statistical ensembles

The statistical systems often appear static from a human perspective (despite microscopic motions, which are not directly observable) and this allows us to describe these systems by a set of macroscopic variables. Such systems may be described by *statistical ensembles*, which are assumed to be in a statistical equilibrium, and given various macroscopic constraints, these ensembles depend on a few observable parameters. Three important thermodynamic ensembles were defined for a fixed volume Ω.

The microcanonical ensemble is a statistical ensemble, where the total energy of the system and the number of particles in the system are known (they each have particular fixed values). The system is totally isolated, no exchange of energy or particles with the surrounding environment is allowed. Each member of the ensemble has the same total energy and the same total number of particles. Following (1.84) the distribution function holds the shape of

$$D_{\mathrm{N}} = \frac{1}{Z_{\mathrm{M}}}. \tag{1.87}$$

The canonical ensemble is a statistical ensemble, where the number of particles is fixed, but the energy is not known exactly. The canonical ensemble is suitable for the description of a system, which exchanges heat with its surrounding environment, however there is no exchange of particles with the surroundings. Each member of the ensemble has the same number of particles, their energies are not defined, however there is a constraint on mean energy, which is usually given by the temperature of the surrounding bath. Following (1.84) the distribution function holds the shape of

$$D_{\mathrm{N}} = \frac{1}{Z_{\mathrm{C}}} \exp\left[-\beta H\right], \tag{1.88}$$

where $\beta \equiv \lambda_H$ is the Lagrange multiplier for the Hamiltonian, which can be expressed by using the temperature T as $\beta = 1/K_{\mathrm{B}}T$, and Z_{C} is the partition function for the canonical ensemble. Note that the distribution function bears the index N because the number of particles is known.

The grand-canonical ensemble is a statistical ensemble, where neither the energy nor the number of particles is fixed. Similarly to the previous case, instead of the total energy we define the temperature, and in the place of the total number of particles we define the chemical potential. The grand-canonical ensemble is appropriate for the description of open systems, which have thermal contact with their surroundings and exchange particles with a reservoir. For members of the grand-canonical ensemble the temperature and the chemical potential of the reservoir place constraints on the mean energy of each member and mean number of particles. Following (1.84) the distribution function takes the form of

$$D = \frac{1}{Z_{\mathrm{G}}} \exp\left[-\beta H + \alpha N\right], \tag{1.89}$$

where $\alpha \equiv -\lambda_N \equiv \beta\mu$ represents the Lagrange multiplier for the number of particles and may also be expressed by using the chemical potential μ.

In the following, we will use the properties of both the canonical and grand-canonical ensembles in order to derive the equilibrium particle distribution function.

1.6.2 Maxwell-Boltzmann distribution

We will derive the particle distribution function for an open system of particles, for which we will use the calculation for a grandcanonical ensemble. To be clear, we remind the reader the usage of such a statistical ensemble means that we do not know the exact total energy of the particles in the volume nor the total number of particles. However, the constraints are placed upon the mean values of energy and the mean number of particles, these quantities are further denoted by a temperature T and a chemical potential μ. Elaborating on eq. (1.84), in this case the distribution function is given by

$$D = \frac{1}{Z_{\mathrm{G}}} \exp\left[-\beta H + \alpha N\right], \tag{1.90}$$

where Z_{G} is the partition function of the grandcanonical ensemble and the two operators connected with the two known constraints are H, the Hamiltonian as an operator of energy, and N as the formal operator of number of particles. These two operators use the Lagrange multipliers β, which represents the temperature, and α, representing the chemical potential. Note that the argument of the exponential is a hybrid function, because the hamiltonian H is continuous, whereas the operator of number of particles N is discrete.

The grandcanonical partition function may be written using its canonical counterpart Z_{C}

$$Z_{\mathrm{G}} = \sum_{N=0}^{\infty} e^{\alpha N} Z_{\mathrm{C}}(\beta, N), \tag{1.91}$$

which basically says that the partition function (also termed as the *sum over states*) simply is a sum of partition functions of the canonical ensembles with a fixed number of particles N with a given temperature contained in β.

For a hamiltonian H_N of N non-interacting particles (point masses) in the external potential $V(\boldsymbol{r})$ given by

$$H_N = \sum_i H_1(\boldsymbol{r}_i, \boldsymbol{p}_i), \tag{1.92}$$

where

$$H_1(\boldsymbol{r}, \boldsymbol{p}) = p^2/(2m) + V(\boldsymbol{r}) \tag{1.93}$$

is the one-particle hamiltonian in the external potential, we may write the canonical partition function as

$$Z_{\mathrm{C}} = \int_{-\infty}^{+\infty} \exp\left(-\beta H_N\right) \mathrm{d}\tau_N, \tag{1.94}$$

where for $\mathrm{d}\tau_N$ we use a slightly modified definition

$$\mathrm{d}\tau_N \equiv \frac{\mathrm{d}^3\boldsymbol{r}_1\mathrm{d}^3\boldsymbol{p}_1\ldots\mathrm{d}^3\boldsymbol{r}_N\mathrm{d}^3\boldsymbol{p}_N}{\hbar^{3N}N!}. \tag{1.95}$$

The advantage of utilising the factor $\hbar^{3N}$ in the denominator is that $\mathrm{d}\tau_N$ is then dimensionless, as is the corresponding distribution function D.

Turning the sum over N particles into a multiplication (it is possible because the functions H_1 are the same) we obtain

$$Z_{\mathrm{C}}(\beta,N) = \int \exp\left[-\beta H_N\right]\mathrm{d}\tau_N = \int_{-\infty}^{+\infty}\prod_{i=1}^{N}\left(e^{-\beta p_i^2/(2m)}e^{-\beta V(\boldsymbol{r}_i)}\mathrm{d}^3\boldsymbol{r}_i\mathrm{d}^3\boldsymbol{p}_i\right)\frac{1}{\hbar^{3N}N!} =$$

$$= \frac{1}{\hbar^{3N}N!}\left[\int_{-\infty}^{+\infty}e^{-\beta p^2/(2m)}e^{-\beta V(\boldsymbol{r})}\mathrm{d}^3\boldsymbol{r}\mathrm{d}^3\boldsymbol{p}\right]^N. \tag{1.96}$$

Thanks to the unification of the description of the energy we may perform integrations both over the positions and momenta. The integration over the momenta is simpler, as it is in fact a Gauss integral $\int_{-\infty}^{+\infty}\exp\left(-x^2\right)\mathrm{d}x = \sqrt{\pi}$. The integration in the spatial domain requires a bit more justification.

In order to be able to perform the integration in the spatial domain, we must define the external potential V. To keep things simple, we would require an integrable potential, which is zero outside the region of interest (outside our volume Ω, where we study the problem), and which has some value inside Ω. It is easy to show that the potential well will have exactly those properties. Then the integration of $\boldsymbol{r}$ leads to a scaled volume of our region of interest.

$$\Omega\Xi \equiv \int_{-\infty}^{+\infty}e^{-\beta V(\boldsymbol{r})}\mathrm{d}^3\boldsymbol{r}. \tag{1.97}$$

Hence, we have

$$Z_{\mathrm{C}}(\beta,N) = \frac{1}{\hbar^{3N}N!}\left(2\pi mK_{\mathrm{B}}T\right)^{3N/2}\Omega^N\Xi^N \equiv \frac{Z_1^N}{N!}, \tag{1.98}$$

where

$$Z_1 \equiv \frac{\left(2\pi mK_{\mathrm{B}}T\right)^{3/2}}{\hbar^{3N}}\Omega\Xi. \tag{1.99}$$

By choosing the potential V in the form of an infinite potential well (i.e. $V(\boldsymbol{r}) = 0$ for $\boldsymbol{r}\in\Omega$ and $V(\boldsymbol{r}) = +\infty$ for $\boldsymbol{r}\notin\Omega$), we set $\Xi = 1$ and the integration of the spatial part in (1.96) leads exactly to the volume of the region of interest. Beware of the conflicting notation, when Ω means both the regions of interest and the volume of this region. Further note that we defined a particle partition function $Z_1(\beta) \equiv Z_{\mathrm{C}}(\beta, N = 1)$.

In the following, we will apply the above discussed approximation of the infinite potential well. By combining (1.91) and (1.98) we have

$$Z_{\mathrm{G}} = \sum_{N=0}^{\infty} \frac{1}{N!} \left(e^{\alpha} Z_1\right)^N = \exp\left[e^{\alpha} Z_1\right], \tag{1.100}$$

where we used the Taylor expansion of the exponential function.

Now we search for the probability of finding a particle in the spatial point $\boldsymbol{r}$ with a momentum of $\boldsymbol{p}$. The procedure is somewhat complicated because we don't know the exact number of particles in our open system. The sought probability f is[4], however, given by

$$f(\boldsymbol{r}, \boldsymbol{p}) \equiv \sum_{N=0}^{\infty} \left[\mathcal{P}(N) \sum_{i=1}^{N} \mathcal{P}_N(i, \boldsymbol{r}, \boldsymbol{p}) \right], \tag{1.101}$$

where $\mathcal{P}(N)$ evaluates the probability that there are exactly N particles in the ensemble and $\mathcal{P}_N(i, \boldsymbol{r}, \boldsymbol{p})$ indicates the probability that the i-th particle (from a fixed number of N particles) can be found around coordinates $(\boldsymbol{r}, \boldsymbol{p})$ in the phase space. $\mathcal{P}(N)$ may be intuitively written as a trace of the full-system distribution function D[5] from (1.90):

$$D \sim \begin{bmatrix} \frac{1}{Z_{\mathrm{G}}} & & & \\ & \frac{1}{Z_{\mathrm{G}}} e^{\alpha} \exp(-\beta H_1) & & \\ & & \frac{1}{Z_{\mathrm{G}}} e^{2\alpha} \exp(-\beta[H_1(\boldsymbol{r}_1, \boldsymbol{p}_1) + H_1(\boldsymbol{r}_2, \boldsymbol{p}_2)]) & \\ & & & \ddots \end{bmatrix} \tag{1.102}$$

with diagonal terms $D_{N,N}$ defined by

$$D_{N,N} = \frac{1}{Z_{\mathrm{G}}} e^{\alpha N} \exp\left[-\beta \sum_{i=1}^{N} H_1(\boldsymbol{r}_i, \boldsymbol{p}_i)\right]. \tag{1.103}$$

Hence (using eq. **??**)

$$\mathcal{P}(N) = \int_{-\infty}^{+\infty} D_{N,N} \mathrm{d}\tau_N = \frac{e^{\alpha N} Z_{\mathrm{C}}(\beta, N)}{Z_{\mathrm{G}}(\alpha, \beta)}. \tag{1.104}$$

Further, $\mathcal{P}_N$ is simply defined for the 1-particle probability (probability of finding the particle carrying flag 1 at phase coordinates $(\boldsymbol{r}, \boldsymbol{p})$) by

$$\mathcal{P}_N(1, \boldsymbol{r}, \boldsymbol{p}) = \int_{-\infty}^{+\infty} D_N \frac{\mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 \ldots \mathrm{d}^3\boldsymbol{r}_N \mathrm{d}^3\boldsymbol{p}_N}{\hbar^{3N} N!} = \frac{1}{N\hbar^3} \int_{-\infty}^{+\infty} D_N \mathrm{d}\tau_{N-1} \tag{1.105}$$

[4]Let us take a short excursion to past chapters. The notation $f(\boldsymbol{r}, \boldsymbol{p})$ was already used when we discussed the reduced distribution functions of N particles – see eq. (1.24). The question is, what is the relation of (1.24) to (1.101). We need to keep in mind that in past chapters our ensemble consisted of a fixed number of particles. Therefore strictly speaking, for the previous case $\mathcal{P}(N) = 1$ and the first sum disappears. Then the two fs have the same meaning.

[5]Note that D does not have a simple dimension. It belongs to the Fock space, which is formed as a direct sum of tensor products of Hilbert spaces with varying dimensions. The proper treatment is beyond the scope of this text, the interested reader is pointed to elsewhere.

where $D_N = e^{-\beta H_N}/Z_{\mathrm{C}}(\beta, N)$ is the distribution function for the canonical ensemble. We further use (1.98) to get

$$\mathcal{P}_N(1, \boldsymbol{r}, \boldsymbol{p}) = \frac{1}{N\hbar^3} \int_{-\infty}^{+\infty} \frac{e^{-\beta H_N} \mathrm{d}\tau_{N-1}}{\frac{1}{N!} Z_1^N(\beta)} = \frac{1}{N\hbar^3} \int_{-\infty}^{+\infty} \frac{N(N-1)! e^{-\beta H_N} \mathrm{d}\tau_{N-1}}{Z_1^N(\beta)} =$$
$$= \frac{(N-1)!}{\hbar^3} e^{-\beta H_1(\boldsymbol{r},\boldsymbol{p})} \frac{1}{Z_1^N} \int e^{-\beta H_{N-1}} \mathrm{d}\tau_{N-1} = \frac{1}{\hbar^3} \frac{e^{-\beta H_1(\boldsymbol{r},\boldsymbol{p})}}{Z_1(\beta)}. \tag{1.106}$$

Note that the factor of $\exp(-\beta H_1(\boldsymbol{r}, \boldsymbol{p}))$ could be taken out of the integral, because we are not integrating over $\boldsymbol{r}$ and $\boldsymbol{p}$ coordinates. Furthermore, in the last step we used

$$\int e^{-\beta H_{N-1}} \mathrm{d}\tau_{N-1} = \frac{1}{(N-1)!} Z_1^{(N-1)}, \tag{1.107}$$

which follows from (1.94) and (1.98). The 1-particle hamiltonian H_1 is defined by (1.93).

Then (1.101) takes the form of

$$f(\boldsymbol{r}, \boldsymbol{p}) = \sum_{N=0}^{\infty} \frac{e^{\alpha N} Z_{\mathrm{C}}(\beta, N)}{Z_{\mathrm{G}}(\beta, \alpha)} N \frac{1}{\hbar^3} \frac{e^{-\beta H_1}}{Z_1}, \tag{1.108}$$

where the second sum over i in (1.101) was turned into multiplication by N, as we have N indistinguishable particles. We further elaborate on (1.108) by using (1.98) and by moving terms not depending on N to the front of the sum to obtain:

$$f(\boldsymbol{r}, \boldsymbol{p}) = \frac{1}{\hbar^3 Z_{\mathrm{G}}} \sum_{N=0}^{\infty} \frac{e^{\alpha N} Z_1^N}{N!} N \frac{e^{-\beta H_1}}{Z_1} = \frac{e^{\alpha} e^{-\beta H_1}}{\hbar^3} \frac{\sum_{N=1}^{\infty} \frac{e^{\alpha(N-1)} Z_1^{N-1}}{(N-1)!}}{\exp\left[e^{\alpha} Z_1\right]}, \tag{1.109}$$

where we used (1.100) and realised that the summation index N must start from 1 instead of the original 0, because the expression $(N-1)!$ is not defined for $N = 0$[6]. Now we realise that the sum in the numerator is the Taylor expansion of the denominator and hence finally

$$f(\boldsymbol{r}, \boldsymbol{p}) = \frac{e^{\alpha - \beta H_1(\boldsymbol{r},\boldsymbol{p})}}{\hbar^3}. \tag{1.110}$$

Now we use the fact that the mean values of physical quantities may be expressed from the partition function (1.85):

$$\langle N \rangle = \frac{\partial \ln Z_{\mathrm{G}}}{\partial \alpha} = \frac{\partial}{\partial \alpha} e^{\alpha} Z_1 = e^{\alpha} Z_1 \tag{1.111}$$

[6] Or alternatively we may say that for $N = 0$ the fraction $N/N! = 0$ and therefore the first term in the sum vanished.

and our previous finding that $\int_{-\infty}^{+\infty} \exp[-\beta V(\boldsymbol{r})] = \Omega$ to finally have

$$f'(\boldsymbol{p}) = \int_{-\infty}^{+\infty} f(\boldsymbol{r},\boldsymbol{p})\mathrm{d}^3\boldsymbol{r} = \int_{-\infty}^{+\infty} \mathrm{d}^3\boldsymbol{r}\frac{1}{\hbar^3}e^{\alpha}\exp\left\{-\beta\left[p^2/(2m)+V(\boldsymbol{r})\right]\right\} =$$
$$= \frac{\Omega}{\hbar^3}\exp\left[\alpha-\beta\frac{p^2}{2m}\right] = \langle N\rangle g(\boldsymbol{p}). \tag{1.112}$$

Note that the expression $\langle N\rangle/\Omega \equiv n$ has the meaning of the particle density n.

Function $g(\boldsymbol{p})$ is what is our final solution to the problem and is usually termed the *Maxwell-Boltzmann distribution function*. Explicitly written by using (1.111) and (1.98) we have

$$g(\boldsymbol{p}) = \frac{\Omega}{\hbar^3}e^{-\frac{\beta p^2}{2m}}\frac{1}{Z_1} = \frac{e^{-\beta p^2/2m}}{(2\pi K_{\mathrm{B}} m T)^{3/2}} = \frac{e^{-\frac{p^2}{2K_{\mathrm{B}}Tm}}}{(2\pi K_{\mathrm{B}} m T)^{3/2}}. \tag{1.113}$$

1.6.3 Chemical equilibrium, Saha's equation

Let's consider a system of three different kinds of particles – free electrons and protons and neutral hydrogen atoms. They play a role in a chemical equation $\mathrm{e}+\mathrm{p} \rightleftarrows \mathrm{H}$, where the double arrow indicates that the system is in the equilibrium, therefore the speed of the reactions to the left is the same as the speed of the reverse reaction. Let us define the system with a simple hamiltonian

$$H = \sum_{i=1}^{\mathrm{N^e}} H_1^{\mathrm{e}}(\boldsymbol{r}_i\,\boldsymbol{p}_i) + \sum_{i=1}^{\mathrm{N^p}} H_1^{\mathrm{p}}(\boldsymbol{r}_i\,\boldsymbol{p}_i) + \sum_{i=1}^{\mathrm{N^H}} H_1^{\mathrm{H}}(\boldsymbol{r}_i\,\boldsymbol{p}_i) \tag{1.114}$$

where

$$H_1^a(\boldsymbol{r},\,\boldsymbol{p}) = \frac{p^2}{2m^a} + V(\boldsymbol{r}) + h_1^a\ ,\quad a = \{\mathrm{e},\,\mathrm{p},\,\mathrm{H}\}\ . \tag{1.115}$$

Hamiltonians h_1^a correspond to the internal degrees of freedom and we will assume that they have a discrete spectrum. The partition function of this grand-canonical ensemble can be written in the form

$$Z_{\mathrm{G}}(\beta,\,\mu^{\mathrm{e}},\,\mu^{\mathrm{p}},\,\mu^{\mathrm{H}}) = \prod_a \left\{\mathrm{Tr}\left[\exp(\beta\mu^a N^a)\right]\,\mathrm{Tr}\left[\exp(-\beta H^a)\right]\right\}\ . \tag{1.116}$$

The partition function of a canonical ensemble of N^a particles of the species a reads:

$$\begin{aligned} Z_{\mathrm{C}}^a(\beta,\,\mathrm{N}^a) &= \frac{1}{\mathrm{N}^a!}\left(\int \exp\left[-\beta\left(\frac{p^2}{2m^a}+V(\boldsymbol{r})\right)\right]\frac{\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{p}}{\hbar^3}\right)^{\mathrm{N}^a}\left(\mathrm{Tr}\left\{\exp\left(-\beta h_1^a\right)\right\}\right)^{\mathrm{N}^a} \\ &= \frac{1}{\mathrm{N}^a!}\left[\frac{1}{\hbar^3}\left(2\pi m^a K_{\mathrm{B}}T\right)^{3/2}\Omega\,\zeta^a(T)\right]^{\mathrm{N}^a} = \frac{1}{\mathrm{N}^a!}\left[Z_1^a(T)\,\zeta^a(T)\right]^{\mathrm{N}^a}\ , \end{aligned} \tag{1.117}$$

(Dat stabni kulturu vnorenym zavorkam) here we again integrated over the spatial domain which defines the fixed volume of the configuration space accessible to the particles (here we assume $V(\boldsymbol{r})$ to represent an infinitely deep potential well),

$$\Omega \equiv \int \exp\left[-\beta\, V(\boldsymbol{r})\right] \mathrm{d}^3\boldsymbol{r}\ , \tag{1.118}$$

Furthermore,

$$\zeta^a(T) \equiv \mathrm{Tr}\left\{\exp\left(-\beta h_1^a\right)\right\} = \sum d_i^a \exp\left[-\beta \mathcal{E}_i^a\right]\ , \tag{1.119}$$

where d_i^a is the degeneracy of the energy level $\mathcal{E}_i^a$. Finally, let us assume for simplicity that each type of particle only has one internal energy level,

$$\begin{aligned} \zeta^{\mathrm{e}} &= d^{\mathrm{e}} \exp\left[-\beta \mathcal{E}_0^{\mathrm{e}}\right] \\ \zeta^{\mathrm{p}} &= d^{\mathrm{p}} \exp\left[-\beta \mathcal{E}_0^{\mathrm{p}}\right] \\ \zeta^{\mathrm{H}} &= d_0 \exp\left[-\beta(\mathcal{E}_0^{\mathrm{e}} + \mathcal{E}_0^{\mathrm{p}} - U_{\mathrm{i}})\right]\ , \end{aligned} \tag{1.120}$$

where $d^{\mathrm{e}} = d^{\mathrm{p}} = 2$ due to the spin degeneracy of free electrons and protons and d_0 denotes the degeneracy of the ground state of the hydrogen atom with (positive) ionisation energy U_{i}. Inserting (1.117) into (1.116) and replacing the first trace term N^a with a sum, we obtain

$$Z_{\mathrm{G}}(\beta,\, \mu^{\mathrm{e}},\, \mu^{\mathrm{p}},\, \mu^{\mathrm{H}}) = \prod_a \sum_{\mathrm{N}^a} \frac{1}{\mathrm{N}^a!} \left[\mathrm{e}^{\beta\mu^a} Z_1^a\, \zeta^a(T)\right]^{\mathrm{N}^a} = \prod_a \exp\left[\mathrm{e}^{\beta\mu^a} Z_1^a\, \zeta^a(T)\right]\ . \tag{1.121}$$

This allows us to evaluate the mean number of particles of species a,

$$\langle \mathrm{N}^a \rangle = \frac{1}{\beta} \frac{\partial \ln Z_{\mathrm{G}}}{\partial \mu^a} = \frac{\left(2\pi m^a K_{\mathrm{B}} T\right)^{3/2}}{\hbar^3} \Omega\, \zeta^a(T) \exp\left(\frac{\mu^a}{K_{\mathrm{B}} T}\right)\ . \tag{1.122}$$

Chemical equilibrium means that the rate of the reaction $\mathrm{e} + \mathrm{p} \rightleftarrows \mathrm{H}$ is equal in both directions, which, in the thermodynamical limit, can be expressed in terms of chemical potentials,

$$\mu^{\mathrm{e}} + \mu^{\mathrm{p}} = \mu^{\mathrm{H}}\ . \tag{1.123}$$

Equality (1.123) can be rewritten in a more complicated way:

$$\prod_a \left[\exp\left(\frac{\mu^a}{K_{\mathrm{B}} T}\right)\right]^{\nu^a} = 1\,, \text{ with } \quad \nu^{\mathrm{e}} = \nu^{\mathrm{p}} = 1,\ \nu^{\mathrm{H}} = -1\ . \tag{1.124}$$

Inserting (1.122) into (1.124) and denoting $n_a \equiv \langle \mathrm{N}^a \rangle / \Omega$, we obtain:

$$\prod_a \left[n^a \frac{\hbar^3}{\left(2\pi m^a K_{\mathrm{B}} T\right)^{3/2}} \frac{1}{\zeta^a(T)}\right]^{\nu^a} = 1 \tag{1.125}$$

which may finally be rewritten with the use of (1.121), $m_{\mathrm{H}} \approx m_{\mathrm{p}}$ and $n_{\mathrm{e}} \approx n_{\mathrm{p}}$ to the so-called Saha's equation for the equilibrium of ionised and neutral atoms,

$$\frac{n_{\mathrm{e}}}{n_{\mathrm{H}}} = \frac{4}{d_0} \frac{1}{n_{\mathrm{e}}} \frac{\left(2\pi m^{\mathrm{e}} K_{\mathrm{B}} T\right)^{3/2}}{\hbar^3} \exp\left(-\frac{U_{\mathrm{i}}}{K_{\mathrm{B}} T}\right)\ . \tag{1.126}$$

Chapter 2

Plasma

Plasma is defined as a *quasi-neutral gas of a mixture of charged and neutral particles showing collective behaviour*. It is characterised by its density n, temperature T (which is usually bound to the width of the velocity distribution, the Maxwell velocity distribution (1.113) for the equilibrium states), and the ionisation degree (usually well described by the Saha equation (1.126)). The *collective behaviour* indicates that the electromagnetic interaction is a long-range one, hence the behaviour of each particle of the ensemble influences the behaviour of all other particles[1]. The *quasineutrality* indicates that on large scales, the possible charge concentrations are efficiently shielded and thus not observable from the outside. The shielding length scale is termed the *Debye length* λ_{D}.

In a system of charged particles, each particle induces an electric potential, which interacts with the electric potential of the surrounding particles. The particles are free to move. Assuming that the system is in thermodynamical equilibrium or close to it, any disbalance of the charge is almost immediately shielded in order to preserve the electrical quasineutrality of the system. An illustrative conception of the shielding process is shown in the following model (see Fig. 2.1). In the system of charged particles, suddenly a concentration of one charge appears. There are random mechanisms that

[1]The Coulomb electric interaction decreases as $1/r^2$, where r is the distance, whereas for the given solid angle (where $\Delta r/r = \mathrm{const}$) the volume of the plasma increases as r^3. Hence altogether the electric force does not vanish to large distances within the plasma.

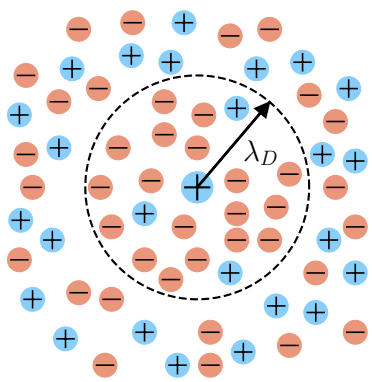

Figure 2.1: The concept of Debye shielding.

can drive such clumping. The particles of the opposite charge then cluster around this clump to remove the charge disbalance. We need to explicitly mention that the cluster of the opposite charge is not stationary. Thermal motions of particles are very large, so particles propagate through the shielding region almost freely, only the electric potential of the initial charge concentration slows them down. The propagating particles spend a longer time in the shielding region, creating an apparent opposite charge clump. In the system there exists a characteristic length scale, on which the charge concentration is shielded.

In order to derive this length scale, we will start with the Poisson equation, which binds the electric intensity $\boldsymbol{E}$ and the charge density ρ_e (a charge density should not be confused with electron density, as the subindex might seem to indicate – in general, there are also other charged species in the plasma than electrons):

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{\rho_\mathrm{e}(\boldsymbol{r})}{\varepsilon_0}, \tag{2.1}$$

where ε_0 is the permitivity of the vacuum. The electric-field intensity may be expressed by using the potential of the electric field ϕ as

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r}), \tag{2.2}$$

hence the Poisson equation gains the form

$$\triangle\phi(\boldsymbol{r}) = -\frac{\rho_\mathrm{e}(\boldsymbol{r})}{\varepsilon_0}. \tag{2.3}$$

Next we search for the ϕ which solves the Poisson equation. Assuming that the plasma is near equilibrium, we may obtain the expression for ρ_e from the Maxwell-Boltzmann distribution function. When the motion of particles of type α in the equilibrium is influenced by an external potential $q_\alpha\phi$, the velocity distribution function gains the form

$$f_\alpha(\boldsymbol{r}, \boldsymbol{v}) = A_\alpha \exp\left[-\frac{\frac{1}{2}m_\alpha v^2 + q_\alpha\phi(\boldsymbol{r})}{K_\mathrm{B}T_\alpha}\right], \tag{2.4}$$

where A_α is a normalisation constant. The integration of (2.4) over the velocity space yields the particle density n_α. In this case, the density may be a function of the electric potential ϕ. We remind the reader that in principle, ϕ may be a function of position:

$$n_\alpha(\phi) = \int_0^\infty f_\alpha(\boldsymbol{v})\mathrm{d}^3\boldsymbol{v} = \exp\left[-\frac{q_\alpha\phi(\boldsymbol{r})}{K_\mathrm{B}T_\alpha}\right] \int_0^\infty A_\alpha \exp\left[-\frac{\frac{1}{2}m_\alpha v^2}{K_\mathrm{B}T_\alpha}\right] \mathrm{d}^3\boldsymbol{v}. \tag{2.5}$$

If there was not any electric potential, i.e., when $\phi = 0$, then the integral in the above equation would directly yield the particle density. In the case $\phi \neq 0$ it can be interpreted as a *background* or unperturbed density. The functional form for the particle density then yields

$$n_\alpha(\phi) = n_{\alpha,0} \exp\left[-\frac{q_\alpha\phi(\boldsymbol{r})}{K_\mathrm{B}T_\alpha}\right]. \tag{2.6}$$

Using the particle density we may now express the charge density as a sum of the respective particle densities multiplied by the particle charges:

$$\rho_{\mathrm{e}}(\phi) \equiv \sum_{\alpha} q_{\alpha} n_{\alpha}(\phi(\boldsymbol{r})). \tag{2.7}$$

Consequently, (2.3) may be written as

$$\triangle\phi(\boldsymbol{r}) = -\frac{1}{\varepsilon_0} \sum_{\alpha} q_{\alpha} n_{\alpha,0} \exp\left[-\frac{q_{\alpha}\phi(\boldsymbol{r})}{K_{\mathrm{B}} T_{\alpha}}\right]. \tag{2.8}$$

We will further assume that the electric potential will only perturb the background density, hence we may expand the exponential in a Taylor expansion:

$$\triangle\phi(\boldsymbol{r}) = -\frac{1}{\varepsilon_0}\left[\sum_{\alpha} q_{\alpha} n_{\alpha,0} - \sum_{\alpha} \frac{q_{\alpha}^2 n_{\alpha,0}\phi(\boldsymbol{r})}{K_{\mathrm{B}} T_{\alpha}} + \dots\right]. \tag{2.9}$$

The constant term in the Taylor expansion must vanish, otherwise the quasineutrality of the plasma cannot be kept. Finally we arrive at a suitable form of the Poisson equation

$$\triangle\phi(\boldsymbol{r}) = \left[\sum_{\alpha} \frac{q_{\alpha}^2 n_{\alpha,0}}{\varepsilon_0 K_{\mathrm{B}} T_{\alpha}}\right] \phi(\boldsymbol{r}). \tag{2.10}$$

In the following, let us assume that we will pick a particular testing particle having the charge Q. We will further assume that the electric potential is spherically symmetric around this selected particle. In that case only the radial part of the Poisson equation is relevant:

$$\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left[r\phi(r)\right] = a\phi(r), \tag{2.11}$$

where we defined

$$a = \sum_{\alpha} \frac{q_{\alpha}^2 n_{\alpha,0}}{\varepsilon_0 K_{\mathrm{B}} T_{\alpha}} \tag{2.12}$$

for simplicity. Then by substituting $\psi(r) \equiv r\phi(r)$ equation (2.11) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\psi(r) = a\psi(r), \tag{2.13}$$

which has a solution in a form

$$\psi(r) = c_1 e^{\sqrt{a}r} + c_2 e^{-\sqrt{a}r}. \tag{2.14}$$

The constant c_1 must be zero, otherwise the solution diverges for large r, which is unphysical. The solution for the electric potencial then has a form

$$\phi(r) = \frac{c_2}{r} e^{-\sqrt{a}r}. \tag{2.15}$$

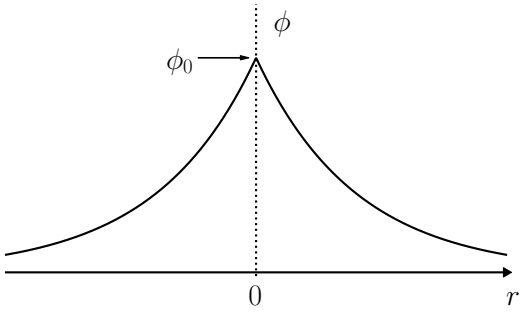

Figure 2.2: The modelled profile of the attenuation of the electric potential around the charge concentration.

The form of c_2 may be expressed in a limit of $r \to 0$, when the potential should converge to the coulombic solution of an isolated charge. Obviously

$$\frac{c_2}{r} \to \frac{Q}{4\pi\varepsilon_0 r}. \tag{2.16}$$

A final solution then takes a form of

$$\phi(r) = \frac{Q}{4\pi\varepsilon_0 r} \exp\left[-\frac{r}{\lambda_\mathrm{D}}\right], \tag{2.17}$$

where

$$\lambda_\mathrm{D} \equiv \left[\sum_\alpha \frac{q_\alpha^2 n_{\alpha,0}}{\varepsilon_0 K_\mathrm{B} T_\alpha}\right]^{-1/2} \tag{2.18}$$

is a definition of the Debye shielding length. This gives the characteristic scale on which the charge of the selected testing particle is damped in plasmas.

By going back to our simplistic model with a positive charge, when assuming that only the electrons will be responsible for the shielding, one gets a simplified definition of the Debye length scale

$$\lambda_\mathrm{D} \equiv \sqrt{\frac{\varepsilon_0 K_\mathrm{B} T_\mathrm{e}}{n_0 e^2}}, \tag{2.19}$$

which is usually used in applications. Thus in such a system with an artificially inserted ion charge the electrons will concentrate so that the inserted charge is exponentially attenuated (see Fig. 2.2).

Note that Debye length increases with temperature, when the thermal motions of electrons are larger and the shielding is thus less effective (the electrons are moving out of the shielding region) and decreases with increasing density (the more electrons, the more effective the shielding is).

A few additional notes.

- In physics, temperature is a measure of the width of the particle velocity distribution and usually is expressed in Kelvins. In plasma physics, a quite often used

unit is the electronvolt, which is expressed by the kinetic energy of the particle having the given temperature. From

$$E_{\mathrm{k}} = \frac{1}{2} m \bar{v}^2 = K_{\mathrm{B}} T \tag{2.20}$$

where $\bar{v}$ is the root-mean-square velocity, we find a useful working relation that 1eV corresponds to 11 600K.

- Plasma may have many temperatures at once. Electrons and ions may have different velocity distributions and thus different temperatures. This is due to the fact that the frequencies of ion-ion and electron-electron collisions are much higher than the frequency of ion-electron collisions. The velocity distribution of a single particle species does not have to be isotropic, e.g., when a magnetic field is present. Then the temperature of the plasma is usually higher along the field than across the field.

- Plasma densities vary over many orders in the universe. The density of 10^6 m^{-3} is the typical plasma density in the interplanetary space around the Earth, it might be many orders less in interstellar or intergalactic space. It is around 10^8 m^{-3} in giant molecular clouds, from which stars are born. The plasma density in the solar photosphere is in the order of 10^{22} m^{-3}, and in the solar core 10^{31} m^{-3}. In laboratory plasmas, densities between 10^{20}–10^{28} m^{-3} are considered for experiments with thermonuclear fusion.

- Various approaches can be practically used to describe plasmas and their dynamics. The most detailed is the kinetic description using the Boltzmann equation. The dynamics of plasmas can also be described using the fluid and magnetohydrodynamic approaches. The motion of individual or test particles is investigated using the drift description. Using some of these approaches, we may state the conditions for the plasma equilibrium, to study instabilities and conditions for the propagation, excitation and damping of waves and other oscillatory motions. We will study all of these aspects in the following chapters.

- The ionisation degree may be estimated using the Saha equation (1.126). It strongly depends on temperature. For example, considering air at room temperature (assuming that the air is composed from nitrogen only), one gets the estimate (assuming $n_0 \sim 3 \times 10^{25}$ m^{-3}, $T \sim 300$ K, and $U_{\mathrm{i}} \sim 14.5$ eV)) of the ionisation degree to be $n_i/n_0 \sim 10^{-122}$. The air in the room certainly is not a plasma.

- The term ionosphere stems from the fact that is contains ions. Let us assess the situation of hydrogen: the temperature of the ionosphere is about $K_{\mathrm{B}} T_{\mathrm{e}} \sim 0.1$ eV, the ionisation potential is $U^{\mathrm{H}} = 13.6$ eV, the particle density is about $n \sim 10^{12}$ m^{-3}. Using the Saha equation one gets the expected ion density of $n_{\mathrm{i}}^2 \sim 10^{-52}$ m^{-3}. The ionisation degree is tiny. However, the ionosphere is not composed of hydrogen. There are various chemical reactions with a smaller ionisation potential,

which indeed fill the ionosphere with ions and electrons. E.g.,

$$O_2^+ + e^- \rightarrow O + O + 6.6\text{eV} \tag{2.21}$$

or

$$O_2^+ + N_2 \rightarrow NO^+ + NO + 0.87\text{eV}. \tag{2.22}$$

Then the matter in the ionosphere may be considered to be a plasma.

- In practice, there are three criteria which help to decide whether the studied system may be considered a plasma:

  1. The system is much larger than the Debye length, hence the quasineutrality holds. $L \gg \lambda_D$, where L is the characteristic size of the system.
  2. Shielding is effective. The number of particles in the Debye sphere is much larger than one, $N_D \gg 1$. In case it is not, the studied system is probably a bound system.
  3. The particle interactions are dominated by electromagnetic forces and not by collisions. $\omega\tau > 1$, where ω is the characteristic oscillation frequency of plasma and τ is the characterictic time between particle collisions. For example, the exhaust of a jet engine cannot be considered plasma despite the huge temperature and high ionisation, because the particle interactions are purely collisional and electromagnetic forces play a negligible role.

Chapter 3

Charged particle motion in electromagnetic fields

In plasma, we sometimes have to consider the motion of each individual particle to capture some of its effects. For instance, when the densities of plasmas are not large enough for it to be allowed to completely ignore the motion of individual particles. The plasmas are usually not dominated by collisions. Hence the motion of individual particles play an important role. In this approach we will study the motion of a test particle in the *prescribed* electric and magnetic fields. In general, the trajectory of such a motion may be unambiguously found by solving the equation of motion

$$m\frac{\mathrm{d}\boldsymbol{v}(\boldsymbol{r},t)}{\mathrm{d}t} = q\left[\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{v}(\boldsymbol{r},t)\times\boldsymbol{B}(\boldsymbol{r},t)\right], \tag{3.1}$$

In the following sections we will discuss some examples by giving particular analytical forms to background fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$. We will derive the relations for some *drifts*, thus systematic motion of charged particles which are not captured when we use other approaches to study plasmas. It is important to study drifts of particles because these drifts often drive instabilities, which is critical e.g. for laboratory plasma in plasma confinement and thermonuclear fusion.

3.1 Homogeneous magnetic field

Let us define the system with a constant homogeneous magnetic field and no electric field, where we define the Cartesian coordinate system so that the z axis is parallel to the direction of magnetic induction. Then we have the vector equation of motion in the form of

$$m\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = q\left[\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}\times\begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}\right], \tag{3.2}$$

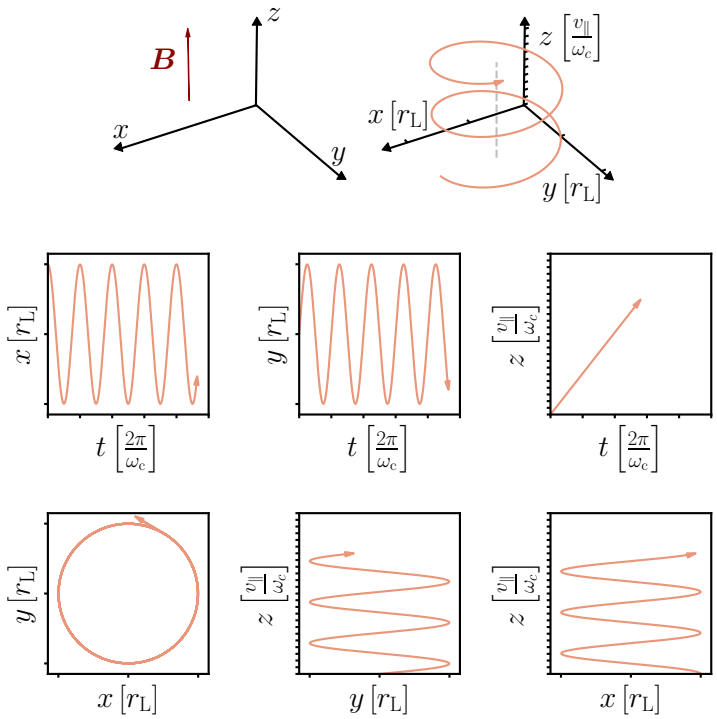

Figure 3.1: Motion of an electron in homogeneous magnetic field. The 3-D view of the motion, trajectories in the coordinate–time and coordinate–coordinate spaces.

where m is the mass of the particle charged with the electrical charge q. This equation becomes a system of scalar differential equations:

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = qBv_y, \tag{3.3}$$

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = -qBv_x, \tag{3.4}$$

$$m\frac{\mathrm{d}v_z}{\mathrm{d}t} = 0. \tag{3.5}$$

We immediately see that the solution for the parallel component of the velocity is trivial, which corresponds to constant motion along the magnetic field. By differentiating the perpendicular components with respect to time we have

$$\frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} = \frac{q}{m}B\frac{\mathrm{d}v_y}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2v_y}{\mathrm{d}t^2} = -\frac{q}{m}B\frac{\mathrm{d}v_x}{\mathrm{d}t}. \tag{3.6}$$

Their combination then gives

$$\frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} = -\left(\frac{q}{m}B\right)^2 v_x \quad \text{and} \quad \frac{\mathrm{d}^2v_y}{\mathrm{d}t^2} = -\left(\frac{q}{m}B\right)^2 v_y, \tag{3.7}$$

which has a trivial solution in the form of harmonic functions. We define

$$\omega_\mathrm{c} \equiv \frac{qB\,\mathrm{sgn}\,q}{m} = \frac{|q|B}{m} \tag{3.8}$$

to be the *cyclotron* frequency. Let's take the solution of the second order ordinary differential equations as

$$v_{x,y} = v_\perp \exp\left(\mathrm{i}\omega_\mathrm{c} t + \mathrm{i}\delta_{x,y}\right), \tag{3.9}$$

where $v_\perp$ is the amplitude of the velocity and it is obviously[1] identical for both perpendicular components and is usually initially driven by thermal motion. Both perpendicular components are allowed to have different phase shifts $\delta_{x,y}$. The trajectory in the phase space obviously is a circle. Let's set the spacetime coordinate system so that $\delta_x = 0$. Then

$$v_x = v_\perp \exp \mathrm{i}\omega_\mathrm{c} t, \tag{3.10}$$

and we consider only the real component to have a physical meaning. Then

$$v_y = \frac{m}{qB}\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{m}{qB} v_\perp \mathrm{i}\omega_\mathrm{c} \exp \mathrm{i}\omega_\mathrm{c} t = \frac{m}{qB} v_\perp \mathrm{i}\frac{qB \operatorname{sgn} q}{m} \exp \mathrm{i}\omega_\mathrm{c} t = v_\perp \mathrm{i} \operatorname{sgn} q \exp \mathrm{i}\omega_\mathrm{c} t. \tag{3.11}$$

We see that the direction of rotation in the phase space depends on the charge of the particle, thus being opposite for electrons and ions.

To obtain the trajectory, one needs to integrate (3.10) and (3.11):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Re[v_x] = \Re[v_\perp \exp \mathrm{i}\omega_\mathrm{c} t] \to x - x_0 = \Re\left[-\mathrm{i}\frac{v_\perp}{\omega_\mathrm{c}} \exp \mathrm{i}\omega_\mathrm{c} t\right] = \frac{v_\perp}{\omega_\mathrm{c}} \sin \omega_\mathrm{c} t \tag{3.12}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \Re[v_y] = \Re[v_\perp \mathrm{i} \exp \mathrm{i}\omega_\mathrm{c} t] \to y - y_0 = \Re\left[\frac{v_\perp}{\omega_\mathrm{c}} \operatorname{sgn} q \exp \mathrm{i}\omega_\mathrm{c} t\right] = \frac{v_\perp}{\omega_\mathrm{c}} \operatorname{sgn} q \cos \omega_\mathrm{c} t \tag{3.13}$$

Of course, we should not forget about the parallel component

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \Re[v_z] = 0 \to z - z_0 = v_\parallel t. \tag{3.14}$$

Then the total trajectory is a helix. The ratio $v_\perp/\omega_\mathrm{c} = r_\mathrm{L}$ is the *Larmor radius*, the typical radius of gyration. Note that

$$r_\mathrm{L} = \frac{v_\perp m}{qB \operatorname{sgn} q} = \frac{v_\perp m}{|q|B} \tag{3.15}$$

depends on the perpendicular velocity and the mass of the particle. A more massive particle has a larger Larmor radius and thus a smaller cyclotron frequency. Also, a stronger magnetic field causes the particles to gyrate with tighter orbits.

Electrons and ions circulate the magnetic field lines in the opposite directions (see Fig. 3.1), but they both follow the field lines (we will generalise this case further). The charge travelling along the helical trajectory induces it's own magnetic field (according to Maxwell's equations), which has the opposite sign to the background magnetic field. Thus the plasma weakens the background magnetic field, it is *diamagnetic*.

[1] It might be shown that if we multiply (3.3) by v_x, (3.4) by v_y and sum the two, we obtain $\frac{\mathrm{d}v_x^2}{\mathrm{d}t} + \frac{\mathrm{d}v_y^2}{\mathrm{d}t} = \frac{\mathrm{d}v^2}{\mathrm{d}t} = 0$, hence the amplitude of the velocity is constant in time.

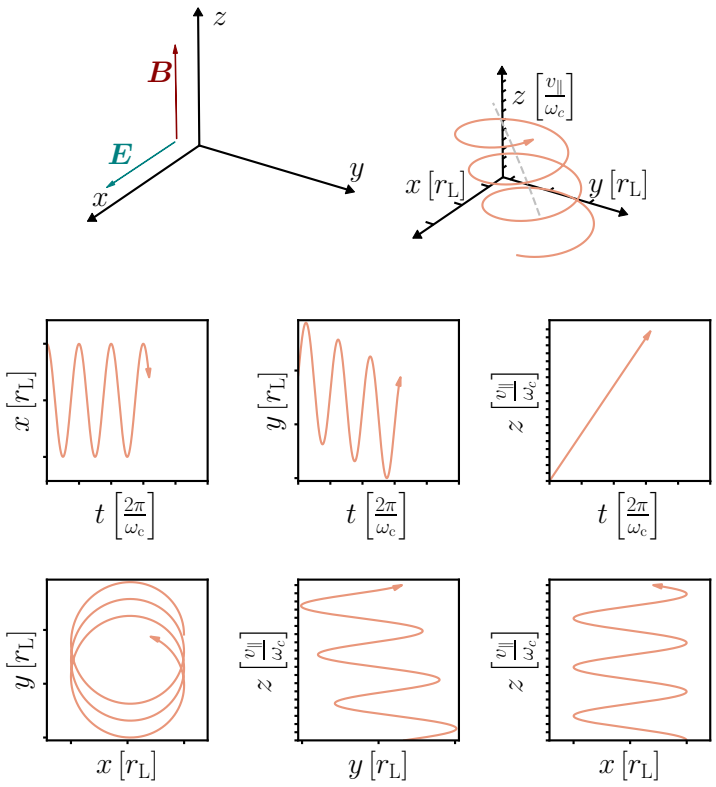

Figure 3.2: Motion of an electron in the crossed electric and magnetic fields.

3.2 Homogeneous electric field

Now we will add a homogeneous constant electric field to the problem, and thus we choose the coordinate system so that the magnetic field is still along the z axis, and the electric field has non-trivial components only in x and z directions (see Fig. 3.2). Then the equation of motion is

$$m\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = q\left[\begin{pmatrix} E_x \\ 0 \\ E_z \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}\right]. \tag{3.16}$$

The parallel component can be solved immediately:

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = \frac{q}{m}E_z \rightarrow z - z_0 = \frac{q}{2m}E_z t^2 + v_{z,0}t. \tag{3.17}$$

The particle is uniformly accelerated along the magnetic field caused by the parallel component of the electric field. By using the definition of the cyclotron frequency, the

remaining two components of the equation of motion are

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{q}{m}E_x + \omega_\mathrm{c}\,\mathrm{sgn}\,q v_y, \tag{3.18}$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\omega_\mathrm{c}\,\mathrm{sgn}\,q v_x. \tag{3.19}$$

Similarly to the previous case, we take the time derivative of (3.18) and (3.19) and combine the two. We have

$$\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} = \omega_\mathrm{c}\,\mathrm{sgn}\,q\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\,\mathrm{sgn}^{\,2} q\omega_\mathrm{c}^2 v_x = -\omega_\mathrm{c}^2 v_x, \tag{3.20}$$

where we assumed that $q \neq 0$, and

$$\frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\,\mathrm{sgn}\,q\omega_\mathrm{c}\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\,\mathrm{sgn}\,q\omega_\mathrm{c}\left(\frac{q}{m}E_x + \omega_\mathrm{c}\,\mathrm{sgn}\,q v_y\right) = -\omega_\mathrm{c}^2\left(\frac{E_x}{B} + v_y\right). \tag{3.21}$$

We solved (3.20) already in the previous case and obtained (3.10):

$$v_x = v_\perp \exp \mathrm{i}\omega_\mathrm{c} t. \tag{3.22}$$

For (3.21) we note that E_x/B does not depend on time, thus we may simply add it to v_y on the left-hand side of the equation which then becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(v_y + \frac{E_x}{B}\right) = -\omega_\mathrm{c}^2\left(v_y + \frac{E_x}{B}\right), \tag{3.23}$$

which has a trivial solution in the form of

$$v_y = v_\perp \mathrm{i}\,\mathrm{sgn}\,q \exp \mathrm{i}\omega_\mathrm{c} t - \frac{E_x}{B}. \tag{3.24}$$

We see that in the solution an additional term, *drift*, appeared. Let's find a general solution in a vector form. The equation of motion is

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \tag{3.25}$$

If we substitute $\boldsymbol{v}' = \boldsymbol{v} - \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}$ (we are transforming to a different coordinate system, which moves with a speed of $-\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}$ with respect to the original one). As we will find out later, in this moving coordinate system the parallel components of the electric field disappear. Thus (3.25) becomes:

$$\begin{aligned} m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t} &= q\left(\boldsymbol{E} + \boldsymbol{v}'\times\boldsymbol{B} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\times\boldsymbol{B}\right) = \\ &= q\left[\cancel{\boldsymbol{E}} + \boldsymbol{v}'\times\boldsymbol{B} - \frac{1}{\cancel{B^2}}\cancel{\boldsymbol{E}B^2} + \frac{\boldsymbol{B}(\boldsymbol{E}\cdot\boldsymbol{B})}{B^2}\right] = \\ &= q\left[\boldsymbol{v}'\times\boldsymbol{B} + \boldsymbol{E}_\parallel\right], \end{aligned} \tag{3.26}$$

where we defined the parallel component of the electric field

$$\boldsymbol{E}_{\parallel} \equiv \frac{\boldsymbol{B}(\boldsymbol{E}\cdot\boldsymbol{B})}{B^2}. \tag{3.27}$$

We transformed the new problem back to the previous case, as in this moving coordinate system the effects of the electric field (causing the acceleration along the magnetic field lines) and the magnetic field (causing the gyration around the magnetic field lines) are separable. Thus we solve the problem in the moving coordinate system and transform back to the original coordinate system

$$\boldsymbol{v} = \boldsymbol{v}' + \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}. \tag{3.28}$$

We see that a new component of the particle velocity appeared,

$$\boldsymbol{v}_E = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}, \tag{3.29}$$

which is the expression for the *drift of the gyration centre* (illustrated in Fig. 3.2), the *E-B drift*. Note that the sign of this drift does neither depend on the particle mass nor on its electric charge. This drift does not cause a separation of the charge.

Note that the E-B drift naturally shows up as a consequence of the Lorentz transform of the electromagnetic field tensor. For details, see section B.1 in the Appendix.

3.2.1 Drift in the field of general force

Let's make a generalisation, which we will use further. In the case of E-B drift, we have an additional force (the force of the electric field) acting on the motion of the particle. This force (the Lorentz force) is simply given by

$$\boldsymbol{F}_E = q\boldsymbol{E}. \tag{3.30}$$

After substitution into (3.29) we have

$$\boldsymbol{v}_E = \frac{1}{q}\frac{\boldsymbol{F}_E\times\boldsymbol{B}}{B^2}. \tag{3.31}$$

Could this expression be generalised for *any* force $\boldsymbol{F}$ other than Lorentz? The answer is yes, if and only if the force in question acts homogeneously, in particular, when the vector of the force does not depend on speed nor position. Then we have a drift of gyration centre $\boldsymbol{v}_{\mathrm{gs}}$

$$\boldsymbol{v}_{\mathrm{gs}} = \frac{1}{q}\frac{\boldsymbol{F}\times\boldsymbol{B}}{B^2}. \tag{3.32}$$

An example? The action of gravity field may be considered homogeneous in laboratory or at least when the small spatial scales (compared to the curvature of the gravity field lines) are in question. Then we have the *gravity drift* $\boldsymbol{v}_g$

$$\boldsymbol{v}_g = \frac{m}{q}\frac{\boldsymbol{g}\times\boldsymbol{B}}{B^2}, \tag{3.33}$$

where $\boldsymbol{g}$ is the vector of gravity acceleration. How about the centrifugal force $\boldsymbol{F}_{\mathrm{c}} = \frac{mv_{\parallel}^2}{R^2}\boldsymbol{R}$ acting in the curved magnetic field? Is the expression for the curvature drift $\boldsymbol{v}_{\mathrm{c}}$

$$\boldsymbol{v}_{\mathrm{c}} = \frac{mv_{\parallel}^2}{qR^2B^2}\boldsymbol{R} \times \boldsymbol{B} \tag{3.34}$$

correct? It is not, because we violated one of the assumptions written above: the vector of the acting force *does depend on position* in this case. The magnetic field is not homogeneous anymore and we must find another correction. (A figure justifying the notation $v_{\parallel}$ in the above equations would be suitable here. It may contain both macroscopic view in which the force as well as B would not be homogeneous and a zoomed view with both of them being approximately homogeneous, i.e. justifying use of (3.32) above.)

3.3 Inhomogeneous magnetic field

Let us study the effects of the non-homogeneous magnetic field. We will keep our coordinate system, thus having a magnetic field in the z-direction, however we will allow for a smooth change in the y-direction. What does the word *smooth* mean? Let us remind the reader that there is a characteristic scale of the particle motion even in the case of homogeneous field: the Larmor radius $r_{\mathrm{L}} = \frac{mv_{\perp}}{|q|B}$. By *smooth* we further mean that the characteristic spatial change of the magnetic field occurs on much larger scales than the Larmor gyration. The characteristic scale of the spatial change of the magnetic field L may be estimated by comparing the gradient of the field to its magnitude, thus

$$L \sim \frac{|\boldsymbol{B}|}{|\nabla B|}. \tag{3.35}$$

We will study the case when $L \gg r_{\mathrm{L}}$. The ratio of the two is a small number, thus can be used as a small parameter to expand the physical functions to the Taylor series to the first order and neglect the higher orders of this small parameter. Hence in our case of the inhomogeneous magnetic field we have

$$\boldsymbol{B} = \boldsymbol{B}_0 + (\boldsymbol{r} \cdot \nabla)\boldsymbol{B} \tag{3.36}$$

and in our coordinate system we have

$$B_z = B_0 + y\frac{\partial B_z}{\partial y}. \tag{3.37}$$

What effect do we expect? By waving our hands we can say that when we study the gyration of the testing particle, we see that the particles still gyrate along the magnetic field lines. However, in the part of the orbit where the magnetic field is stronger, the Larmor radius is smaller and vice versa. We should see a drift in the direction perpendicular to both $\boldsymbol{B}$ and $\nabla|\boldsymbol{B}|$, which would be in the x direction. We would

expect the effect to increase with the gradient of the field, when the difference between the effective radii in the "upper" and "lower" part of the orbit is larger.

In our case, only the y component of the Lorentz force is perturbed

$$F_y = -qv_x B_z = -qv_x \left(B_0 + y\frac{\partial B_z}{\partial y} \right). \tag{3.38}$$

The first term on the right-hand side is responsible for the normal Larmor rotation, the second term is a correction caused by the inhomogeneous magnetic field and thus causes the expected drift. The approach we will take is that we will separate the effects of the B_0 causing the Larmor rotation and the effect of the small correction from $\nabla|\boldsymbol{B}|$. To investigate the correction, we will average the particle motion over one Larmor rotation. Let's remind again that this is possible if and only if the Larmor rotation is the dominant motion of the studied particle and thus when $|\partial B_z/\partial y| \ll |B_0|$. Then we use the solution for v_x from (3.10) and for y from (3.13). We have

$$F_y = -qv_\perp \cos\omega_\mathrm{c} t \left(B_0 + r_\mathrm{L}\ \mathrm{sgn}\, q \cos\omega_\mathrm{c} t \frac{\partial B_z}{\partial y} \right). \tag{3.39}$$

The averaged (over the Larmor orbit) Lorentz force is thus

$$\begin{aligned}
\langle F_y \rangle &= \frac{1}{2\pi}\int_0^{2\pi} F_y \mathrm{d}(\omega_\mathrm{c} t) = -\frac{1}{2\pi}\int_0^{2\pi} qv_\perp \cos\omega_\mathrm{c} t \left(B_0 + r_\mathrm{L}\ \mathrm{sgn}\, q \cos\omega_\mathrm{c} t \frac{\partial B_z}{\partial y} \right) \mathrm{d}(\omega_\mathrm{c} t) = \\
&= -qv_\perp B_0 \frac{1}{2\pi}\int_0^{2\pi} \cos\omega_\mathrm{c} t \mathrm{d}(\omega_\mathrm{c} t) - qv_\perp r_\mathrm{L}\ \mathrm{sgn}\, q \frac{\partial B_z}{\partial y}\frac{1}{2\pi}\int_0^{2\pi} \cos^2\omega_\mathrm{c} t\, \mathrm{d}(\omega_\mathrm{c} t) = \\
&= 0 - \frac{1}{2} v_\perp r_\mathrm{L}\, q\, \mathrm{sgn}\, q \frac{\partial B_z}{\partial y}.
\end{aligned} \tag{3.40}$$

We leave it to the reader to prove that the integral over one period of the sine or cosine function vanishes and that the integral over one period of the sine squared or cosine squared equals to π.

We found a correction term. Since the averages of the remaining components of the Lorentz force vanish, we may generalise

$$\langle \boldsymbol{F} \rangle = -\frac{1}{2} v_\perp r_\mathrm{L}\, q\, \mathrm{sgn}\, q \nabla|\boldsymbol{B}|, \tag{3.41}$$

and use it in the expression for the drift in the field of general force (3.32). Note that under the assumptions we took the additional force defined by (3.41) is homogeneous and does not depend on velocity. Hence we have a drift of the gyration centre, the *grad-B drift* $\boldsymbol{v}_{\nabla B}$:

$$\boldsymbol{v}_{\nabla B} = \frac{1}{2}\,\mathrm{sgn}\, q v_\perp r_\mathrm{L}\, \frac{\boldsymbol{B} \times \nabla|\boldsymbol{B}|}{B^2}. \tag{3.42}$$

Note that the drift direction does depend on the sign of the particle charge and thus the grad-B drift may cause the separation of charges and drive an additional electric field. See Fig. 3.3.

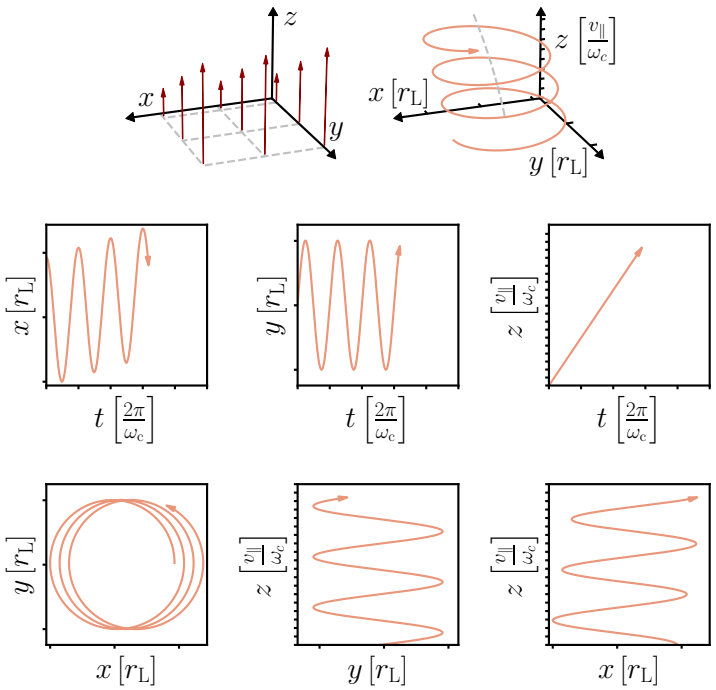

Figure 3.3: Drift of an electron in an inhomogeneous magnetic field.

At the beginning we used the assumption of small spatial changes of the magnetic field to derive this relation and we should check that the assumption holds and thus the solution is consistent. We will simply compute how much the particle moves due to the drift after one Larmor rotation, hence after one period $\tau = 2\pi/\omega_\mathrm{c}$. We will further use (3.35) and $v_\perp/\omega_\mathrm{c} = r_\mathrm{L}$ to evaluate the displacement, Δ, of the gyration centre over one rotation:

$$\Delta = \tau|\boldsymbol{v}_{\nabla B}| = \frac{2\pi}{\omega_\mathrm{c}}\frac{1}{2}v_\perp r_\mathrm{L}\frac{|\nabla B|}{|B|} = \pi r_\mathrm{L}\frac{r_\mathrm{L}}{L} \ll r_\mathrm{L} \tag{3.43}$$

under the assumption $L \gg r_\mathrm{L}$. The solution is consistent with the assumptions, the dominant motion still is the Larmor rotation.

Now we may continue to investigate the motion of particles in the curved magnetic field. Obviously, additionally to the curvature drift (3.34), we need to account for the inherent gradient of the field. Let's make a qualitative estimate. The Cartesian coordinates are no longer convenient to use when investigating this case, cylindrical coordinates (R, ϑ, z) are more appropriate (they naturally incorporate the "curvature" of the field). In any case, we assumed the changes of the magnetic field to be smooth, thus we may always introduce *local* cylindrical coordinates to approximate the real configuration of the field lines. In a vacuum we have

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} = 0, \tag{3.44}$$

hence when we assume that the dominant direction of the magnetic field lines is along the azimuthal ϑ direction, $\nabla|\boldsymbol{B}|$ must have only the component in R direction. Hence

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{R}\frac{\partial}{\partial R}(RB_\vartheta) = 0 \tag{3.45}$$

and therefore

$$B_\vartheta = \frac{\mathcal{C}}{R}, \tag{3.46}$$

where $\mathcal{C}$ is the integration constant. We further obtain

$$\nabla B = \frac{\partial}{\partial R}\frac{\mathcal{C}}{R} = -\mathcal{C}\frac{1}{R^2}\frac{\boldsymbol{R}}{R} \tag{3.47}$$

and when expressing $\mathcal{C}$ from (3.46) we finaly have

$$\nabla B = -\frac{|B|}{R^2}\boldsymbol{R}. \tag{3.48}$$

Now we can ingest this expression for grad-B into (3.42) and (when also explicitly writing the expression for r_L) have

$$\boldsymbol{v}_{\nabla B} = -\frac{1}{2}\operatorname{sgn} q\frac{v_\perp r_\mathrm{L}}{B^2}\boldsymbol{B} \times |\boldsymbol{B}|\frac{\boldsymbol{R}}{R^2} = \frac{1}{2}\frac{mv_\perp^2}{qB^2R^2}\boldsymbol{R} \times \boldsymbol{B}. \tag{3.49}$$

After adding the curvature drift (3.34), we finally have an expression for the drift in the curved magnetic field:

$$\boldsymbol{v}_\mathrm{gc} = \boldsymbol{v}_\mathrm{c} + \boldsymbol{v}_{\nabla B} = \frac{m}{q}\frac{\boldsymbol{R} \times \boldsymbol{B}}{R^2B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right). \tag{3.50}$$

This has important consequences, especially for laboratory plasmas. It shows that the plasma cannot be confined in the curved magnetic field. Both important drifts, the curvature drift and the grad-B drift, sum up and thus act to displace the plasma. Both these drifts are drivers for *kink instability*. Let's imagine that the plasma flows along the cylindrical tube of the magnetic field (the flux tube) with a prevalently poloidal configuration of the magnetic field, which is generated by the current. When this flux tube is displaced (e.g. by some random motions), the tube gets curved with the gradient of the magnetic field pointing against the displacement. The drifts in the curved field cause the charge separation in the direction perpendicular to the plane of the kink. The separated changes introduce a small electric field and the E-B drift cause the instability to grow.

When the configuration of the magnetic field in the flux tube is rather helical than cylindrical, E-B drift may cancel out the drift in the curved magnetic field and the plasma may remain confined within this tube. This is the famous tokamak configuration.

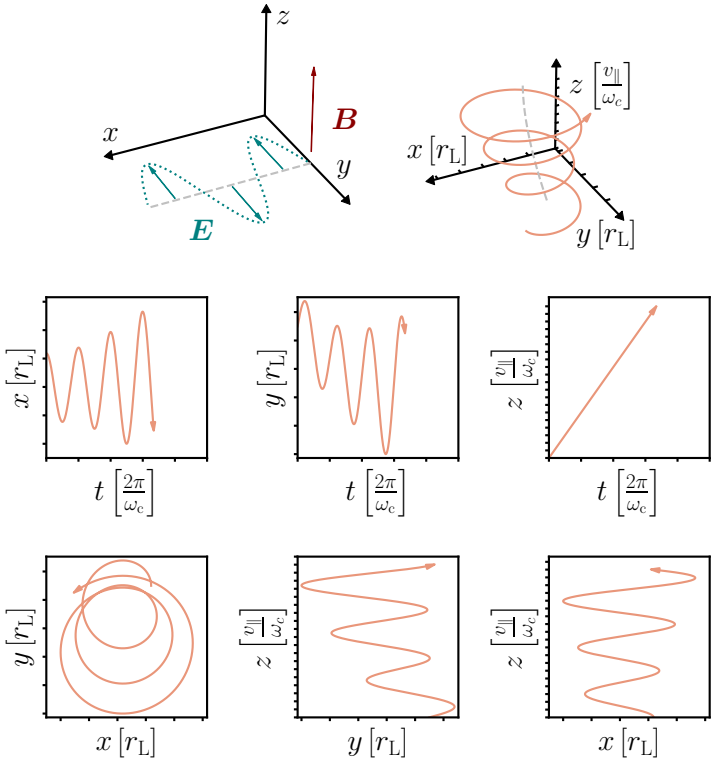

Figure 3.4: Drift of electrons in the inhomogeneous electric field.

3.4 Inhomogeneous electric field

The inhomogeneous electric field also requires a correction to the normal E-B drift of particles. Let us investigate one example only, when the electric field has a harmonic character. Such a configuration is not uncommon, later on we will study plasma oscillations and waves propagating through plasmas, where we will see that there are many harmonic motions in plasmas, which may be approximated as a superposition of simple harmonic functions. Let us assume that we have a periodic electric field in the x direction (see Fig. 3.4),

$$\boldsymbol{E} = E_0 \cos ky \boldsymbol{e}_x, \tag{3.51}$$

where k is the wave number, which is related to the wavelength ($\lambda = 2\pi/k$), which is the characteristic spatial scale of the perturbation. Otherwise we will still keep the same configuration of the coordinate system as we did in the previous cases. Hence the perpendicular components of the equation of motion are

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{qB}{m} v_y + \frac{q}{m} E_x(y) \quad \text{and} \quad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -\frac{qB}{m} v_x. \tag{3.52}$$

The standard way to solve this system is to take the time derivative of these equations and combine the two, hence

$$\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} = -\omega_\mathrm{c}^2 v_x + \omega_\mathrm{c}\frac{\operatorname{sgn} q}{B}\frac{\mathrm{d}E_x}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\omega_\mathrm{c}^2 v_y - \omega_\mathrm{c}^2\frac{E_x(y)}{B}. \tag{3.53}$$

This is a system of implicit ordinary differential equations, which cannot be trivially solved. We would need to proceed iteratively, since in order to find the solution, we need to know $\boldsymbol{E}$, which is the function of the solution. Let's focus on the case of small electric fields and apply the averaging formalism again in order to find the correction. In this approach we simply use the unperturbed solution and average over one Larmor rotation.

By substituting (3.13) into (3.53) we have

$$\frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\omega_\mathrm{c}^2 v_y - \omega_\mathrm{c}^2\frac{E_0}{B}\cos\left[k(y_0 + r_\mathrm{L}\ \operatorname{sgn} q\cos\omega_\mathrm{c} t)\right], \tag{3.54}$$

which we average over one Larmor orbit to have

$$\left\langle\frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2}\right\rangle = -\omega_\mathrm{c}^2\langle v_y\rangle - \omega_\mathrm{c}^2\frac{E_0}{B}\langle\cos\left[k(y_0 + r_\mathrm{L}\ \operatorname{sgn} q\cos\omega_\mathrm{c} t)\right]\rangle. \tag{3.55}$$

The action of the additional force oscillates (we do not expect a secular trend in time; this assumption should be validated in the solution), therefore when averaged over one Larmor orbit, it vanishes, thus

$$\left\langle\frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2}\right\rangle = 0. \tag{3.56}$$

We approximate the cosine term by the Taylor series expansion of the parameter kr_L, which we assume to be small[2]. Essentially again the electric field is perturbed on scales much larger than the Larmor radius

$$\begin{aligned}
\cos\left[k(y_0\ + r_\mathrm{L}\ \operatorname{sgn} q\cos\omega_\mathrm{c} t)\right] &= \\
&= \cos ky_0\cos\left(kr_\mathrm{L}\ \operatorname{sgn} q\cos\omega_\mathrm{c} t\right) - \sin ky_0\sin\left(kr_\mathrm{L}\ \operatorname{sgn} q\cos\omega_\mathrm{c} t\right) = \\
&= \cos ky_0\cos\left(kr_\mathrm{L}\ \cos\omega_\mathrm{c} t\right) - \operatorname{sgn} q\sin ky_0\sin\left(kr_\mathrm{L}\ \cos\omega_\mathrm{c} t\right) \\
&= \cos ky_0\left[1 - \frac{1}{2}k^2 {r_\mathrm{L}}^2\cos^2\omega_\mathrm{c} t\right] - \operatorname{sgn} q\sin ky_0(kr_\mathrm{L}\ \cos\omega_\mathrm{c} t)
\end{aligned} \tag{3.57}$$

Hence after averaging, the term with the sine vanishes, while the term with the cosine squared becomes 1/2. Finally

$$\langle v_y\rangle = -\frac{E_0}{B}\cos ky_0\left(1 - \frac{1}{4}k^2 {r_\mathrm{L}}^2\right) = -\frac{E_x(y)}{B}\left(1 - \frac{1}{4}k^2 {r_\mathrm{L}}^2\right). \tag{3.58}$$

We may generalise this expression to

$$\boldsymbol{v}_E = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\left(1 - \frac{1}{4}k^2 {r_\mathrm{L}}^2\right) \tag{3.59}$$

[2]We remind the reader that $\cos\varepsilon \approx 1 - \frac{1}{2}\varepsilon^2$ and $\sin\varepsilon \approx \varepsilon$ for $\varepsilon \to 0$.

and we leave it to the reader to prove that in our case it is the correct generalisation. In the section dealing with waves in plasmas we will often use the solution as a superposition of various modes, when for each mode we may conveniently solve the equations in the Fourier space. The Fourier representation is convenient because differential operators become multiplications. We ask the reader to see (5.13) for a short moment and apply it to the equation we just obtained. We may generalise this expression to

$$\boldsymbol{v}_E = -\frac{\boldsymbol{B}\times}{B^2}\left(1+\frac{1}{4}{r_\mathrm{L}}^2\triangle\right)\boldsymbol{E}. \tag{3.60}$$

The correction term captures the effect of the finite Larmor radius. Since r_L is larger for ions, ions "feel" the changes in the background electric field stronger than electrons. When an electric field appears in plasma, the *drift instability* causes this electric field to grow. Note that the effect of the inhomogeneous electric field is proportional to $(kr_\mathrm{L})^2$, whereas the effect of the inhomogeneous magnetic field is proportional to kr_L (when applying the Fourier transform formalism to (3.42)). Hence, the effects of the non-uniform E-field are relatively more important at large k, therefore at small spatial scales.

3.5 Drift currents

Some of the drifts discussed above and summarised graphically in Fig. 3.6 depend on the mass or charge of the particles, thus lead to the separation of charges and give rise to *drift currents*. In plasmas we have to consider k different particle species, each having particle density n_k, mass m_k and q_k. From the general equation for the drift velocity (3.32), we derive the relation for the drift current

$$\boldsymbol{j}_\mathrm{drift} = \sum_k n_k q_k \boldsymbol{v}_{\mathrm{drift},k} = \frac{\sum_k n_k \boldsymbol{F}_k}{B^2}\times\boldsymbol{B}. \tag{3.61}$$

For instance, the drift current due to the grad-B drift is

$$\boldsymbol{j}_{\nabla B} = \frac{1}{2}\sum_k \frac{m_k n_k v_{\perp,k}^2}{B^2}(\boldsymbol{B}\times\nabla B). \tag{3.62}$$

This current leads to the charge separation. An example of the natural drift current is the ring current in the Earth's magnetosphere (see Fig. 3.5).

3.6 Guiding centre motion

Let us show that it is possible to derive all the above discussed drifts and much more in a very elegant and general way[3]. We will correctly solve the equations of motion

[3]Following de Blank 2012: *Guiding center motion*, Fusion Science and Technology 61(2T), 61–68

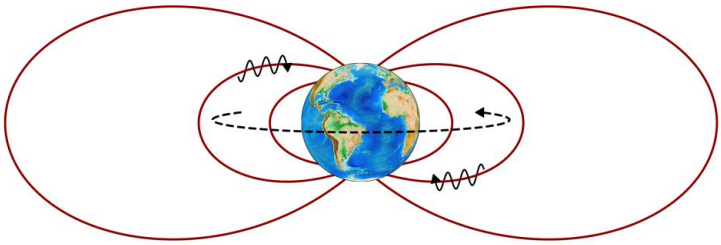

Figure 3.5: A combination of drifts in the curved magnetic field in the Earth's magnetosphere results in the formation of the *ring current* in the equatorial plane (the dashed line indicates the drift of the electrons).

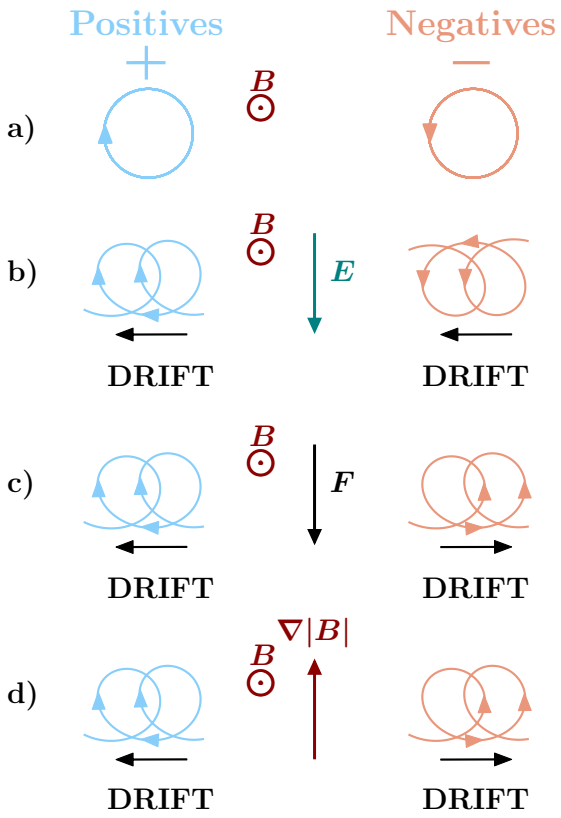

Figure 3.6: Charged particle drifts in a homogeneous magnetic field. (A) No disturbing force, (B) With an electric field, E (C) With an independent force, F (e.g. gravity) (D) In an inhomogeneous magnetic field, grad B. Based on Hannes Alfvén's, Cosmical Electrodynamics (1950); (cc) Stannered, redesigned by Ian Tresman.

and will find the corrections to the normal gyration under the approximation of slowly varying fields. That means that we limit ourselves to the case when

$$\frac{r_{\mathrm{L}}}{L} = \frac{v_\perp}{\omega_{\mathrm{c}} L} \ll 1 \quad \text{and} \quad \frac{1}{\omega_{\mathrm{c}}\tau} \ll 1, \tag{3.63}$$

where L and τ are the characteristic spatial and temporal changes in the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively. We see that $\omega_{\mathrm{c}}^{-1} \propto m/q$, which is small[4]. Thus we define $\epsilon \equiv m/q$ to be a small parameter of the system and we will expand the equations above to the Taylor series using this parameter. Then $r_{\mathrm{L}} \propto \epsilon L$ and $\tau^{-1} \propto \epsilon\omega_{\mathrm{c}}$. We want to solve a system of ordinary differential equations

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}\,, \tag{3.64}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{q}{m}\left(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\right)\,. \tag{3.65}$$

We will expand these equations in ϵ. First, we will introduce the coordinate transformation $(\boldsymbol{x}, \boldsymbol{v}) \to (\boldsymbol{R}, \boldsymbol{u})$. Coordinates $(\boldsymbol{R}, \boldsymbol{u})$ are functions of time.

$$\boldsymbol{R}(t) = \boldsymbol{x}(t) - \boldsymbol{\rho}(t), \tag{3.66}$$

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_{\mathrm{E}}. \tag{3.67}$$

Here $\boldsymbol{R}$ is the instantaneous position of the gyration center, $\boldsymbol{x}$ is the instantaneous position of the given particle (with respect to the fixed reference coordinate system), $\boldsymbol{v}_{\mathrm{E}} \equiv \boldsymbol{E}\times\boldsymbol{B}/B^2$ is the velocity of the E-B drift [see (3.29)], and $\boldsymbol{\rho}$ is the instantaneous position of the particle with respect to the gyration centre. From the previous section [see e.g. (3.15)] we may see that

$$\boldsymbol{\rho} \equiv \frac{\epsilon}{B}\boldsymbol{b}\times\boldsymbol{u}, \tag{3.68}$$

where $\boldsymbol{b}(\boldsymbol{x}, t) = \boldsymbol{B}(\boldsymbol{x}, t)/|\boldsymbol{B}(\boldsymbol{x}, t)|$ is the unity vector along the field lines. Vectors $\boldsymbol{b}$ and $\boldsymbol{v}_{\mathrm{E}}$ are functions of both space $\boldsymbol{x}$ and time t and their values are measured with respect to the particle. Note that

$$\omega_{\mathrm{c}}(t) = \frac{B(\boldsymbol{x}, t)}{\epsilon} \tag{3.69}$$

is the instantaneous cyclotron frequency.

We further introduce the local cylindrical coordinates in the plane perpendicular to the magnetic field with two direction vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2)$, so that

$$\boldsymbol{u} = v_\parallel\boldsymbol{b} + u_\perp\boldsymbol{e}_\perp\,, \quad \text{where} \quad \boldsymbol{e}_\perp \equiv \boldsymbol{e}_1\cos\phi + \boldsymbol{e}_2\sin\phi\,, \quad \text{and} \quad \phi = \phi_0 - \omega_{\mathrm{c}}t. \tag{3.70}$$

Note again that the base vectors of this local coordinate system $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{b})$ vary in both the space and time. Unit vectors $\boldsymbol{e}_\perp$ and

$$\boldsymbol{e}_\rho \equiv \boldsymbol{e}_2\cos\phi - \boldsymbol{e}_1\sin\phi \tag{3.71}$$

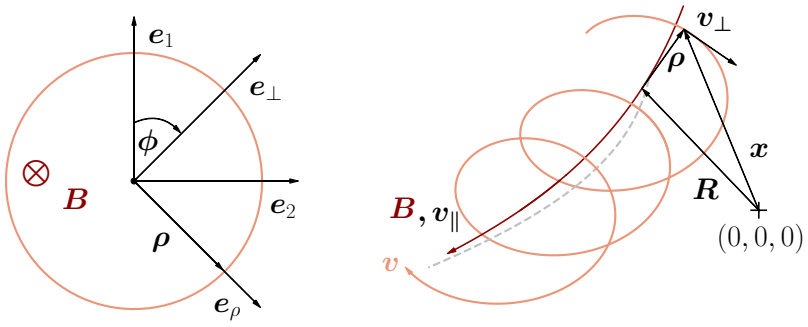

Figure 3.7: A sketch of the coordinate system of the guiding centre variables. The dashed line indicates the position of the gyration centre which deviates from the magnetic field line in general.

are projections of the particle's velocity $\boldsymbol{u}$ and relative position $\boldsymbol{\rho}$, respectively, to the plane perpendicular to $\boldsymbol{b}$.

Hence we derived useful relations defining the guiding centre variables (see also Fig. 3.7).

$$\boldsymbol{x} = \boldsymbol{R} + \boldsymbol{\rho}, \tag{3.72}$$

$$\boldsymbol{\rho} = -\frac{m}{qB^2}\boldsymbol{u} \times \boldsymbol{B} = \rho \operatorname{sgn} q\, \boldsymbol{e}_\rho, \tag{3.73}$$

$$\boldsymbol{u}_\perp = u_\perp \boldsymbol{e}_\perp \quad \text{and} \tag{3.74}$$

$$\boldsymbol{v} = v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E}. \tag{3.75}$$

Now we transform (3.64) and (3.65) to the newly introduced coordinates. From (3.66) we have:

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} &= \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} - \frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t} = \boldsymbol{v} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\epsilon}{B}\boldsymbol{b} \times \boldsymbol{u}\right) = \\
&= v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E} + \epsilon \boldsymbol{u} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{b}}{B}\right) - \frac{\epsilon}{B}\boldsymbol{b} \times \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \\
&= v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E} + \epsilon \boldsymbol{u} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{b}}{B}\right) - \frac{\epsilon}{B}\boldsymbol{b} \times \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} - \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right) = \\
&= v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \epsilon \boldsymbol{u} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{b}}{B}\right) + \frac{\epsilon}{B}\boldsymbol{b} \times \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t},
\end{aligned} \tag{3.76}$$

[4] As m/q is not a dimensionless quantity, its value depends on the system of units. It's smallness needs to be understood in terms of equation (3.63) in which it parametrises ω_c with respect to the characteristic time scale(s) of changes of the fields.

where we used the equation of motion (3.65) to evaluate the term $-\frac{\epsilon}{B}\boldsymbol{b}\times\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$:

$$\begin{aligned}-\frac{\epsilon}{B}\boldsymbol{b}\times\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= -\frac{\epsilon}{B}\boldsymbol{b}\times\left[\frac{1}{\epsilon}\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)\right] = \\ &= -\frac{1}{B}\boldsymbol{b}\times\boldsymbol{E}-\frac{1}{B}(\boldsymbol{v}\,\boldsymbol{b}\cdot\boldsymbol{B}-\boldsymbol{B}\,\boldsymbol{b}\cdot\boldsymbol{v}) = -\frac{1}{B}\boldsymbol{b}\times\boldsymbol{E}-\boldsymbol{v}\frac{\boldsymbol{B}\cdot\boldsymbol{B}}{B^2}+v_\parallel\frac{\boldsymbol{B}}{B} = \\ &= \boldsymbol{v}_\mathrm{E}-\boldsymbol{v}+v_\parallel\boldsymbol{b} = -u_\perp\boldsymbol{e}_\perp. \end{aligned} \tag{3.77}$$

For the time derivative of velocity, we expand $\boldsymbol{v}$ using (3.75) to get

$$\begin{aligned}\frac{\mathrm{d}(v_\parallel\boldsymbol{b})}{\mathrm{d}t}+\frac{\mathrm{d}(u_\perp\boldsymbol{e}_\perp)}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t} &= \frac{1}{\epsilon}\left[\boldsymbol{E}+(v_\parallel\boldsymbol{b}+u_\perp\boldsymbol{e}_\perp+\boldsymbol{v}_\mathrm{E})\times\boldsymbol{B}\right] = \\ &= \frac{1}{\epsilon}\left[\boldsymbol{E}+v_\parallel\boldsymbol{b}\times\boldsymbol{B}+u_\perp\boldsymbol{e}_\perp\times\boldsymbol{B}+\boldsymbol{v}_\mathrm{E}\times\boldsymbol{B}\right] = \\ &= \frac{1}{\epsilon}\left[\cancel{\boldsymbol{E}}+u_\perp\boldsymbol{e}_\perp\times\boldsymbol{B}-\frac{E}{\cancel{B^2}}\cancel{\boldsymbol{B}\cdot\boldsymbol{B}}+\frac{\boldsymbol{E}\cdot\boldsymbol{B}}{B^2}\boldsymbol{B}\right] = \\ &= \frac{1}{\epsilon}\left[E_\parallel\boldsymbol{b}+u_\perp\boldsymbol{e}_\perp\times\boldsymbol{B}\right], \end{aligned} \tag{3.78}$$

where we defined $E_\parallel\equiv\frac{\boldsymbol{E}\cdot\boldsymbol{B}}{|\boldsymbol{B}|}$. Next, we project this equation into the directions of $\boldsymbol{e}_\perp$, $\boldsymbol{b}$, and $\boldsymbol{e}_\rho$ separately. For the first one, we have

$$\frac{1}{\epsilon}\left[E_\parallel\boldsymbol{b}\cdot\boldsymbol{e}_\perp+u_\perp(\boldsymbol{e}_\perp\times\boldsymbol{B})\cdot\boldsymbol{e}_\perp\right] = \left[\frac{\mathrm{d}(v_\parallel\boldsymbol{b})}{\mathrm{d}t}+\frac{\mathrm{d}(u_\perp\boldsymbol{e}_\perp)}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right]\cdot\boldsymbol{e}_\perp. \tag{3.79}$$

The left-hand side is zero since $\boldsymbol{b}\perp\boldsymbol{e}_\perp$ and $(\boldsymbol{e}_\perp\times\boldsymbol{B})\cdot\boldsymbol{e}_\perp = (\boldsymbol{e}_\perp\times\boldsymbol{e}_\perp)\cdot\boldsymbol{B} = 0$. Thus we have

$$0 = \boldsymbol{e}_\perp\cdot v_\parallel\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}+\boldsymbol{e}_\perp\cdot\boldsymbol{b}\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}+\boldsymbol{e}_\perp\cdot\boldsymbol{e}_\perp\frac{\mathrm{d}u_\perp}{\mathrm{d}t}+u_\perp\boldsymbol{e}_\perp\cdot\frac{\mathrm{d}\boldsymbol{e}_\perp}{\mathrm{d}t}+\boldsymbol{e}_\perp\cdot\frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}. \tag{3.80}$$

The second term vanishes because $\boldsymbol{e}_\perp\perp\boldsymbol{b}$ and the fourth term is

$$\boldsymbol{e}_\perp\cdot\frac{\mathrm{d}\boldsymbol{e}_\perp}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{e}_\perp\cdot\boldsymbol{e}_\perp) = 0, \tag{3.81}$$

because $\boldsymbol{e}_\perp\cdot\boldsymbol{e}_\perp = 1$. Finally, we have

$$\frac{\mathrm{d}u_\perp}{\mathrm{d}t} = -\boldsymbol{e}_\perp\cdot\left(v_\parallel\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right). \tag{3.82}$$

For the projection into the $\boldsymbol{b}$ direction we have:

$$\frac{1}{\epsilon}\left[E_\parallel\boldsymbol{b}\cdot\boldsymbol{b}+u_\perp(\boldsymbol{e}_\perp\times\boldsymbol{B})\cdot\boldsymbol{b}\right] = \left[\frac{\mathrm{d}(v_\parallel\boldsymbol{b})}{\mathrm{d}t}+\frac{\mathrm{d}(u_\perp\boldsymbol{e}_\perp)}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right]\cdot\boldsymbol{b}. \tag{3.83}$$

The second term on the left-hand side of (3.83) equals to zero because $(\boldsymbol{e}_\perp\times\boldsymbol{B})\cdot\boldsymbol{b} = -(\boldsymbol{b}\times\boldsymbol{B})\cdot\boldsymbol{e}_\perp = 0$. Writing out the terms on the right-hand side explicitly, we obtain

$$\frac{E_\parallel}{\epsilon} = \boldsymbol{b}\cdot\boldsymbol{b}\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}+v_\parallel\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{e}_\perp\frac{\mathrm{d}u_\perp}{\mathrm{d}t}+u_\perp\boldsymbol{b}\cdot\frac{\mathrm{d}\boldsymbol{e}_\perp}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}\cdot\boldsymbol{b}}{\mathrm{d}t}-\boldsymbol{v}_\mathrm{E}\cdot\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}. \tag{3.84}$$

The second term on the right-hand side of (3.84) is zero from the same arguments as in (3.81). The third term vanishes as $\boldsymbol{b} \perp \boldsymbol{e}_\perp$ and the fifth term also vanishes, because $\boldsymbol{v}_\mathrm{E} \perp \boldsymbol{b}$. The fourth term may be rewritten as

$$u_\perp \boldsymbol{b} \cdot \frac{\mathrm{d}\boldsymbol{e}_\perp}{\mathrm{d}t} = u_\perp \frac{\mathrm{d}\boldsymbol{e}_\perp \cdot \boldsymbol{b}}{\mathrm{d}t} - u_\perp \boldsymbol{e}_\perp \cdot \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} = -u_\perp \boldsymbol{e}_\perp \cdot \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}. \tag{3.85}$$

Finally, we have

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{E_\parallel}{\epsilon} + (u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E}) \cdot \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}. \tag{3.86}$$

The most intense algebra concerns the projection into the $\boldsymbol{e}_\rho$ direction, where we will evaluate the individual terms one by one.

$$\frac{1}{\epsilon}\left[E_\parallel \overset{①}{\boldsymbol{b}} \cdot \boldsymbol{e}_\rho + u_\perp (\boldsymbol{e}_\perp \overset{②}{\times} \boldsymbol{B}) \cdot \boldsymbol{e}_\rho\right] = \boldsymbol{e}_\rho \cdot \overset{③}{\frac{\mathrm{d}(v_\parallel \boldsymbol{b})}{\mathrm{d}t}} + \boldsymbol{e}_\rho \cdot \overset{④}{\frac{\mathrm{d}(u_\perp \boldsymbol{e}_\perp)}{\mathrm{d}t}} + \boldsymbol{e}_\rho \cdot \overset{⑤}{\frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}}\,. \tag{3.87}$$

① :

$$\boldsymbol{b} \perp \boldsymbol{e}_\rho \Rightarrow \boldsymbol{b} \cdot \boldsymbol{e}_\rho = 0 \tag{3.88}$$

② :

$$u_\perp (\boldsymbol{e}_\perp \times \boldsymbol{B}) \cdot \boldsymbol{e}_\rho = u_\perp (\boldsymbol{e}_\rho \times \boldsymbol{e}_\perp) \cdot \boldsymbol{B} = u_\perp \left[\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}\right] \cdot \boldsymbol{B} =$$

$$= u_\perp \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} = -u_\perp B \tag{3.89}$$

③ :

$$\boldsymbol{e}_\rho \cdot \frac{\mathrm{d}(v_\parallel \boldsymbol{b})}{\mathrm{d}t} = \boldsymbol{e}_\rho \cdot v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \cancel{\boldsymbol{e}_\rho \cdot \boldsymbol{b} \frac{\mathrm{d}v_\parallel}{\mathrm{d}t}} \tag{3.90}$$

④ :

$$\begin{aligned}
\boldsymbol{e}_\rho \cdot \frac{\mathrm{d}(u_\perp \boldsymbol{e}_\perp)}{\mathrm{d}t} &= u_\perp (\boldsymbol{e}_2 \cos\phi - \boldsymbol{e}_1 \sin\phi) \cdot \frac{\mathrm{d}(\boldsymbol{e}_1 \cos\phi + \boldsymbol{e}_2 \sin\phi)}{\mathrm{d}t} + \cancel{\boldsymbol{e}_\rho \cdot \boldsymbol{e}_\perp \frac{\mathrm{d}u_\perp}{\mathrm{d}t}} = \\
&= u_\perp \left[(\boldsymbol{e}_2 \cos\phi - \boldsymbol{e}_1 \sin\phi) \cdot (\cos\phi \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \boldsymbol{e}_1 \sin\phi \frac{\mathrm{d}\phi}{\mathrm{d}t} + \sin\phi \frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t} + \boldsymbol{e}_2 \cos\phi \frac{\mathrm{d}\phi}{\mathrm{d}t})\right] = \\
&= u_\perp \left[\cos^2\phi\, \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \cancel{\sin\phi \cos\phi\, \boldsymbol{e}_1 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}} + \cancel{\sin\phi \cos\phi \frac{\mathrm{d}\phi}{\mathrm{d}t} \boldsymbol{e}_1 \cdot \boldsymbol{e}_2} + \sin^2\phi \frac{\mathrm{d}\phi}{\mathrm{d}t} \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 + \right. \\
&\left. + \cancel{\sin\phi \cos\phi\, \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t}} - \sin^2\phi\, \boldsymbol{e}_1 \cdot \frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t} + \cos^2 \frac{\mathrm{d}\phi}{\mathrm{d}t} \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 - \cancel{\sin\phi \cos\phi \frac{\mathrm{d}\phi}{\mathrm{d}t} \boldsymbol{e}_1 \cdot \boldsymbol{e}_2}\right] = \\
&= u_\perp \left[(\sin^2\phi + \cos^2\phi)\left(\frac{\mathrm{d}\phi}{\mathrm{d}t} + \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}\right) - \sin^2\phi \left(\boldsymbol{e}_1 \cdot \frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t} + \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}\right)\right] = \\
&= u_\perp \left[\frac{\mathrm{d}\phi}{\mathrm{d}t} + \boldsymbol{e}_2 \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \sin^2\phi \left(\cancel{\frac{\mathrm{d}(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)}{\mathrm{d}t}} - \cancel{\boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}} + \cancel{\boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}}\right)\right],
\end{aligned} \tag{3.91}$$

where in the penultimate step we added and subtracted the term $\sin^2\phi\, \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}$.

(5) remains unchanged

Hence, altogether we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{B}{\epsilon} - \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \frac{1}{u_\perp}\boldsymbol{e}_\rho \cdot \left(v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t} \right). \tag{3.92}$$

Let us repeat that now we again have six differential equations for modified variables that consist of the position of the gyration center $\boldsymbol{R}$, parallel component of velocity, and variables in the plane perpendicular to the field lines. Thus we have to solve the following set of equations:

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \epsilon \boldsymbol{u} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{b}}{B}\right) + \frac{\epsilon}{B}\boldsymbol{b} \times \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}, \tag{3.93}$$

$$\frac{\mathrm{d}u_\perp}{\mathrm{d}t} = -\boldsymbol{e}_\perp \cdot \left(v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t} \right), \tag{3.94}$$

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{E_\parallel}{\epsilon} + (u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E}) \cdot \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}, \tag{3.95}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{B}{\epsilon} - \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \frac{1}{u_\perp}\boldsymbol{e}_\rho \cdot \left(v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t} \right). \tag{3.96}$$

To proceed further, we need to replace the total time derivative by the derivative along the trajectory

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp \right) \cdot \boldsymbol{\nabla} \tag{3.97}$$

Let us perform the proper derivation for (3.93). Let us first decompose $\boldsymbol{u}$ according to formula (3.70) and use relation (3.97) to express $\mathrm{d}/\mathrm{d}t$ in (3.93):

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = {} & v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \frac{\epsilon}{B}\left(v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp \right) \times \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp \right) \cdot \boldsymbol{\nabla} \right] \boldsymbol{b} + \\ & \epsilon \left(v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp \right) \times \boldsymbol{b} \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp \right) \cdot \boldsymbol{\nabla} \right] \left(\frac{1}{B} \right) + \\ & \frac{\epsilon}{B}\boldsymbol{b} \times \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} \right) \cdot \boldsymbol{\nabla} \right] \boldsymbol{v}_\mathrm{E} + \frac{\epsilon}{B}\boldsymbol{b} \times \left(u_\perp \boldsymbol{e}_\perp \cdot \boldsymbol{\nabla} \right) \boldsymbol{v}_\mathrm{E} \,. \end{aligned} \tag{3.98}$$

Next, we add a zero term

$$\frac{\epsilon}{B}\boldsymbol{b} \times \boldsymbol{b} \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} \right) \cdot \boldsymbol{\nabla} \right] v_\parallel = 0 \tag{3.99}$$

to the right hand side of (3.98). Then, we drop several terms containing $\boldsymbol{b} \times \boldsymbol{b}$ and rearrange the equation to obtain

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = {} & v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \frac{\epsilon}{B}\boldsymbol{b} \times D_t \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} \right) + \frac{\epsilon}{B} v_\parallel \boldsymbol{b} \times \left[u_\perp \boldsymbol{e}_\perp \cdot \boldsymbol{\nabla} \right] \boldsymbol{b} + \\ & \frac{\epsilon u_\perp}{B}\boldsymbol{e}_\perp \times \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp \right) \cdot \boldsymbol{\nabla} \right] \boldsymbol{b} - \\ & \frac{\epsilon u_\perp}{B^2}\boldsymbol{e}_\perp \times \boldsymbol{b} \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp \right) \cdot \boldsymbol{\nabla} \right] B + \frac{\epsilon}{B}\boldsymbol{b} \times \left(u_\perp \boldsymbol{e}_\perp \cdot \boldsymbol{\nabla} \right) \boldsymbol{v}_\mathrm{E} \,, \end{aligned} \tag{3.100}$$

where we denote

$$D_t \equiv \frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E}\right) \cdot \boldsymbol{\nabla} \,. \tag{3.101}$$

Equation (3.100) contains a lot of terms varying on different time scale. Obviously, the terms containing factors $\boldsymbol{e}_\perp$ or $\boldsymbol{e}_\rho$ will change with the gyration of the particle. Therefore we may call these terms as *fast* terms. Recalling back the "classical" derivation of the ∇B drift, we treated a similar problem by "averaging" over one Larmor rotation. In the case of (3.100) we will proceed by analogy. We will average (3.100) over ϕ.

The method of averaging has a mathematically correct basis. The set of equations (3.93) to (3.96) can symbolically be written in the compact form

$$\frac{\mathrm{d}z}{\mathrm{d}t} = f_z(z) = \langle f_z \rangle + \tilde{f}_z, \tag{3.102}$$

where $z \in \{\boldsymbol{R}, v_\parallel, u_\perp, \phi\}$. The angled brackets denote the average over the gyroangle ϕ, hence $\langle \bullet \rangle = (2\pi)^{-1} \int \mathrm{d}\phi(\bullet)$. The function $\tilde{f}_z$ is periodic in the fast-oscillating gyroangle ϕ. Such separation may always be done. We will show that the oscillating component can be removed to the leading order by means of the small parameter, ϵ, by the redefinition of the variable z to

$$\bar{z} = z + \frac{\epsilon}{B} \int_0^\phi \tilde{f}_z(\phi')\mathrm{d}\phi'. \tag{3.103}$$

From (3.96) we have:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{B}{\epsilon} - \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \frac{1}{u_\perp}\boldsymbol{e}_\rho \cdot \left(v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right) = -\frac{B}{\epsilon} + \mathcal{O}(\epsilon^0). \tag{3.104}$$

The total time derivative (3.97) in terms of the guiding centre variables reads

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} \cdot \frac{\partial}{\partial \boldsymbol{R}} + \frac{\mathrm{d}v_\parallel}{\mathrm{d}t}\frac{\partial}{\partial v_\parallel} + \frac{\mathrm{d}u_\perp}{\mathrm{d}t}\frac{\partial}{\partial u_\perp} + \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\partial}{\partial \phi}. \tag{3.105}$$

The last term is the leading order term, where for $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ we use (3.104). Note that the third term on the right-hand side is actually a leading order term ($\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} \sim \frac{E_\parallel}{\epsilon} \sim \mathcal{O}(\epsilon^{-1})$), however we assume here that $E_\parallel \sim \mathcal{O}(\epsilon)$, otherwise the assumption of small changes would be violated. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} = -\frac{B}{\epsilon}\frac{\partial}{\partial \phi} + \mathcal{O}(\epsilon^0). \tag{3.106}$$

Note that such an approximation is far from being general and is valid only for the system we investigate.

By applying the leading order time derivative (3.106) to the transformed variables (3.103) we have

$$\begin{aligned}\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} &= \frac{\mathrm{d}z}{\mathrm{d}t} - \frac{B}{\epsilon}\frac{\partial}{\partial \phi}\frac{\epsilon}{B}\int_\phi \tilde{f}_z(\phi')\mathrm{d}\phi' = \\ &= \frac{\mathrm{d}z}{\mathrm{d}t} - \tilde{f}_z + \mathcal{O}(\epsilon) = \langle f_z(z)\rangle + \tilde{f}_z - \tilde{f}_z + \mathcal{O}(\epsilon) = \langle f_z(z)\rangle + \mathcal{O}(\epsilon).\end{aligned} \tag{3.107}$$

Note that the final expression contains z, e.g., non-averaged variables as arguments. Therefore we have a mixture of the $\bar{z}$ averaged (guiding-centre based) arguments on the left-hand side and non-averaged (particle-based) arguments on the right-hand side. Fortunately, the right-hand side can approximately be written in the guiding centre variables $(\boldsymbol{R}, v_\parallel, u_\perp, \phi)$. First, the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ (and consequently vectors $\boldsymbol{b}$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{v}_\mathrm{E}$) at the particle position are expanded in a Taylor series around the position of the guiding center. For instance:

$$\boldsymbol{b}(\boldsymbol{x},t) = \boldsymbol{b}(\boldsymbol{R},t) + \boldsymbol{\rho}\cdot\nabla\boldsymbol{b}(\boldsymbol{R},t) + \cdots = \boldsymbol{b}(\boldsymbol{R},t) + \epsilon\frac{u_\perp}{B}\boldsymbol{e}_\rho\cdot\nabla\boldsymbol{b}(\boldsymbol{R},t) + \ldots \tag{3.108}$$

For the averaged variables even the first order term vanishes, because $\langle\boldsymbol{e}_\rho\rangle = 0$ and thus for (3.108) we have

$$\langle\boldsymbol{b}(\boldsymbol{x},t)\rangle = \boldsymbol{b}(\boldsymbol{R},t) + \mathcal{O}(\epsilon). \tag{3.109}$$

Then, finally:

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} = \langle f_z(\bar{z})\rangle + \mathcal{O}(\epsilon), \quad \text{Q.E.D.} \tag{3.110}$$

Let us go back to averaging (3.100) over ϕ. For the terms independent of $\boldsymbol{e}_\perp$, this step does not lead to any change. On the other hand, terms linear in $\boldsymbol{e}_\perp$ are zero as $\langle\sin\psi\rangle = \langle\cos\psi\rangle = 0$. The terms quadratic in $\boldsymbol{e}_\perp$ are generally non-zero and have to be treated carefully.

$$\left\langle\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t}\right\rangle = \left\langle v_\parallel\boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \frac{\epsilon}{B}\boldsymbol{b}\times D_t\left(v_\parallel\boldsymbol{b}+\boldsymbol{v}_\mathrm{E}\right) + \frac{\epsilon u_\perp^2}{B}\boldsymbol{e}_\perp\times\left(\boldsymbol{e}_\perp\cdot\boldsymbol{\nabla}\right)\boldsymbol{b} - \frac{\epsilon u_\perp^2}{B^2}\boldsymbol{e}_\perp\times\boldsymbol{b}\left(\boldsymbol{e}_\perp\cdot\boldsymbol{\nabla}\right)B\right\rangle . \tag{3.111}$$

Now, let us consider the last term in (3.111). Using the notation $\nabla_1 \equiv \boldsymbol{e}_1\cdot\boldsymbol{\nabla}$ and $\nabla_2 \equiv \boldsymbol{e}_2\cdot\boldsymbol{\nabla}$ we may write:

$$\left(\boldsymbol{e}_\perp\times\boldsymbol{b}\right)\left(\boldsymbol{e}_\perp\cdot\boldsymbol{\nabla}\right)B = \left(\boldsymbol{e}_1\sin\phi - \boldsymbol{e}_2\cos\phi\right)\left(\cos\phi\nabla_1 + \sin\phi\nabla_2\right)B \tag{3.112}$$

Considering $\langle\sin\phi\cos\phi\rangle = 0$ and $\langle\sin^2\phi\rangle = \langle\cos^2\phi\rangle = \frac{1}{2}$, averaging of (3.112) over ϕ gives:

$$\langle\left(\boldsymbol{e}_\perp\times\boldsymbol{b}\right)\left(\boldsymbol{e}_\perp\cdot\boldsymbol{\nabla}\right)B\rangle = \tfrac{1}{2}\left(\boldsymbol{e}_1\nabla_2 B - \boldsymbol{e}_2\nabla_1 B\right) = -\tfrac{1}{2}\left(\boldsymbol{b}\times\boldsymbol{\nabla}B\right) . \tag{3.113}$$

Note that an alternative derivation of this relation can be found in Appendix B.2.

A similar approach will be applied to the fourth term of the right-hand side of eq. (3.111). In this case, we have to keep in mind that the vector field $\boldsymbol{b}$ has components (0, 0, 1) at the point $\boldsymbol{R}$, but derivatives of all of its components are non-zero in general. Hence,

$$\begin{aligned}
\boldsymbol{e}_\perp\times\left(\boldsymbol{e}_\perp\cdot\boldsymbol{\nabla}\right)\boldsymbol{b} &= \left(\boldsymbol{e}_1\cos\phi + \boldsymbol{e}_2\sin\phi\right)\times\left(\cos\phi\nabla_1 + \sin\phi\nabla_2\right)\boldsymbol{b} \\
&= \left(\boldsymbol{e}_1\cos\phi + \boldsymbol{e}_2\sin\phi\right)\times\left[\boldsymbol{e}_1\left(\cos\phi\nabla_1 + \sin\phi\nabla_2\right)b_1 + \right. \\
&\quad \left.\boldsymbol{e}_2\left(\cos\phi\nabla_1 + \sin\phi\nabla_2\right)b_2 + \boldsymbol{e}_3\left(\cos\phi\nabla_1 + \sin\phi\nabla_2\right)b_3\right] \\
&= \boldsymbol{e}_1\left[\sin\phi\left(\cos\phi\nabla_1 + \sin\phi\nabla_2\right)b_3\right] - \boldsymbol{e}_2\left[\cos\phi\left(\cos\phi\nabla_1 + \sin\phi\nabla_2\right)b_3\right] + \\
&\quad \boldsymbol{e}_3\left[\cos\phi\left(\cos\phi\nabla_1 + \sin\phi\nabla_2\right)b_2 - \sin\phi\left(\cos\phi\nabla_1 + \sin\phi\nabla_2\right)\right]b_1
\end{aligned} \tag{3.114}$$

Averaging over ϕ then leads to

$$\langle \boldsymbol{e}_\perp \times (\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla})\, \boldsymbol{b} \rangle = \tfrac{1}{2} (\boldsymbol{e}_1 \nabla_2 - \boldsymbol{e}_2 \nabla_1)\, b_3 + \tfrac{1}{2} \boldsymbol{e}_3 (\nabla_1 b_2 - \nabla_2 b_1) \ . \tag{3.115}$$

The first term on the right hand side of eq. (3.115) may be neglected as $\nabla_i b_3 = \mathcal{O}((\nabla_i b_{[1,2]})^2)$.[5]

The second term on the right-hand side of eq. (3.115) can be rewritten in vector notation:

$$\langle \boldsymbol{e}_\perp \times (\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla})\, \boldsymbol{b} \rangle = \tfrac{1}{2} \boldsymbol{e}_3 (\boldsymbol{e}_3 \cdot \nabla \times \boldsymbol{b}) = \tfrac{1}{2} \boldsymbol{b} (\boldsymbol{b} \cdot \nabla \times \boldsymbol{b}) \ , \tag{3.118}$$

where we used the identity $\boldsymbol{b} \equiv \boldsymbol{e}_3$. Note that an alternative derivation of this relation can be found in Appendix B.2. Inserting (3.113) and (3.118) into (3.100) gives

$$\left\langle \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} \right\rangle = \overset{①}{v_\parallel \boldsymbol{b}} + \overset{②}{\boldsymbol{v}_\mathrm{E}} + \frac{\epsilon}{B} \boldsymbol{b} \times \overset{③}{D_t} \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} \right) + \overset{④}{\frac{\epsilon u_\perp^2}{2B}} (\boldsymbol{b} \cdot \boldsymbol{\nabla} \times \boldsymbol{b})\, \boldsymbol{b} + \overset{⑤}{\frac{\epsilon u_\perp^2}{2B^2}} \boldsymbol{b} \times \boldsymbol{\nabla} B + \mathcal{O}(\epsilon^2) \ . \tag{3.119}$$

This is the equation for the guiding centre motion. It contains all of the drifts we discussed in the previous chapters and much more.

① This term represents the movement along the field lines. Local vector $\boldsymbol{b}$ is curved in $(\boldsymbol{x}, t)$ space, nevertheless, the motion of the particle is always dominantly parallel to the local vector of the magnetic induction.

② This is the E-B drift.

③ This term contains the generalised curvature drift (term $\sim \frac{\epsilon}{B} \boldsymbol{b} \times (v_\parallel \boldsymbol{b} \cdot \boldsymbol{\nabla})(v_\parallel \boldsymbol{b})$[6],

[5] The fact that $\nabla_i b_3 = \mathcal{O}((\nabla_i b_{[1,2]})^2)$ is a consequence of normalisation $|\boldsymbol{b}| = 1$. Vector $\boldsymbol{b}$ at a small displacement $\boldsymbol{\delta R}$ from $\boldsymbol{R}$ can be expressed by means of the Taylor series:

$$\boldsymbol{b}(\boldsymbol{R} + \boldsymbol{\delta R}) = \boldsymbol{b}(\boldsymbol{R}) + \boldsymbol{\delta R} \cdot \boldsymbol{\nabla} \boldsymbol{b}(\boldsymbol{R}) + \mathcal{O}(\delta R^2) \ . \tag{3.116}$$

Denoting $\boldsymbol{b}' \equiv \boldsymbol{b}(\boldsymbol{R} + \boldsymbol{\delta R})$ and $\delta b_i \equiv \boldsymbol{\delta R} \cdot \boldsymbol{\nabla} b_i(\boldsymbol{R})$, we may write $\boldsymbol{b}' = \boldsymbol{b} + \delta b_1 \boldsymbol{e}_1 + \delta b_2 \boldsymbol{e}_2 + \delta b_3 \boldsymbol{e}_3$. We further consider $\boldsymbol{b} = (0,\, 0,\, 1)$. Both $\boldsymbol{b}$ and $\boldsymbol{b}'$ to be normalised, because they consist of base unity vectors. Hence $|\boldsymbol{b}'|^2 = 1$ which implies

$$|\boldsymbol{b}'|^2 = 1 = \left| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \delta b_1 \\ \delta b_2 \\ \delta b_3 \end{pmatrix} \right|^2 = \delta b_1^2 + \delta b_2^2 + (1 + \delta b_3)^2 \sim \delta b_1^2 + \delta b_2^2 + \delta b_3^2 + 2\delta b_3 + 1. \tag{3.117}$$

and, therefore, $2\delta b_3 \approx -\delta b_1^2 - \delta b_2^2 - \delta b_3^2 \sim \mathcal{O}(\delta b_{[1,2]}^2)$, Q.E.D.

[6] Deriving it more explicitly, let's assume the parallel velocity to be constant (or weakly variable), then

$$\begin{aligned} \boldsymbol{v}_\mathrm{c} &= \frac{m v_\parallel^2}{qB} [\boldsymbol{b} \times (\boldsymbol{b} \cdot \boldsymbol{\nabla}) \boldsymbol{b}] = \frac{m v_\parallel^2}{qB^2} \boldsymbol{B} \times [\boldsymbol{b} \cdot (\boldsymbol{e}_1 \nabla b_1 + \boldsymbol{e}_2 \nabla b_2 + \boldsymbol{b} \nabla b_3)] = \\ &= \frac{m v_\parallel^2}{qB^2} \boldsymbol{B} \times \nabla b_3 = -\frac{m v_\parallel^2}{qB^2 R^2} \boldsymbol{B} \times \boldsymbol{R}, \end{aligned} \tag{3.120}$$

where we noted that in the local cylindrical coordinates, b_3 is along the constant radius R and in order to fulfill $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ in a vaccuum, the field must decrease as $1/R$ and hence the gradient of the field must decrease as $-1/R^2 \boldsymbol{R}$. This relation is exactly the same as (3.34).

the polarisation drift[7] and the effects of the non-homogeneous electric field.

④ This is the correction to the parallel current: $\boldsymbol{b}\cdot\boldsymbol{\nabla}\times\boldsymbol{b}=\mu_0 j_\parallel/B$.

⑤ This is the grad-B drift.

Similar equations might be derived for mean values of time derivatives of velocity components (see Appendix B.3), namely:

$$\left\langle\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}\right\rangle=\frac{E_\parallel}{\epsilon}-\frac{u_\perp^2}{2B}\nabla_\parallel B+\boldsymbol{v}_\mathrm{E}\cdot D_t\boldsymbol{b}+\mathcal{O}(\epsilon) \tag{3.122}$$

$$\left\langle\frac{\mathrm{d}u_\perp}{\mathrm{d}t}\right\rangle=\frac{v_\parallel u_\perp}{2B}\nabla_\parallel B-\frac{u_\perp}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{v}_\mathrm{E}-\boldsymbol{b}\cdot\nabla_\parallel\boldsymbol{v}_\mathrm{E})+\mathcal{O}(\epsilon) \tag{3.123}$$

$$\left\langle\frac{\mathrm{d}\phi}{\mathrm{d}t}\right\rangle=-\frac{B}{\epsilon}-\boldsymbol{e}_2\cdot D_t\boldsymbol{e}_1-\frac{v_\parallel}{2}\boldsymbol{b}\cdot\boldsymbol{\nabla}\times(v_\parallel\boldsymbol{b}+\boldsymbol{v}_\mathrm{E})+\mathcal{O}(\epsilon) \tag{3.124}$$

3.7 Magnetic mirrors

We deliberately did not study the motion of a testing particle in the case of an inhomogeneous magnetic field with $\nabla|\boldsymbol{B}|\parallel\boldsymbol{B}$. We will derive some useful properties of this configuration of the magnetic field now. Let us consider the axisymmetric configuration in cylindrical coordinates (such as that indicated in Fig. 3.8), where $\boldsymbol{B}=(B_r,B_\varphi,B_z)$ and $B_\varphi=0$ and $\partial/\partial\varphi=0$. From Gauss's law for magnetism

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=\frac{1}{r}\frac{\partial}{\partial r}rB_r+\frac{1}{r}\frac{\partial B_\varphi}{\partial\varphi}+\frac{\partial B_z}{\partial z}=0 \tag{3.125}$$

we immediately see that

$$rB_r=-\int_0^r r'\frac{\partial B_z}{\partial z}\mathrm{d}r'. \tag{3.126}$$

Assuming that we know $\frac{\partial B_z}{\partial z}$ on the axis and consider only particle motions close enough to the axis, we may neglect the radial dependence of this term and obtain

$$rB_r\sim-\left[\frac{\partial B_z}{\partial z}\right]_{r=0}\int_0^r r'\mathrm{d}r'=-\frac{1}{2}r^2\left[\frac{\partial B_z}{\partial z}\right]_{r=0}, \tag{3.127}$$

[7] Let us handle the third term of 3.119 a little more in detail, in each step by always focusing on one particular term, while leaving the remaining terms unevaluated.

$$\frac{\epsilon}{B}\boldsymbol{b}\times D_t\left(v_\parallel\boldsymbol{b}+\boldsymbol{v}_\mathrm{E}\right)=\frac{\epsilon}{B}\boldsymbol{b}\times\frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}+\mathcal{A}=\frac{m}{qB^2}\boldsymbol{B}\times\frac{\mathrm{d}}{\mathrm{d}t}\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}+\mathcal{A}=\frac{m}{qB^4}\boldsymbol{B}\times\left(\frac{\partial\boldsymbol{E}}{\partial t}\times\boldsymbol{B}\right)+\mathcal{B}=$$
$$=\frac{m}{qB^4}\left(\boldsymbol{B}\cdot\boldsymbol{B}\frac{\partial\boldsymbol{E}}{\partial t}-\boldsymbol{B}\cdot\frac{\partial\boldsymbol{E}}{\partial t}\boldsymbol{B}\right)+\mathcal{B}=\frac{m}{qB^2}\frac{\partial\boldsymbol{E}}{\partial t}+\mathcal{C}=\frac{\operatorname{sgn}q}{\omega_\mathrm{c}B}\frac{\partial\boldsymbol{E}}{\partial t}+\mathcal{C}, \tag{3.121}$$

where the first term is the polarisation drift.

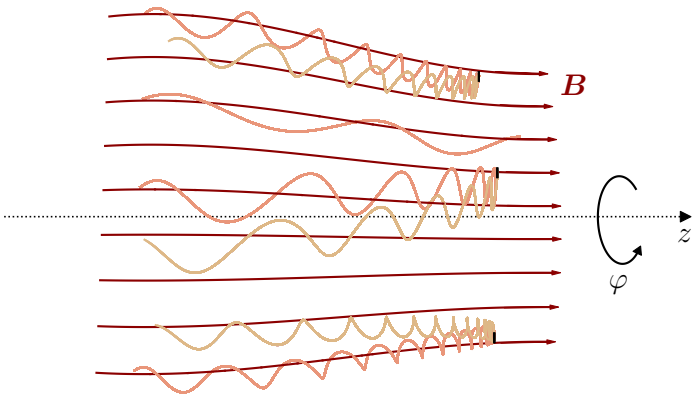

Figure 3.8: An illustration of the gyration of charged particles in the magnetic field with the configuration of the magnetic mirror. The particles gyrate along the field line, however they also undergo a drift between field-lines thanks to the drift in the curved magnetic field. As the particle approaches the strong-field region, its motion in the parallel direction slows down. When it reaches zero, the particle reflects back to the region where the field is weak. The reflection positions are indicated by vertical lines. Note that there is one particle trajectory indicated which has a large initial parallel velocity, which does not reflect and in principle passes through the magnetic mirror to the other side.

and hence

$$B_r \sim -\frac{1}{2} r \left[\frac{\partial B_z}{\partial z}\right]_{r=0}. \tag{3.128}$$

Thus we have an additional component to the Lorentz force and we may expect a drift. It will be a grad-B drift, but not the radial one, since $\partial B/\partial\varphi = 0$. The Lorentz force is

$$F = q\boldsymbol{v}\times\boldsymbol{B} = q\begin{pmatrix} v_r \\ v_\varphi \\ v_z \end{pmatrix}\times\begin{pmatrix} B_r \\ B_\varphi \\ B_z \end{pmatrix} = q\begin{pmatrix} v_\varphi B_z - v_z B_\varphi \\ v_z B_r - v_r B_z \\ v_r B_\varphi - v_\varphi B_r \end{pmatrix} =$$
$$= q\begin{pmatrix} \textcircled{1}\, v_\varphi B_z \\ \textcircled{2}\, v_z B_r - \textcircled{3}\, v_r B_z \\ \textcircled{4}\, -v_\varphi B_r \end{pmatrix}. \tag{3.129}$$

Terms ① and ③ are not interesting, as they are responsible for the Larmor rotation. Term ② vanishes around the axis, however even when we had relaxed the assumptions, this term would only cause the normal grad-B drift. Term ④ is something new.

Hence we have an additional force

$$F_z = -qv_\varphi B_r = \frac{1}{2} q v_\varphi r \frac{\partial B_z}{\partial z}, \tag{3.130}$$

which we (in agreement with the procedure already used before) average over one Larmor rotation. By taking into account that $v_\varphi = -\operatorname{sgn} q v_\perp$, $r = r_\mathrm{L} = v_\perp/\omega_\mathrm{c}$ and $\omega_\mathrm{c} =$

$q \operatorname{sgn} qB/m$ we have

$$F_z = -\frac{1}{2} q v_\perp \operatorname{sgn} q \frac{v_\perp m}{q \operatorname{sgn} qB} \frac{\partial B_z}{\partial z} = -\frac{1}{2} \frac{m v_\perp^2}{B} \frac{\partial B_z}{\partial z} = -\mu \frac{\partial B_z}{\partial z}, \tag{3.131}$$

where $\mu = \frac{1}{2} m v_\perp^2 / B$ is the *magnetic moment*. This equation may be generalised, when we identify z direction with the *parallel direction*, hence

$$F_\parallel = -\mu \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{s}} = -\mu \nabla_\parallel \boldsymbol{B}, \tag{3.132}$$

where $\mathrm{d}\boldsymbol{s}$ is the trajectory element.

We will further show that during the particle's motion, its r_L changes, but its magnetic moment remains invariant. First, we will handle the parallel component of the equation of motion

$$m \frac{\mathrm{d}\boldsymbol{v}_\parallel}{\mathrm{d}t} = -\mu \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{s}} \tag{3.133}$$

by multiplying it by $\boldsymbol{v}_\parallel$:

$$m \boldsymbol{v}_\parallel \cdot \frac{\mathrm{d}\boldsymbol{v}_\parallel}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} m v_\parallel^2 \right) = -\mu \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{s}} \cdot \boldsymbol{v}_\parallel = -\mu \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{s}} \cdot \frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = -\mu \frac{\mathrm{d}B}{\mathrm{d}t}. \tag{3.134}$$

The energy conservation law gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} m v_\parallel^2 + \frac{1}{2} m v_\perp^2 \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} m v_\parallel^2 + \mu B \right) = 0, \tag{3.135}$$

where we used $\frac{1}{2} m v_\perp^2 = \mu B$. Finally we subtract (3.134) and (3.135) to get

$$0 = -\mu \frac{\mathrm{d}B}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t} \mu B = -\mu \frac{\mathrm{d}B}{\mathrm{d}t} + \mu \frac{\mathrm{d}B}{\mathrm{d}t} + B \frac{\mathrm{d}\mu}{\mathrm{d}t} \tag{3.136}$$

and hence

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0. \tag{3.137}$$

The magnetic moment remains invariant. Unfortunately, this conclusion cannot be considered general, as we took very restrictive assumptions at the beginning. It is interesting to study to what extent extent the conservation of the magnetic moment is valid. Here we will go back to the guiding centre equations we derived in the previous section. Let's work with (3.123) in the form

$$\frac{\mathrm{d}u_\perp}{\mathrm{d}t} = \frac{v_\parallel u_\perp}{2B} \boldsymbol{b} \cdot \nabla_\parallel B - \frac{u_\perp}{2} (\boldsymbol{\nabla} \cdot \overset{\text{A}}{\boldsymbol{v}}_\mathrm{E} - \boldsymbol{b} \cdot \overset{\text{B}}{\nabla}_\parallel \boldsymbol{v}_\mathrm{E}) + \mathcal{O}(\epsilon), \tag{3.138}$$

where $\nabla_\parallel \bullet \equiv \boldsymbol{b} \cdot \boldsymbol{\nabla} \bullet$. Let's trim the terms one-by-one:

Ⓐ

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{v}_{\mathrm{E}} &= \boldsymbol{\nabla}\cdot\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} = \frac{1}{B^2}\left[\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) - \boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})\right] + (\boldsymbol{E}\times\boldsymbol{B})\cdot\nabla\frac{1}{B^2} = \\ &= -\frac{1}{B^2}\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t} - \frac{\boldsymbol{E}_{\parallel}+\boldsymbol{E}_{\perp}}{B^2}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) - \frac{2(\boldsymbol{E}\times\boldsymbol{B})}{B^2}\cdot\frac{\nabla B}{B} = \\ &= -\frac{B}{B^2}\boldsymbol{b}\cdot\left[B\frac{\partial\boldsymbol{b}}{\partial t} + \boldsymbol{b}\frac{\partial B}{\partial t}\right] - \frac{\boldsymbol{E}_{\parallel}}{B^2}\cdot\left[(\nabla B)\times\boldsymbol{b} + B(\boldsymbol{\nabla}\times\boldsymbol{b})\right] - \\ &- \frac{\boldsymbol{E}_{\perp}}{B^2}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) - 2\boldsymbol{v}_{\mathrm{E}}\cdot\frac{\nabla B}{B} = -\boldsymbol{b}\cdot\frac{\partial\boldsymbol{b}}{\partial t} - \frac{1}{B}\frac{\partial B}{\partial t} - \\ &- \frac{E_{\parallel}}{B^2}\boldsymbol{b}\cdot[(\nabla B)\times\boldsymbol{b}] - \frac{E_{\parallel}}{B}\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b}),\end{aligned} \tag{3.139}$$

where we used the fact that $\boldsymbol{b}\cdot\frac{\partial\boldsymbol{b}}{\partial t} = 0$, defined the parallel amplitude of the electric intensity by $E_{\parallel} = \boldsymbol{E}_{\parallel}\cdot\boldsymbol{b}$, and also assumed $(\boldsymbol{\nabla}\times\boldsymbol{B}) \parallel \boldsymbol{B}$ (force-free approximation). We also applied Faraday's law $\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$.

Ⓑ

$$\begin{aligned}\boldsymbol{b}\cdot\nabla_{\parallel}\boldsymbol{v}_{\mathrm{E}} &= \nabla_{\parallel}(\boldsymbol{b}\cdot\boldsymbol{v}_{\mathrm{E}}) - \boldsymbol{v}_{\mathrm{E}}\cdot(\nabla_{\parallel}\boldsymbol{b}) = -\boldsymbol{v}_{\mathrm{E}}\cdot[(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{b}] = \\ &- \frac{\boldsymbol{v}_{\mathrm{E}}}{B}\cdot[(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{B}] - (\boldsymbol{v}_{\mathrm{E}}\cdot\boldsymbol{B})(\boldsymbol{b}\cdot\boldsymbol{\nabla})\frac{1}{B} = -\frac{\boldsymbol{v}_{\mathrm{E}}}{B}\cdot[(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{B}],\end{aligned} \tag{3.140}$$

where we used $\boldsymbol{v}_{\mathrm{E}} \perp \boldsymbol{b}$. When adjusting the term $(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{B}$ let us evaluate the term ∇B^2:

$$\nabla B^2 = \nabla(\boldsymbol{B}\cdot\boldsymbol{B}) = 2\boldsymbol{B}\cdot\nabla\boldsymbol{B} = 2B\boldsymbol{b}\cdot\nabla\boldsymbol{B} \quad \text{and} \quad \nabla B^2 = 2B\nabla B. \tag{3.141}$$

Hence $\boldsymbol{b}\cdot\boldsymbol{\nabla}B = \nabla B$.

We will further apply the assumption that $E_{\parallel} \sim \mathcal{O}(\epsilon)$, where we remind the reader $\epsilon = m/q$ is a small parameter. Hence altogether (3.138) reads

$$\begin{aligned}\frac{\mathrm{d}u_{\perp}}{\mathrm{d}t} &= \frac{v_{\parallel}u_{\perp}}{2B}\boldsymbol{b}\cdot\nabla_{\parallel}B + \frac{u_{\perp}}{2}\frac{1}{B}\frac{\partial B}{\partial t} + \frac{u_{\perp}}{2}\boldsymbol{v}_{\mathrm{E}}\cdot\frac{\nabla B}{B} + \mathcal{O}(\epsilon) = \\ &= \frac{u_{\perp}}{2}\left[\frac{\partial}{\partial t} + (v_{\parallel}\boldsymbol{b} + \boldsymbol{v}_{\mathrm{E}})\cdot\boldsymbol{\nabla}\right]B + \mathcal{O}(\epsilon) = \frac{u_{\perp}}{2B}D_tB + \mathcal{O}(\epsilon),\end{aligned} \tag{3.142}$$

where again D_t is the full time derivative along the guiding centre trajectory.

Hence by taking the time derivative of the magnetic moment $\mu = u_{\perp}^2/2B$ and by using (3.142) we have

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{u_{\perp}\frac{\mathrm{d}}{\mathrm{d}t}u_{\perp}}{B} + \frac{u_{\perp}^2}{2B}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{B} = \frac{u_{\perp}^2}{2B^2}D_tB + \mathcal{O}(\epsilon) - \frac{u_{\perp}^2}{2B^2}\frac{\mathrm{d}B}{\mathrm{d}t} = \mathcal{O}(\epsilon). \tag{3.143}$$

Hence the magnetic moment is invariant when the changes of the fields are small (the assumption of $\boldsymbol{E} \sim \mathcal{O}(\epsilon)$.

A measurable consequence of the conservation of the magnetic moment is the effect of *magnetic mirrors*. Let's imagine that a charged particle moves (for example due to thermal motion) from a weak into a stronger magnetic field. It "sees" the stronger field, hence its Larmor radius decreases and perpendicular velocity increases, so that μ remains constant. However, the total kinetic energy must also be conserved, hence the parallel velocity decreases as the particle enters the stronger field. For large enough B the parallel velocity drops to zero and the particle suffers from reflection back to the weaker field (see Fig. 3.8).

The reflection is not perfect. For example particles originally with $v_\perp = 0$ have null magnetic moments and thus penetrate the "magnetic trap". The Lorentz force does not act on them along the magnetic field lines. Similar behaviour may be expected for particles with a small ratio of $v_\perp/v_\parallel$. Is there a limit to this ratio?

Let's assume that in the region of the weak field B_0, the particle has the following values of velocity components: $v_\perp = v_{\perp,0}$ and $v_\parallel = v_{\parallel,0}$. In the place of reflection we have $v_\perp = v'_\perp$ and $v_\parallel = 0$ and the field has the induction B'. Since

$$\mu = \text{const} = \frac{1}{2} m v_{\perp,0}^2 / B_0 = \frac{1}{2} m v'^2_\perp / B' \quad \text{and} \quad v'^2_\perp = v_{\perp,0}^2 + v_{\parallel,0}^2 \equiv v_0^2, \tag{3.144}$$

we have

$$\frac{B_0}{B'} = \frac{v_{\perp,0}^2}{v'^2_\perp} = \frac{v_{\perp,0}^2}{v_0^2} = \sin^2 \vartheta, \tag{3.145}$$

where ϑ is the *pitch angle* in the region of the weak field.

The smaller ϑ is, the stronger magnetic field needs to be reached at the place of reflection. Considering that there is some maximum finite value of the induction of the magnetic field, $B_{\max}$, in the real setup, particles with sufficiently small values of ϑ may not get reflected. From the equation above it follows that the maximal pitch angle for the particle to reflect is

$$\sin^2 \vartheta_{\max} = \frac{B_0}{B_{\max}}. \tag{3.146}$$

If we plot all three components of velocity for the particles (Fig. 3.9), we find that $\vartheta_{\max}$ creates a cone in the phase space. Particles located inside this cone will escape the reflection. Hence the name *loss cone*. Note that the shape of the loss cone depends neither on the charge of the particle nor on its mass.

3.8 Adiabatic invariants

Let us take the hamiltonian $\mathcal{H} = \mathcal{H}(\boldsymbol{p}, \boldsymbol{q})$ and its corresponding system of equations of motion

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \quad \text{and} \quad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}. \tag{3.147}$$

Should any of the coordinates q_i by *cyclic*, thus

$$q_i(t) = q_i(t + T), \tag{3.148}$$

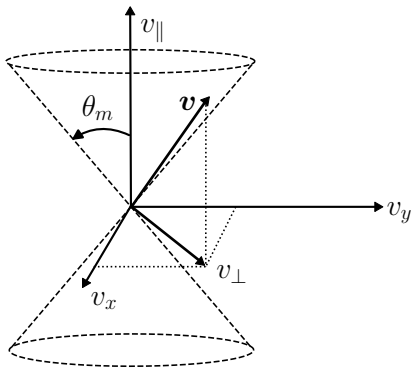

Figure 3.9: An illustration of the loss cone.

where T is the period, then

$$\oint p_i \mathrm{d}q_i = \text{const} \tag{3.149}$$

is the *integral of motion*. For example, in the case of the mathematical pendulum, the deviation $\varphi(t) = \varphi_0 \cos \omega t$ is cyclic with a period of $T = 2\pi/\omega$. Let's define the canonical pair $q = \varphi$ and $p = ml^2\dot{\varphi}$, where m and l are the mass and the length of the pendulum, respectively. Then we have

$$\begin{aligned} \int_0^T p \mathrm{d}q &= \int_0^T (-ml^2\varphi_0\omega \sin \omega t)(-\varphi_0 \omega \sin \omega t)\mathrm{d}t = \\ &= ml^2\varphi_0^2\omega \int_0^{2\pi} \sin^2 \omega t \, \mathrm{d}(\omega t) = \pi m l^2 \varphi_0^2 \omega = \text{const.} \end{aligned} \tag{3.150}$$

When the integral of motion contains a parameter which is slowly variable (slowly means that the time scale of the change of this parameter is much larger than period T of the cyclic coordinate), it turns into an *adiabatic invariant*. In the case of the pendulum, the invariant is conserved when we slowly change the length of the pendulum. Clearly, when l increases, ω must decrease in order for the invariant to remain constant. That is true only if the time scale on which the length of the pendulum changes is much longer than the period T of the cyclic coordinate. Should the time scale of the length changes be too short, the system behaves in a completely different regime.

3.8.1 First adiabatic invariant

Charged particles moving in a background electric and magnetic field have the coordinate ϕ, which is obviously cyclic and connected to Larmor rotation, see (3.70). The canonically conjugated momentum is the *angular momentum* $p = mv_\perp r_\mathrm{L}$. Hence

$$J_1 = \oint p_i \mathrm{d}q_i = \int_0^{2\pi} m v_\perp r_\mathrm{L} \, \mathrm{d}\phi = 2\pi v_\perp \frac{m^2 v_\perp}{q \operatorname{sgn} qB} = \frac{4\pi m}{q \operatorname{sgn} q}\mu = \text{const.} \tag{3.151}$$

It basically proves what we have derived in previous chapters: for small changes the magnetic moment μ is an adiabatic invariant.

There are some important situations in which the magnetic moment is not invariant:

- *Magnetic pumping*: If the collision frequency is larger than the pump frequency, μ is no longer conserved. In particular, collisions allow net heating by transferring some of the perpendicular energy to parallel energy.
- *Cyclotron heating*: If $\boldsymbol{B}$ is oscillating with the cyclotron frequency, the condition for adiabatic invariance is violated and heating is possible. In particular, the induced electric field rotates in phase with some of the particles and continuously accelerates them.
- *Magnetic cusps*: The magnetic field at the centre of a cusp vanishes, so the cyclotron frequency is automatically smaller than the rate of *any* changes. Thus the magnetic moment is not conserved and particles are relatively easily scattered into the loss cone.

3.8.2 Second adiabatic invariant

Let us consider a system with two magnetic mirrors and particles outside of the loss cone, thus cycling from one mirror to another. This motion is cyclic, thus in a normal system this motion would result in the motion integral $I = \oint_a^b mv_\parallel \mathrm{d}s$. In plasmas the motions of particles are subjected to drifts, hence the absolute periodicity is violated. We would assume that the motion integral becomes an adiabatic invariant when the effects of the drifts are much smaller than the dominant motion of the system (reflection between two mirrors).

We will derive the *longitudinal invariant* J, which is only defined between two consecutive reflections, hence

$$J_2 = \int_a^b v_\parallel \mathrm{d}s \tag{3.152}$$

and we will show that J is invariant in a stationary magnetic field. Further, we will generalise this case to $\frac{\partial B}{\partial t}$ is small. When the distance between the two mirrors shortens, the reflecting particles gain kinetic energy, which is the basis of *Fermi acceleration*.

Let's study the system of an asymmetrical magnetosphere, when due to drifts the particle occupies different geometries ("a different field line"). The particle drifts from one field line to another, which has a different curvature R_k in time Δt (see Fig. 3.10). Then for small angles we have

$$\frac{\delta s}{R_k} = \frac{\delta s'}{R_k'}. \tag{3.153}$$

The fractional change of the trajectory computed by subtracting 1 from both sides of (3.153) and dividing them by Δt is

$$\frac{\delta s' - \delta s}{\Delta t \delta s} = \frac{R_k' - R_k}{\Delta t R_k} = \boldsymbol{v}_{\mathrm{gc}} \cdot \frac{\boldsymbol{R}_k}{R_k^2} = \frac{1}{\delta s}\frac{\Delta \delta s}{\Delta t} = \frac{1}{\delta s}\frac{\mathrm{d}\delta s}{\mathrm{d}t}, \tag{3.154}$$

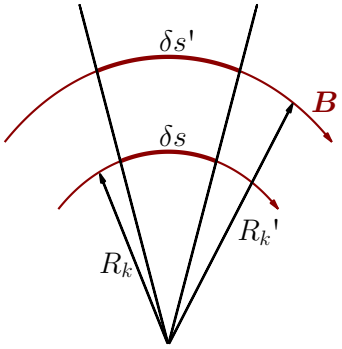

Figure 3.10: A scheme of the coordinate system used in the derivation of the second adiabatic invariant.

where we used the radial component of the curvature drift to estimate the speed of the jump between the field lines

$$\left|\frac{R_k' - R_k}{\Delta t}\right| = \boldsymbol{v}_{\mathrm{gc}} \cdot \frac{\boldsymbol{R}_k}{R_k}. \tag{3.155}$$

We studied the drift speed in the curved magnetic field earlier, thus we only apply results from (3.50) here:

$$\boldsymbol{v}_{\mathrm{gc}} = \frac{1}{2} \operatorname{sgn} q v_\perp r_{\mathrm{L}} \frac{\boldsymbol{B} \times \nabla B}{B^2} + \frac{m v_\parallel^2}{q} \frac{\boldsymbol{R_k} \times \boldsymbol{B}}{R_k^2 B^2}. \tag{3.156}$$

The second term does not have any component along $\boldsymbol{R}_k$, thus we do not have to consider it further. Hence

$$\frac{1}{\delta s}\frac{\mathrm{d}\delta s}{\mathrm{d}t} = \frac{1}{2}\frac{m}{q}\frac{v_\perp^2}{B^3}(\boldsymbol{B} \times \nabla B) \cdot \frac{\boldsymbol{R}_k}{R_k^2}, \tag{3.157}$$

which represents the fractional change of δs from the particle's point of view. Further, we will investigate the change of $v_\parallel$. Let's use the total energy of the particle W,

$$W = \frac{1}{2} m v_\parallel^2 + \frac{1}{2} m v_\perp^2 = \frac{1}{2} m v_\parallel^2 + \mu B = W_\parallel + W_\perp. \tag{3.158}$$

The parallel velocity is then $v_\parallel = \sqrt{\frac{2}{m}(W - \mu B)}$ and its time derivative

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{1}{2} \frac{1}{\sqrt{\frac{2}{m}(W - \mu B)}} \left(-\mu \frac{\mathrm{d}B}{\mathrm{d}t}\right) \frac{2}{m} \tag{3.159}$$

and the fractional change of $v_\parallel$ is

$$\frac{\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}}{v_\parallel} = -\frac{1}{2} \frac{\mu \frac{\mathrm{d}B}{\mathrm{d}t}}{W - \mu B} = -\frac{\mu \frac{\mathrm{d}B}{\mathrm{d}t}}{m v_\parallel^2}, \tag{3.160}$$

where we used (3.158) to express $W - \mu B$. The field $\boldsymbol{B}$ is stationary, but its derivative (seen by the moving particle) generally does not vanish due to the drift:

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\mathrm{d}B}{\mathrm{d}R}\frac{\mathrm{d}R}{\mathrm{d}t} = \boldsymbol{v}_{\mathrm{gc}} \cdot \nabla B = \frac{mv_\parallel^2}{q}\frac{\boldsymbol{R}_k \times \boldsymbol{B}}{R_k^2 B^2} \cdot \nabla B. \tag{3.161}$$

We insert this useful relation into (3.160) and get

$$\frac{\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}}{v_\parallel} = -\frac{\mu}{q}\frac{(\boldsymbol{R}_k \times \boldsymbol{B}) \cdot \nabla B}{R_k^2 B^2} = -\frac{1}{2}\frac{mv_\perp^2}{Bq}\frac{(\boldsymbol{B} \times \nabla B) \cdot \boldsymbol{R}_k}{R_k^2 B^2}. \tag{3.162}$$

The goal is to evaluate the fractional change of $v_\parallel \delta s$ during the change of trajectory:

$$\frac{1}{v_\parallel \delta s}\frac{\mathrm{d}(v_\parallel \delta s)}{\mathrm{d}t} = \frac{1}{\delta s}\frac{\mathrm{d}\delta s}{\mathrm{d}t} + \frac{1}{v_\parallel}\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}. \tag{3.163}$$

We already evaluated the first term in (3.157) and the second in (3.162). They are both equal except for the sign, hence

$$\frac{1}{v_\parallel \delta s}\frac{\mathrm{d}(v_\parallel \delta s)}{\mathrm{d}t} = 0 \tag{3.164}$$

and $v_\parallel \delta s$ remains constant when the particle exchanges the field line. This is not the claim we wanted to prove, which was that $J = \int_a^b v_\parallel \delta s$ is constant (as indicated in Fig. 3.11). However, we may split this integral into three terms,

$$J_2 = \int_a^{a'} v_\parallel \delta s + \int_{a'}^{b'} v_\parallel \delta s + \int_{b'}^{b} v_\parallel \delta s, \tag{3.165}$$

representing the different reflection points for different field lines. When a and a' are close and b and b' are also close, then the contributions of the first and third terms are negligible and J remains approximately invariant. The smallness of these boundary contributions is emphasised by the fact that $v_\parallel$ is small near the reflection points (where the total velocity amplitude is dominated by the perpendicular component).

By replacing the local longitudinal velocity by its average $\langle v_\parallel \rangle$ over the distance between the two reflection points L, we have

$$J_2 = \langle v_\parallel \rangle L, \tag{3.166}$$

which is invariant. When L decreases, $\langle v_\parallel \rangle$ increases, which is the basis of Fermi acceleration. High-energy particles from cosmic radiation of galactic origin were mostly accelerated by this mechanism, e.g. by repeated reflections between the magnetic field embedded in interstellar gaseous clouds, which moved towards each other.

In an axially asymmetric magnetosphere – such as the magnetosphere of the Earth – the gravitation drift drives the ring current, which in turn drives the "jumping of the particles between different field lines". The second adiabatic invariant is conserved on both sides of the magnetosphere.

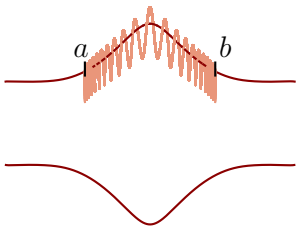

Figure 3.11: A scheme of gyrating particles trapped between two reflecting magnetic mirrors.

3.8.3 Third adiabatic invariant

This adiabatic invariant is defined for an axially symmetric magnetic field when the curvature drift is also cyclic. Then the total magnetic flux enclosed by a drift surface

$$J_3 = \Phi = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \tag{3.167}$$

is invariant. It is associated with the periodic motion of mirror-trapped particles drifting around the axis of the system. The particle moves along the surface which adjusts itself to variations in the magnetic field so that the flux enclosed by this surface remains constant.

When the waves in the Earth's ionosphere (usually having long periods) are in resonance with the drifting particles, the energy from their drifts is transferred to the energy of these waves. The drift is relatively slow, thus in practice, Φ is not conserved in real systems.

3.9 van Allen radiation belts

We will apply what we have just learned to a realistic problem, which is the case of particles being trapped in an axially symmetric magnetosphere. This is the reason why there are radiation belts (known as van Allen belts) in the Earth's magnetosphere and also in the magnetospheres of other planets. We do not aim to solve the full problem, we merely aim to show the basic principles using a rather simplistic approach. We will search for the allowed trajectories of the particles.

Let us assume an axially symmetric magnetic field with the components

$$\boldsymbol{B} = (B_R(R,z), 0, B_z(R,z))\,, \tag{3.168}$$

where B_R and B_z are the radial and vertical components of the magnetic induction vector of the field. The Gauss law must also be fulfilled in this case, hence

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 = \frac{1}{R}\frac{\partial}{\partial R} R B_R + \frac{\partial B_z}{\partial z}. \tag{3.169}$$

Following the recipe exploited further in Section 4.4.2, such a condition is fulfilled when the radial and vertical components of the magnetic field are given (4.69) as

$$B_z = \frac{1}{R}\frac{\partial F}{\partial R} \quad \text{and} \quad B_R = -\frac{1}{R}\frac{\partial F}{\partial z}. \tag{3.170}$$

The function F is called the flux function and it can be explicitly computed for the given configuration of the magnetic field. E.g., for a dipole (see Appendix C.1)

$$F = \mathcal{M}\frac{R^2}{(R^2+z^2)^{3/2}}, \tag{3.171}$$

where $\mathcal{M}$ is the magnetic moment of the dipole[8]. For now, we can take F as a given function, and by taking its spatial derivatives, we obtain the magnetic induction components. It may be easily shown that definition (3.170) fulfills the Gauss's law.

Now, let us deal with the particle equation of motion in the cylindrical coordinates and we will be interested in its azimuthal component:

$$\left(m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_{\varphi} = q\left(v_z B_R - v_R B_z\right) = -\frac{q}{R}\left(v_z\frac{\partial F}{\partial z} + v_R\frac{\partial F}{\partial R}\right) = -\frac{q}{R}\frac{\mathrm{d}F}{\mathrm{d}t}, \tag{3.172}$$

where we used $v_z = \frac{\mathrm{d}z}{\mathrm{d}t}$ and $v_R = \frac{\mathrm{d}R}{\mathrm{d}t}$ and assumed that $\frac{\partial F}{\partial t} = \frac{\partial F}{\partial \varphi} = 0$. Due to the cylindrical symmetry, the remaining terms of the full time derivative $\frac{\mathrm{d}F}{\mathrm{d}t}$ related to the φ-component vanish.

Let us look at the coordinate system we have. We defined two linear coordinates R (radial coordinate, distance from the axis) and z (height) and one angular (the azimuthal angle) φ. The studied particle has the position vector $\boldsymbol{r}$. Obviously, the unit vector which is tangential to the azimuth φ at the given point is parallel to the cross-product of the position vector $\boldsymbol{r}$ and the unit vector in the z-direction, hence

$$\boldsymbol{e}_\varphi \propto \boldsymbol{e}_z \times \boldsymbol{r}. \tag{3.173}$$

But what is its amplitude? We can easily determine it from the amplitude of the cross product, which is

$$|z||r|\sin\vartheta = R, \tag{3.174}$$

where ϑ is the angle between $\boldsymbol{e}_z$ and $\boldsymbol{r}$ (also termed as the co-latitude). Altogether we have

$$R\boldsymbol{e}_\varphi = \boldsymbol{e}_z \times \boldsymbol{r}. \tag{3.175}$$

Then

$$Rv_\varphi = R\boldsymbol{e}_\varphi \cdot \boldsymbol{v} = (\boldsymbol{e}_z \times \boldsymbol{r}) \cdot \boldsymbol{v}. \tag{3.176}$$

Now, we will compute the time derivative of this expression. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}Rv_\varphi = (\boldsymbol{e}_z \times \boldsymbol{r}) \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + (\boldsymbol{e}_z \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}) \cdot \boldsymbol{v}. \tag{3.177}$$

[8]For instance, the dipole magnetic moment of the Earth is $\mathcal{M}_\mathrm{E} = 8 \times 10^{15}\ \mathrm{T\,m^3}$.

The second term vanishes as $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v}$ and the result of the cross product is perpendicular to $\boldsymbol{v}$. By using (3.175) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} R v_\varphi = R\boldsymbol{e}_\varphi \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = R\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_\varphi . \tag{3.178}$$

Finally, let us investigate the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(R v_\varphi + \frac{qF}{m}\right) = R\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} + \frac{q}{m}\frac{\mathrm{d}F}{\mathrm{d}t} = 0, \tag{3.179}$$

where we used (3.178) and (3.172). Hence

$$R v_\varphi + \frac{qF}{m} = I_1 \tag{3.180}$$

is a motion integral.

Another motion integral is the total kinetic energy (we assumed that no dissipation takes place), which may be written in a simplistic form as

$$v_R^2 + v_\varphi^2 + v_z^2 = v^2 = I_2^2 \tag{3.181}$$

Certainly, $v_\varphi^2 \leq v^2$, as v_φ is only component of $\boldsymbol{v}$.

Then

$$R|v_\varphi| \leq |v|R \quad \rightarrow \quad \left|I_1 - \frac{qF}{m}\right| \leq R I_2. \tag{3.182}$$

We will not solve this inequation explicitly but will rather point out some general properties of it by drawing a graph (see Fig. 3.12). Let us represent the radial coordinate on the horizontal axis, R, on the vertical axis then the value of the expression $\frac{qF}{m}(R, \vartheta)$, where ϑ again represents the co-latitude (see Appendix C.2). The solutions for a given combination of q, F, and m will result in hyperbola-like curves for varying ϑ on the canvas. The degenerate limiting $\vartheta = 0$ curve coincides with the coordinate axes.

The remaining terms in the expression will create a sector on the canvas, the apex of which will be at a coordinate $(0, I_1)$ and the opening angle will correspond to I_2 (the larger, the more the sector is opened). Particles with a given kinetic energy and a value of I_1 are now allowed to move freely along their corresponding $\frac{qF}{m}$-line for a given ϑ, but only in the limits determined by the sector.

This simplistic approach gave us some important physical predictions. Particles with larger speeds are allowed to more within the larger range of radial distances. At lower co-latitudes (hence near the poles) the particles are in general located nearer the surface (lower R) and the extent of their allowed radial coordinate is lower (the belt is thinner) than near the equator. These predictions are fully consistent with the observations of the real van Allen radiation belts.

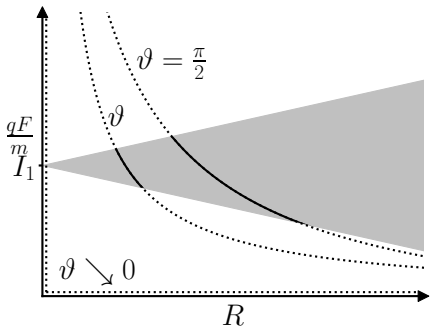

Figure 3.12: A graphical solution of the trapped particles in the axially symmetric magnetosphere. In the shadowed regions. Due to the inequality (3.182), the particles are allowed in the shadowed regions. For a given ϑ, the particle may move along the given field line. Hence the resulting allowed region is given by the intersection of the two (denoted by the solid thick line). Such a region obviously depends on the co-latitude and the energy of the particle.

Chapter 4

Plasma as fluid

In Section 1.5, we derived the moments of the Boltzmann equation by neglecting the particle distribution function and by considering the physical parameters averaged over the ensemble of the particles. Plasma may be treated in the same way, however, we have to bear in mind that the individual particles are charged and thus must follow an additional set of physical laws – the Maxwell equations.

4.1 Double-fluid model in physics of plasmas

When we consider electrons and ions to be non-interacting or at most weakly interacting, we may separate the particle distribution function into two terms, independently describing electrons and ions. Hence we have two independent "fluids" in the plasma, the electron fluid (described by n_e, $\boldsymbol{u}_\mathrm{e}$, p_e,...) and the ion fluid (fully described by n_i, $\boldsymbol{u}_\mathrm{i}$, p_i,...). For each type of particles α, where $\alpha \in \{\mathrm{e}, \mathrm{i}\}$, we have the equation of continuity, the Euler equation, and the equation of state, hence

$$\frac{\partial n_\alpha}{\partial t} + \boldsymbol{\nabla} \cdot (n_\alpha \boldsymbol{u}_\alpha) = 0 \tag{4.1}$$

$$m_\alpha n_\alpha \left[\frac{\partial \boldsymbol{u}_\alpha}{\partial t} + (\boldsymbol{u}_\alpha \cdot \boldsymbol{\nabla}) \boldsymbol{u}_\alpha \right] = q_\alpha n_\alpha (\boldsymbol{E} + \boldsymbol{u}_\alpha \times \boldsymbol{B}) - \nabla p_\alpha \tag{4.2}$$

$$p_\alpha = C_\alpha (m_\alpha n_\alpha)^{\gamma_\alpha}, \tag{4.3}$$

where γ_α is the ratio of specific heats and C_α is a constant. This set of equations must be accompanied by the Maxwell equations

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = n_\mathrm{i} q_\mathrm{i} + n_\mathrm{e} q_\mathrm{e} \tag{4.4}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{4.5}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{4.6}$$

$$\frac{1}{\mu_0} \boldsymbol{\nabla} \times \boldsymbol{B} = n_\mathrm{i} q_\mathrm{i} \boldsymbol{u}_\mathrm{i} + n_\mathrm{e} q_\mathrm{e} \boldsymbol{u}_\mathrm{e} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}. \tag{4.7}$$

Note that in the general expression on the right-hand side of (4.4) for a mixture of different charges α stands $\sum_\alpha n_\alpha q_\alpha$. Similarly, the sum of the first two terms on the right-hand side of (4.7) for a general mixture of charges turns into $\sum_\alpha n_\alpha q_\alpha \boldsymbol{u}_\alpha$. Also, additional force terms may be added on the right hand side of (4.5), such as the action of the gravity force $m_\alpha n_\alpha \boldsymbol{g}$ or others.

The above given set of equations altogether represents 18 equations (by considering the vector equations as triplets) for 16 variables (for electrons and ions each we have we have n, u_x, u_y, u_z, and p and we must add the components of the magnetic and electric fields). This disharmony is only apparent. Equations (4.4) and (4.6) play the role of boundary conditions and are not independent from (4.5) and (4.7). For instance, when we apply the operator of divergence to (4.5), we have

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{B}. \tag{4.8}$$

The left-hand side is a vanishing vector identity, hence

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \text{const}, \tag{4.9}$$

and our universe is set up so that this constant is equal to zero. The physical meaning of this boundary condition is that there are no magnetic charges.

The equation of state (4.3) requires some commentary. Note that when dropping the index α, we define the mass density as $\rho = mn$. The *polytropic* equation of state in the new variable reads

$$p = C\rho^\gamma \quad \text{or} \quad \frac{\nabla p}{p} = \gamma \frac{\nabla \rho}{\rho}. \tag{4.10}$$

The value of γ varies with the type of the process in question. E.g. for an isothermic process, we have $\gamma = 1$. For an adiabatic process, $\gamma = (N + 2)/N$, where N is the number of degrees of freedom. Note that $\gamma = 5/3$ for an adiabatic process concerning ideal gas.

4.1.1 Plasma approximation

When solving problems using the fluid approximation, we often use the *plasma approximation*. That essentially means that we neglect the possible concentration of charges occurring on small scales, and consider large scales only. Due to the requirement of the bulk neutrality of plasma, we must assume that the particle density of both fluids is equal, hence $n_\mathrm{e} = n_\mathrm{i} = n$. In the process of deriving useful relations, we then do not use the Poisson equation (4.4), which would otherwise lead to a discrepancy, because in detail, $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$, which is not true due to collective behaviour. We will show later that the plasma approximation is valid only when scales larger than Debye shielding length are considered.

4.2 Highly ionised plasmas

The need to capture the particle interactions on scales smaller than the Debye length naturally leads to additional term in the Euler equation. Such a term is termed *col-*

lisional term, even though strictly speaking it is not the same as the collisional term in the Boltzmann equation we derived earlier. The Euler equation for electrons then reads

$$m_\mathrm{e} n_\mathrm{e} \frac{\mathrm{d}\boldsymbol{u}_\mathrm{e}}{\mathrm{d}t} = -e n_\mathrm{e}(\boldsymbol{E} + \boldsymbol{u}_\mathrm{e} \times \boldsymbol{B}) - \nabla p_\mathrm{e} + \boldsymbol{P}_\mathrm{ei}. \tag{4.11}$$

The term P_ei represents the momentum change due to collisions with ions. Due to the momentum conservation law, it is clear that

$$\boldsymbol{P}_\mathrm{ei} = -\boldsymbol{P}_\mathrm{ie}. \tag{4.12}$$

Heuristically, the collisional term may be written as

$$\boldsymbol{P}_\mathrm{ei} = \underbrace{\underbrace{\eta}_{\textcircled{5}}\underbrace{e^2}_{\textcircled{1}}\underbrace{n_\mathrm{e}}_{\textcircled{2}}\underbrace{n_\mathrm{i}\underbrace{(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e})}_{\textcircled{3}}}_{\textcircled{4}}}_{} \sim \eta e^2 n^2 (\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}), \tag{4.13}$$

where we may notice the construction of the expression from the following

① The interaction is coulombic; hence, proportional to the multiplication of charges of both species.

② The rate of collisions is proportional to the electron flux.

③ Relative speed of the fluids.

④ Describes how often the electrons meet ions.

⑤ The scaling parameter (describing the efficiency of the collisions), which has a physical meaning of the *specific resistivity*.

Alternatively,

$$\boldsymbol{P}_\mathrm{ei} = m_\mathrm{e} n (\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}) \nu_\mathrm{ei}, \tag{4.14}$$

where ν_ei is the collisional frequency, with which the electrons hit the ions. Such an expression may also be viewed phenomenologically: the expression $m_\mathrm{e} n(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e})$ physically means the momentum exchanged due to the collisions, which is "normalised" by the collisional frequency (as the Euler equation evaluates the change of the momentum in time). Note that the particle density n does not hold the indication of species as the plasma approximation is assumed here. By comparing the two formulations of P_ei we obtain the expression for the collisional frequency as

$$\nu_\mathrm{ei} = \frac{n e^2}{m_\mathrm{e}} \eta = \omega_\mathrm{p}^2 \varepsilon_0 \eta, \tag{4.15}$$

where ω_p is the plasma frequency (to be defined in Section 5.4.1).

Let us estimate the specific resistivity η from the model of an electron squeezing around an ion and hence losing some portion of its momentum. See Fig. 4.1. The

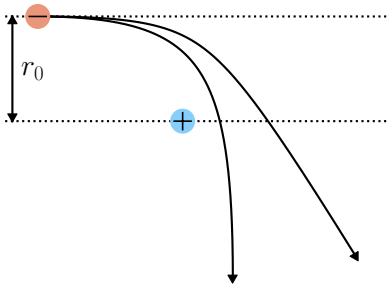

Figure 4.1: The geometry to estimate the specific resistivity

interaction is driven by the electric force, hence the force is

$$F = -\frac{e^2}{4\pi\varepsilon_0 r^2}, \tag{4.16}$$

where r is the distance between the ion (which is considered fixed in this coordinate system) and the passing electron. We roughly estimate that the interaction takes the total time Δt, where

$$\Delta t = \frac{r_0}{v}, \tag{4.17}$$

where r_0 is the impact parameter and v is the mutual speed of the electron and ion. This approximation assumes that the interaction is most effective for short distances and we will estimate the limit by r_0. Here we assumed that the essential interaction occurs on the scales of the impact parameter. During the interaction, the electron's momentum changes by

$$\Delta(m_\mathrm{e} v) \sim |F\,\Delta t| = \frac{e^2}{4\pi\varepsilon_0 r_0 v}. \tag{4.18}$$

Let us consider that during the interaction, the electron loses all of its momentum, hence $\Delta(m_\mathrm{e} v) = m_\mathrm{e} v$. This is the case when the electron is scattered to the direction perpendicular to the direction of its original motion.

$$\Delta(m_\mathrm{e} v) = m_\mathrm{e} v = \frac{e^2}{4\pi\varepsilon_0 r_0 v}, \tag{4.19}$$

which may be used to express the corresponding value of the impact parameter r_0. For this special case of perpendicular scattering, we will denote this special value of the impact parameter as r_{90}:

$$r_{90} = \frac{e^2}{4\pi\varepsilon_0 m_\mathrm{e} v^2}. \tag{4.20}$$

Let's define the *effective cross-section* σ of the interaction as

$$\sigma = \pi r_{90}^2 = \frac{e^4}{16\pi\varepsilon_0^2 m_\mathrm{e}^2 v^4}. \tag{4.21}$$

Then the collisional frequency (how many collisions during a unit of time) may be expressed as

$$\nu_{\mathrm{ei}} = \sigma n v = \frac{n e^4}{16\pi\varepsilon_0^2 m_{\mathrm{e}}^2 v^3}. \tag{4.22}$$

If we further relate the typical speed of the electrons and their temperature by $m_{\mathrm{e}} v_{\mathrm{e}}^2 \sim K_{\mathrm{B}} T_{\mathrm{e}}$, for the specific resistivity we obtain

$$\eta = \frac{\nu_{\mathrm{ei}} m_{\mathrm{e}}}{n e^2} = \frac{e^2}{16\pi\varepsilon_0^2 m_{\mathrm{e}} v^3} = \frac{e^2 m_{\mathrm{e}}^{1/2}}{16\pi\varepsilon_0^2 (K_{\mathrm{B}} T_{\mathrm{e}})^{3/2}}. \tag{4.23}$$

So far we have only considered perpendicular scattering. To account for all impact parameters, we must integrate over all of them, $\int_0^\infty 1/r \,\mathrm{d}r$. Unfortunately, such a function diverges, and the workaround is to limit the integration by the lower and upper bounds from a physical view. The lower bound is obviously the perpendicular scattering. The upper bound is the Debye length, as on larger scales, the plasma is neutral. Then we have

$$\int_{r_{90}}^{\lambda_{\mathrm{D}}} \frac{1}{r} \mathrm{d}r = [\ln r]_{r_{90}}^{\lambda_{\mathrm{D}}} = \ln \frac{\lambda_{\mathrm{D}}}{r_{90}} = \ln \Lambda, \tag{4.24}$$

where Λ is the coulombic logarithm. This brings a correction to what we have previously derived, hence the full expression of the specific resistivity reads

$$\eta = \frac{e^2 m_{\mathrm{e}}^{1/2}}{16\pi\varepsilon_0^2 (K_{\mathrm{B}} T_{\mathrm{e}})^{3/2}} \ln \Lambda. \tag{4.25}$$

Note that

- η does not depend on density. A larger density of the charge carriers is balanced by a higher rate of collisions.
- $\eta \propto T^{-3/2}$, hence for large temperatures the specific resistivity is very small. As a consequence, if we want to heat the plasma via Joule heating from the electric current flowing through the plasma, the processes get very inefficient when large temperatures are reached, hence one cannot obtain the temperatures necessary for thermonuclear fusion using such energy pumping. A different mechanism must be used.
- For $K_{\mathrm{B}} T_{\mathrm{e}} \sim 100$ eV the plasma specific resistivity is 5×10^{-7} Ωm, which is of the same order as the resistivity of metals such as copper.

Also, note that

$$\boldsymbol{P}_{\mathrm{ei}} = \eta e^2 n^2 (\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}) = e n_{\mathrm{e}} \boldsymbol{E}. \tag{4.26}$$

Then

$$\boldsymbol{E} = \eta e n (\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}) = \eta \boldsymbol{j}, \tag{4.27}$$

which is Ohm's law.

4.3 Single-fluid model

Another descrition of the system of two fluids, a *single-fluid approximation*, drops explicit information about the two charged species – electrons (having the particle mass m) and ions (having mass M). Instead, it considers their linear combinations which are directly related to measurable quantities. The governing equations of the single-fluid approximation are obtained by linearly combining the equations for the double-fluid approximation. These equations are replenished with the collision terms $P_{\mathrm{ei}} = \eta e^2 n^2 (\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}})$, where η is the specific resistivity, and an ad-hoc term of the gravity force. We further neglect the convective derivative $(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}$ by assuming the speeds of fluids are generally small and hence this is a quadratic term. This term is also negligible when velocity does not change much along the streamlines, which is a reasonable approximation. We further define the bulk density

$$\rho \equiv n_{\mathrm{i}} M + n_{\mathrm{e}} m \sim n(M+m) \sim nM, \tag{4.28}$$

the bulk velocity

$$\boldsymbol{u} \equiv \frac{1}{\rho}(n_{\mathrm{i}} M \boldsymbol{u}_{\mathrm{i}} + n_{\mathrm{e}} m \boldsymbol{u}_{\mathrm{e}}) \sim \frac{n(M\boldsymbol{u}_{\mathrm{i}} + m\boldsymbol{u}_{\mathrm{e}})}{n(M+m)} = \frac{M\boldsymbol{u}_{\mathrm{i}} + m\boldsymbol{u}_{\mathrm{e}}}{M+m}, \tag{4.29}$$

the bulk current density

$$\boldsymbol{j} \equiv e(n_{\mathrm{i}} \boldsymbol{u}_{\mathrm{i}} - n_{\mathrm{e}} \boldsymbol{u}_{\mathrm{e}}) \sim ne(\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}), \tag{4.30}$$

and the bulk pressure

$$p = p_{\mathrm{i}} + p_{\mathrm{e}}. \tag{4.31}$$

Let us consider the sum of Euler equations for both fluids, where we *ad-hoc* added the term of the acceleration by gravity:

$$Mn \frac{\partial \boldsymbol{u}_{\mathrm{i}}}{\partial t} = en(\boldsymbol{E} + \boldsymbol{u}_{\mathrm{i}} \times \boldsymbol{B}) - \nabla p_{\mathrm{i}} + Mn\boldsymbol{g} + \boldsymbol{P}_{\mathrm{ie}}, \tag{4.32}$$

$$mn \frac{\partial \boldsymbol{u}_{\mathrm{e}}}{\partial t} = -en(\boldsymbol{E} + \boldsymbol{u}_{\mathrm{e}} \times \boldsymbol{B}) - \nabla p_{\mathrm{e}} + mn\boldsymbol{g} + \boldsymbol{P}_{\mathrm{ei}} \tag{4.33}$$

by using the symmetry $P_{\mathrm{ie}} = -P_{\mathrm{ei}}$ and the definitions above we have

$$n(M+m)\frac{\partial}{\partial t}\left(\frac{M\boldsymbol{u}_{\mathrm{i}} + m\boldsymbol{u}_{\mathrm{e}}}{M+m}\right) = n\frac{\partial}{\partial t}(M\boldsymbol{u}_{\mathrm{i}} + m\boldsymbol{u}_{\mathrm{e}}) = en(\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}) \times \boldsymbol{B} - \nabla p + n(M+m)\boldsymbol{g}. \tag{4.34}$$

By using the definitions of the bulk density and the bulk current density we have

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p + \rho \boldsymbol{g}, \tag{4.35}$$

which is the equation of motion for a single-fluid plasma. Note that in general $\boldsymbol{j} \nparallel \boldsymbol{u}$ because the bulk velocity $\boldsymbol{u}$ is weighted by masses, whereas the current density $\boldsymbol{j}$ is weighted by charges. In principle, discrepancies may appear.

Let's consider a different combination of the same equations. Let's multiply (4.32) by m and (4.33) by M

$$mMn\frac{\partial \boldsymbol{u}_\mathrm{i}}{\partial t} = emn(\boldsymbol{E} + \boldsymbol{u}_\mathrm{i} \times \boldsymbol{B}) - m\nabla p_\mathrm{i} + mMn\boldsymbol{g} + m\boldsymbol{P}_\mathrm{ie}, \tag{4.36}$$

$$mMn\frac{\partial \boldsymbol{u}_\mathrm{e}}{\partial t} = -eMn(\boldsymbol{E} + \boldsymbol{u}_\mathrm{e} \times \boldsymbol{B}) - M\nabla p_\mathrm{e} + mMn\boldsymbol{g} + M\boldsymbol{P}_\mathrm{ei}. \tag{4.37}$$

Now subtract one from the other. We have

$$\begin{aligned} nmM\frac{\partial}{\partial t}(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}) = en(M+m)\boldsymbol{E} + en(m\boldsymbol{u}_\mathrm{i} + M\boldsymbol{u}_\mathrm{e}) \times \boldsymbol{B} - \\ - m\nabla p_\mathrm{i} + M\nabla p_\mathrm{e} - (M+m)\boldsymbol{P}_\mathrm{ei}. \end{aligned} \tag{4.38}$$

Now $\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e} = \boldsymbol{j}/(ne)$ may be expressed using the bulk electric current and

$$m\boldsymbol{u}_\mathrm{i} + M\boldsymbol{u}_\mathrm{e} = M\boldsymbol{u}_\mathrm{i} + m\boldsymbol{u}_\mathrm{e} + M(\boldsymbol{u}_\mathrm{e} - \boldsymbol{u}_\mathrm{i}) + m(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}) = \frac{\rho}{n}\boldsymbol{u} - (M-m)\frac{\boldsymbol{j}}{ne}, \tag{4.39}$$

and also

$$P_\mathrm{ei} = \eta e^2 n^2(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}) = \eta en\boldsymbol{j}. \tag{4.40}$$

By further dividing (4.38) by $en(M+m) = e\rho$ we have

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} - \eta\boldsymbol{j} = \frac{1}{e\rho}\left[\frac{Mmn}{e}\frac{\partial}{\partial t}\frac{\boldsymbol{j}}{n} + (M-m)\boldsymbol{j} \times \boldsymbol{B} + m\nabla p_\mathrm{i} - M\nabla p_\mathrm{e}\right]. \tag{4.41}$$

For $m/M \to 0$ and when

$$\frac{Mmn}{e}\frac{\partial}{\partial t}\frac{\boldsymbol{j}}{n} = MnB\frac{m}{eB}\frac{\partial}{\partial t}\frac{\boldsymbol{j}}{n} = MnB\omega_\mathrm{c}^{-1}\frac{\partial}{\partial t}\frac{\boldsymbol{j}}{n} \ll (M-m)\boldsymbol{j} \times \boldsymbol{B}, \tag{4.42}$$

which is fullfilled when the time variations of the current density are much slower than the cyclotron frequency. In that case

$$\frac{1}{\omega_\mathrm{c}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{j}}{n}\right) \ll \frac{\boldsymbol{j}}{n} \tag{4.43}$$

and we obtain

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} - \eta\boldsymbol{j} = \frac{1}{en}\left(\boldsymbol{j} \times \boldsymbol{B} - \nabla p_\mathrm{e}\right), \tag{4.44}$$

which is *generalised Ohm's law*. The first term on the right-hand side is the expression for Hall's current. In various applications, the right-hand side is often completely neglected.

Similarly (and we leave the derivation to the reader) one may obtain the continuity equation for density and charge density $\rho_\mathrm{e} = e(n_\mathrm{i} - n_\mathrm{e})$ for a single-fluid plasma by taking the sum of and subtracting the continuity equations for both fluids:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho\boldsymbol{u} = 0 \tag{4.45}$$

and

$$\frac{\partial \rho_{\mathrm{e}}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0. \tag{4.46}$$

Hence the full set of equations for a single-fluid plasma reads

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p + \rho \boldsymbol{g}, \tag{4.47}$$

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j}, \tag{4.48}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \boldsymbol{u} = 0, \tag{4.49}$$

$$\frac{\partial \rho_{\mathrm{e}}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0. \tag{4.50}$$

This system must be accompanied by the Maxwell equations to fully describe the plasma in equilibrium, where the definitions for ρ_{e} and $\boldsymbol{j}$ are simply used in Equations (4.4)-(4.7). The single-fluid approximation is usually used when resistivity play a non-negligible role in the plasma and is also often used in astrophysics. The system of single-fluid equations is most often understood under the term *magnetohydrodynamic equations* or *MHD* equations.

4.4 Approximations to plasma fluid

Additional approximations to plasma fluid are used in some cases.

4.4.1 Force-free and ideal MHD

The *force-free approximation* is valid for systems where locally the plasma does not "feel" the Lorentz force, i.e.

$$f_{\mathrm{Lorentz}} = \rho_e \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} = 0. \tag{4.51}$$

Note that the above given equation represents an "effective" or "bulk" Lorentz force acting on both species in the mixture.

This approximation is also used in the case when the Lorentz force is negligible compared to other considered forces. Another view to this approximation may also be formulated in such a way that the electromagnetic field induced by the motion of plasma fluid exactly and immediately cancels out the external forces.

For small E, the force-free approximation turns to

$$\boldsymbol{j} \times \boldsymbol{B} = 0. \tag{4.52}$$

The current density is related to the curl of the magnetic induction via the Maxwell equations by

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu \boldsymbol{j}. \tag{4.53}$$

Combining the two we have

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = 0, \tag{4.54}$$

which indicates that the curl of the magnetic induction is parallel to the magnetic induction, i.e.,

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha \boldsymbol{B}. \tag{4.55}$$

This approximation is usually referred to as *linear force free*. Note that the constant of proportionality α may change with the position in space.

The *ideal MHD approximation* is somewhat different. It assumes that in the comoving frame the electric field disappears. I.e.

$$\boldsymbol{E'} = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{0}. \tag{4.56}$$

By using generalised Ohm's law it turns out that the ideal MHD approximation is equivalent to the case when the plasma is ideally conductive, therefore when the resistivity is zero. This may immediately be seen from

$$\boldsymbol{E'} = \eta \boldsymbol{j}, \tag{4.57}$$

where the left-hand side is zero because of (4.56) and this relation may be valid in trivial cases when $\boldsymbol{j} = 0$ or when $\eta = 0$ (which represents infinite conductivity).

The two approximations mentioned in this subsection are identical when the electromagnetic field is degenerated, i.e. when $\boldsymbol{E} \cdot \boldsymbol{B} = 0$. In such a case, it also appears that $\boldsymbol{j} = \rho_e \boldsymbol{u}$.

Frozen field

A consequence of the ideal MHD approximation is something called *frozen field*. It is a description of observed behaviour of highly-conductive plasmas, where a strong coupling between the plasma and the field exists. It was found responsible for strengthening the magnetic field in stars or temporal changes of the Earth's magnetosphere under the influence of the varying solar wind.

Let us start by taking the curl of Ampère's law:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \nabla \boldsymbol{\nabla} \cdot \boldsymbol{B} - \triangle \boldsymbol{B} = \boldsymbol{\nabla} \times \left(\mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right). \tag{4.58}$$

The term $\nabla \boldsymbol{\nabla} \cdot \boldsymbol{B}$ vanishes due to Gauss's law for magnetism. By using the ideal MHD approximation (4.56), Eq. (4.57), and the definition of $\sigma \equiv 1/\eta$ we have

$$\sigma \mu_0 \boldsymbol{\nabla} \times (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) = -\varepsilon_0 \mu_0 \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{E}}{\partial t} - \triangle \boldsymbol{B}. \tag{4.59}$$

Assuming the right-hand-side of eq. (4.59) is finite, $\boldsymbol{\nabla} \times (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$ must go to zero when $\sigma \to \infty$ so that the left-hand-side is finite as well. By additionally using the Faraday's law we then have

$$-\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = 0 \quad \to \quad \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = \frac{\partial \boldsymbol{B}}{\partial t}. \tag{4.60}$$

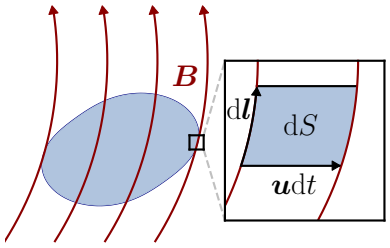

Figure 4.2: Magnetic flux through a closed curve in the frozen-field approximation.

Let us note at this point that $\boldsymbol{\nabla} \times (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) = 0$ does not imply $\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = 0$. A mathematical object having null value does not necessarily have to have vanishing derivatives!

The frozen-field approximation may also be understood as a condition when the magnetic flux through a closed curve bound to the plasma does not change with time. Let us prove this statement. Let's start by defining the system as in Fig. 4.2. By zooming in it is obvious that the change of the surface vector element is given by a cross-product of the plasma velocity and the element of length along the closed curve, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{S} = \boldsymbol{u} \times \mathrm{d}\boldsymbol{l}$.

We will now derive the change of the magnetic flux Ψ with time:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \int \boldsymbol{B} \cdot \frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} + \int \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}. \tag{4.61}$$

We apply the relation between the surface and length elements from above and after using the cyclic exchange of a mixed product we have

$$\oint_l \boldsymbol{B} \cdot (\boldsymbol{u} \times \mathrm{d}\boldsymbol{l}) + \int \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} = -\oint_l (\boldsymbol{u} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{l} + \int \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}. \tag{4.62}$$

We further use the Stokes mathematical theorem and obtain

$$-\int \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{S} + \int \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} = -\int \left[\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) - \frac{\partial \boldsymbol{B}}{\partial t}\right] \cdot \mathrm{d}\boldsymbol{S} = \boldsymbol{0}, \tag{4.63}$$

where we used (4.60). Q.E.D.

4.4.2 Axisymmetrical systems, stream functions

Another class of approximations comes with the symmetries and/or degenerations of the system. Let us consider an axially symmetric stationary system represented in the cylindrical coordinates (R, φ, z). I.e. $\frac{\partial}{\partial \varphi} = 0$ and $\frac{\partial}{\partial t} = 0$. The curl of $\boldsymbol{E}$ has the components

$$\boldsymbol{\nabla} \times \boldsymbol{E} = \left(\frac{1}{R}\frac{\partial E_z}{\partial \varphi} - \frac{\partial E_R}{\partial z}, \frac{\partial E_R}{\partial z} - \frac{\partial E_z}{\partial R}, \frac{1}{R}\frac{\partial}{\partial R}RE_\varphi - \frac{1}{R}\frac{\partial E_R}{\partial \varphi}\right) = \boldsymbol{0}. \tag{4.64}$$

From the third component it turns out that $E_\varphi = 0$.

By using the ideal MHD approximation, we further have

$$E_\varphi = -(\boldsymbol{u} \times \boldsymbol{B})_\varphi = -u_R B_z + u_z B_R = 0, \tag{4.65}$$

and therefore

$$\frac{u_R}{u_z} = \frac{B_R}{B_z}. \tag{4.66}$$

Thus the poloidal components are bound with an unknown scalar function ξ

$$\boldsymbol{u}_\mathrm{p} = \xi(R, z)\boldsymbol{B}_\mathrm{p}, \tag{4.67}$$

which is a consequence of the axial symmetry, stationarity, and the used ideal MHD approximation. Such a result is therefore not valid generally!

Let us only remind the reader that the poloidal component of a vector $\vec{a}$ is simply a combination of the radial and height components, i.e., $\boldsymbol{a}_\mathrm{p} = a_R \boldsymbol{e}_R + a_z \boldsymbol{e}_z$.

Because $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, where $\boldsymbol{A}$ is the magnetic vector-potential, we may write

$$\Psi(R, z) = R A_\varphi, \tag{4.68}$$

where Ψ is the scalar magnetic flux function. Then simply

$$B_R = -\frac{1}{R}\frac{\partial \Psi}{\partial z} \quad \text{and} \quad B_z = \frac{1}{R}\frac{\partial \Psi}{\partial R}. \tag{4.69}$$

By using the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}_\mathrm{p}) + \frac{\partial \rho v_\varphi}{\partial \varphi} = \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}_\mathrm{p}) = 0, \tag{4.70}$$

we have

$$\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}_\mathrm{p}) = \boldsymbol{\nabla} \cdot (\rho\xi \boldsymbol{B}_\mathrm{p}) = \nabla \rho\xi \cdot \boldsymbol{B}_\mathrm{p} + \rho\xi \boldsymbol{\nabla} \cdot \boldsymbol{B}_\mathrm{p} = 0. \tag{4.71}$$

The last term vanishes due to Gauss's law for magnetism. By decomposing the divergence operator into non-vanishing components and using (4.69)

$$\boldsymbol{\nabla} \cdot (\rho\xi \boldsymbol{B}_\mathrm{p}) = \nabla_R(\rho\xi) B_R + \nabla_z(\rho\xi) B_z = -\frac{1}{R}\nabla_R(\rho\xi)\nabla_z \Psi + \frac{1}{R}\nabla_z(\rho\xi)\nabla_R \Psi = 0, \tag{4.72}$$

we see that this is the φ component of the expression $\frac{1}{R}\nabla(\rho\xi) \times \nabla\Psi$. Due to the assumptions the arguments of the gradient operators do not depend on φ. The only non-trivial component of the gradient is the component perpendicular to both $\boldsymbol{e}_R$ and $\boldsymbol{e}_z$. That is the φ-component, which also vanished as we proved in (4.72). Thus

$$\nabla(\rho\xi) \times \nabla\Psi = \boldsymbol{0}. \tag{4.73}$$

It may be interpretted as $\nabla(\rho\xi) \parallel \nabla\Psi$ and therefore the scalar function ξ may be related to Ψ as

$$4\pi\rho\xi = F_1(\Psi). \tag{4.74}$$

Function F_1 is termed the *stream function*, which is obviously constant on a constant surface of Ψ. Its particular mathematical expression may be determined by the boundary conditions. The application of the stream function allows us to reduce the number of variables that need to be solved for in the given system as it introduces a relation between $\boldsymbol{u}$ and $\boldsymbol{B}$.

Similarly, using the poloidal and azimuthal (toroidal) decomposition and equations (4.67) and (4.69) we have

$$\begin{aligned}\boldsymbol{u}\times\boldsymbol{B} &= u_\varphi \boldsymbol{e}_\varphi \times \boldsymbol{B}_\mathrm{p} + \boldsymbol{u}_\mathrm{p}\times B_\varphi \boldsymbol{e}_\varphi = u_\varphi \boldsymbol{e}_\varphi \times \boldsymbol{B}_\mathrm{p} + \xi \boldsymbol{B}_\mathrm{p}\times B_\varphi \boldsymbol{e}_\varphi \\ &= (u_\varphi - \xi B_\varphi)(B_z - B_R)\boldsymbol{e}_z = (u_\varphi - \xi B_\varphi)\frac{1}{R}\left(\frac{\partial\Psi}{\partial R} - \frac{\partial\Psi}{\partial z}\right)\boldsymbol{e}_z \\ &= \frac{u_\varphi - \xi B_\varphi}{R}\nabla\Psi. \end{aligned} \tag{4.75}$$

In the stationary frozen-field approximation we have

$$\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) = \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}, \tag{4.76}$$

and therefore

$$\nabla\times\left(\frac{u_\varphi - \xi B_\varphi}{R}\nabla\Psi\right) = \nabla\left(\frac{u_\varphi - \xi B_\varphi}{R}\right)\times\nabla\Psi = \boldsymbol{0}, \tag{4.77}$$

where we proceeded similarly to the previous case. Again, the surfaces of the arguments of the gradient operators must coalign, and hence we may define a second stream function as

$$\frac{u_\varphi - \xi B_\varphi}{R} = F_2(\Psi). \tag{4.78}$$

By taking the projection of the Euler equation to $\boldsymbol{e}_\varphi$, we can show that

$$B_\varphi \boldsymbol{e}_\varphi \cdot \nabla(RB_\varphi - F_1 R v_\varphi) = 0, \tag{4.79}$$

and hence define the third stream function as

$$RB_\varphi - F_1(\Psi)Rv_\varphi = F_3(\Psi). \tag{4.80}$$

And finally, by taking the projection of the Euler equation onto $\boldsymbol{B}_\mathrm{p}$, we obtain the definition for a fourth stream function as

$$\frac{1}{2}u^2 + \int_{\Psi=\mathrm{const}}\frac{\mathrm{d}P}{\rho} + \Phi - Ru_\varphi F_3(\Psi) = F_4(\Psi). \tag{4.81}$$

Altogether the stream functions provide a "technology" for seeking the solution of the system in a given geometry.

4.5 Drifts in the plasma fluid

Let's discuss the drifts in the plasma fluid.

4.5.1 Drifts perpendicular to $\boldsymbol{B}$

When investigating the drifts in the direction perpendicular to $\boldsymbol{B}$, we solve the perpendicular component of the Euler equation, which for collision-less plasma reads:

$$mn\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = qn(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - \nabla p \tag{4.82}$$

Let us consider systems where the left-hand side is negligible compared to to the right-hand side.[1] Hence we can solve a much simpler equation

$$0 = qn(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - \nabla p \tag{4.83}$$

by applying the vector multiplication by $\boldsymbol{B}$:

$$\begin{aligned} 0 &= qn[\boldsymbol{E}\times\boldsymbol{B} + (\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B}] - \nabla p\times\boldsymbol{B} = \\ &= qn(\boldsymbol{E}\times\boldsymbol{B} - \boldsymbol{u}B^2 + \boldsymbol{B}\boldsymbol{u}\cdot\boldsymbol{B}) - \nabla p\times\boldsymbol{B}. \end{aligned} \tag{4.84}$$

The perpendicular component of this equation is

$$qn(\boldsymbol{E}\times\boldsymbol{B} - \boldsymbol{u}_\perp B^2) - \nabla p\times\boldsymbol{B} = 0 \tag{4.85}$$

and thus

$$\boldsymbol{u}_\perp = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} - \frac{\nabla p\times\boldsymbol{B}}{qnB^2} = \boldsymbol{v}_\mathrm{E} + \boldsymbol{v}_\mathrm{D}. \tag{4.86}$$

We found the expression for the perpendicular drift in the plasma fluid. The first term resembles the already known E-B drift, the second term is new and represents the *diamagnetic drift* $\boldsymbol{v}_\mathrm{D}$. The direction of $\boldsymbol{v}_\mathrm{D}$ is perpendicular to ∇p, hence the assumption of $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ being negligible holds only if $\boldsymbol{E} = 0$. The assumption further holds if $\nabla\phi$ and ∇p are parallel, then $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ is still zero for $\boldsymbol{E} = -\nabla\phi$. In other cases the situation is more complicated and the solution may involve the necessity to deal with the $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ term.

4.5.2 Drifts parallel to $\boldsymbol{B}$

Now let's look at the remaining component of the drifts in a plasma fluid. Let's consider the coordinate system so that the axis $\boldsymbol{e}_z$ is parallel to the magnetic field. Then we study the z-component of the equation of motion

$$mn\left(\frac{\partial u_z}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_z\right) = qnE_z - \frac{\partial p}{\partial z} \tag{4.87}$$

[1] To estimate this, let's use the Fourier transform of Euler equation, for now blindly applying the rules (5.11) to (5.13). By simply waving our hands we see that the left-hand side is proportional to $\mathrm{i}\omega + \mathrm{i}\frac{1}{L}$, where ω is the mode frequency and L is the characteristic scale of spatial change, while the right-hand side is proportional to $\mathrm{i}\omega_\mathrm{c} \propto \frac{1}{r_\mathrm{L}}$. Usually, $\omega \ll \omega_\mathrm{c}$ and $r_\mathrm{L} \ll L$, hence the left-hand side is negligible.

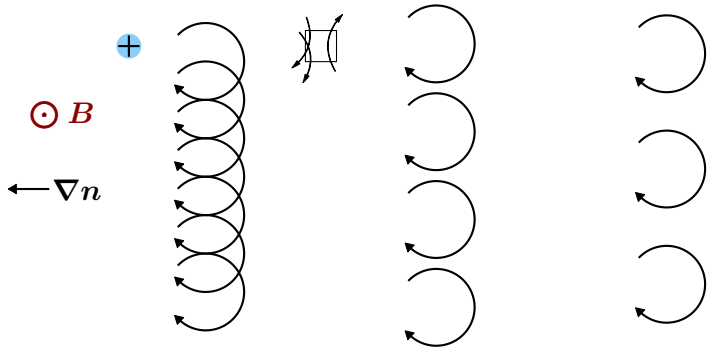

Figure 4.3: A scheme of the origin of the diamagnetic drift. In the "zooming" square one sees that on the side with the larger density, the arrows pointing down are more abundant than on the opposite side. Thus, on average, the downwards motion prevails which is represented by the drift.

with the equation of state in the form

$$\frac{\nabla p}{p} = \gamma \frac{\nabla n}{n}, \tag{4.88}$$

where we further use $\gamma = 1$, because we will solve the equation only for electrons that are fast and may be considered isothermic. Additional assumptions apply: $(\boldsymbol{u} \cdot \nabla) u_z \to 0$, u_z homogeneous, and $\nabla p = K_\mathrm{B} T \nabla n$. Then

$$mn \frac{\partial u_z}{\partial t} = qnE_z - K_\mathrm{B} T \frac{\partial n}{\partial z}. \tag{4.89}$$

For electrons we have $q = -e$, $m \to 0$, and $T \to T_\mathrm{e}$. By using the Poisson equation $E_z = -\partial\phi/\partial z$ we have

$$e \frac{\partial \phi}{\partial z} = \frac{K_\mathrm{B} T_\mathrm{e}}{n} \frac{\partial n}{\partial z} \tag{4.90}$$

and hence

$$e\phi = K_\mathrm{B} T_\mathrm{e} \ln n + \mathcal{C} \tag{4.91}$$

and finally

$$n = n_0 \exp\left[\frac{e\phi}{K_\mathrm{B} T_\mathrm{e}}\right], \tag{4.92}$$

which is the *Boltzmann relation*. The Boltzmann relation describes the spatial distribution of electrons around a random concentration of ions. In a local concentration of ions, the pressure gradient pushes electrons away from this region. The consecutive charge separation gives rise to a local electric field, which acts against the pressure gradient. In equilibrium, both the pressure and electrostatic forces are balanced. The Boltzmann relation is a very useful relation when we further solve the system of equations for ions. Due to the mass difference between the two species the spatial distribution of electrons may be approximated by the Boltzmann relation.

Chapter 5

Waves in plasmas

5.1 Linear waves

It is not trivial to comprehensively describe solutions of systems supporting waves in plasmas. We will investigate a few examples of only *linear waves*. In practice, we will decompose the variables describing both plasma and magnetic and electric fields into two components: a *background component* (having index 0) and a *fluctuating component* (having index 1), e.g., $\rho = \rho_0 + \rho_1$. In (silently) applying this decomposition, we will assume that:

1. The background components automatically fulfill the full set of magnetohydrodynamic equations. The background is of the zeroth order of the smallness variable.
2. The fluctuating components are small compared to the background (e.g., $\rho_1/\rho_0 \ll 1$). When averaging the fluctuating component (both in space and/or time), the averaged value is zero (e.g., $\langle \rho_1 \rangle = 0$). Implicitly, we assume the same about both temporal and spatial derivatives of the fluctuating component. The fluctuating part is of the first order of the smallness variable.

We will solve the set (or subset) of magnetohydrodynamic equations to the first order.

Let's stop for a while and make this decomposition crystal clear forever. It might get confusing especially in the case of velocity. When we computed moments of the Boltzmann equation, we also performed a decomposition having a fluctuating part. Let's compare the two:

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}, \tag{5.1}$$

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1. \tag{5.2}$$

These two are crucially different. While in Eq. (5.1) the *particle* velocity $\boldsymbol{v}$ is decomposed into the *mean* (or bulk) velocity $\boldsymbol{u}$ and the component $\boldsymbol{w}$ fluctuating about this mean value, where the amplitude of the fluctuations may be (and typically is) much larger than the amplitude of the bulk motion, in Eq. (5.2) the *bulk* velocity $\boldsymbol{u}$ is decomposed into the background fulfilling the MHD equations $\boldsymbol{u}_0$ and a small correction to this background value. Hence $\boldsymbol{u}$ from Eq. (5.2) indeed is equal to $\boldsymbol{u}$ from Eq. (5.1).

5.2 Example: Acoustic waves in fluids

To illustrate the linearisation approach, let us derive the dispersion relation for waves in a fluid, fully described by the Euler equation in the form

$$\rho\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] = -\nabla p = -\frac{\gamma p}{\rho}\nabla\rho, \tag{5.3}$$

where on the right-hand side we assumed the adiabatic equation of state $p\rho^{-\gamma} = \text{const}$, and the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0. \tag{5.4}$$

We will assume the following:

Pressure waves in fluids

- Strictly hydrodynamical case (no fields)
- No viscosity
- No bulk motion of the background, $\boldsymbol{u}_0 = 0$
- The background is homogeneous

By introducing the fluctuating part to the velocity and density, i.e., $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1$, $p = p_0 + p_1$, and $\rho = \rho_0 + \rho_1$, equations (5.3) and (5.4) take the form of

$$(\rho_0+\rho_1)\left\{\frac{\partial \boldsymbol{u}_0}{\partial t} + \frac{\partial \boldsymbol{u}_1}{\partial t} + \left[(\boldsymbol{u}_0+\boldsymbol{u}_1)\cdot\boldsymbol{\nabla}\right](\boldsymbol{u}_0+\boldsymbol{u}_1)\right\} = -\frac{\gamma(p_0+p_1)}{(\rho_0+\rho_1)}\nabla(\rho_0+\rho_1) \tag{5.5}$$

$$\frac{\partial(\rho_0+\rho_1)}{\partial t} + \nabla\cdot[(\rho_0+\rho_1)(\boldsymbol{u}_0+\boldsymbol{u}_1)] = 0. \tag{5.6}$$

Now, let's take the continuity equation (5.6) as an example. It can be explicitly written out as

$$\overset{①}{\frac{\partial\rho_0}{\partial t}} + \overset{②}{\frac{\partial\rho_1}{\partial t}} + \nabla\cdot(\overset{③}{\rho_0\boldsymbol{u}_0}) + \rho_0\overset{④}{\nabla\cdot}\boldsymbol{u}_1 + \boldsymbol{u}_1\overset{⑤}{\cdot}\nabla\rho_0 + \nabla\cdot(\overset{⑥}{\rho_1\boldsymbol{u}_0}) + \nabla\cdot(\overset{⑦}{\rho_1\boldsymbol{u}_1}) = 0. \tag{5.7}$$

Let's discuss the terms in the equation above:

- Terms ① and ③ automatically fulfill the continuity equation for the background variables: $\frac{\partial\rho_0}{\partial t} + \nabla\cdot(\rho_0\boldsymbol{u}_0) = 0$.
- Term ⑤ is zero because we assumed a homogeneous background, i.e. $\nabla\rho_0 = 0$.
- Term ⑥ is zero because we assumed $\boldsymbol{u}_0 = 0$.

- Term (7) is a second order term, thus it can be neglected.

Thus only relation

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_1 = 0 \tag{5.8}$$

remains. This is the *linearised continuity equation*.

Similarly[1], we obtain the *linearised Euler equation*:

$$\rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} = -\gamma \frac{p_0}{\rho_0} \nabla \rho_1. \tag{5.9}$$

Equations (5.8) and (5.9) need to be solved. We are looking for a solution supporting waves, thus, we may search for a solution in a form

$$A_1 = \tilde{A}_1 \exp\left[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)\right], \tag{5.10}$$

where $\tilde{A}_1$ is the Fourier image of A_1 or the amplitude, $\boldsymbol{k}$ is a wave vector in the Fourier space and ω is a frequency. There are two possibilities for how to see this approach:

1. We solve these equations in the Fourier space. At this step, a careful reader should open the mathematical textbook and repeat all the assumptions of the Fourier transform.

2. We view the solution as a superposition of modes, each mode[2] is described by a unique combination of $\boldsymbol{k}$ and ω. Thus we solve the equations for one general mode and a complete solution is a superposition of all modes with various amplitudes.

Either way, the beauty of the formulation of the equations in the Fourier space is that the differential operators turn into multiplication. One can easily prove from Eq. (5.10) that

$$\frac{\partial \bullet}{\partial t} \to -\mathrm{i}\omega \bullet \tag{5.11}$$

$$\nabla \bullet \to \mathrm{i}\boldsymbol{k} \bullet \tag{5.12}$$

$$\triangle \bullet \to -k^2 \bullet \tag{5.13}$$

Our goal is to find a *dispersion relation*, thus a relation which couples $\boldsymbol{k}$ and ω. Dispersion relation $D(\boldsymbol{k}, \omega) = 0$ fully describes the given mode of the wave. Equations (5.8) and (5.9) written in the Fourier space take the following form:

$$-\mathrm{i}\omega\rho_1 + \mathrm{i}\rho_0 \boldsymbol{k} \cdot \boldsymbol{u}_1 = 0, \tag{5.14}$$

$$-\mathrm{i}\omega\rho_0 \boldsymbol{u}_1 + \mathrm{i}\gamma \frac{p_0}{\rho_0} \boldsymbol{k} \rho_1 = 0. \tag{5.15}$$

[1] For clarity, let's handle the fraction on the right-hand side explicitly:

$$\frac{p_0 + p_1}{\rho_0 + \rho_1} = \frac{\frac{p_0 + p_1}{\rho_0}}{\frac{\rho_0 + \rho_1}{\rho_0}} = \frac{\frac{p_0}{\rho_0} + \frac{p_1}{\rho_0}}{1 + \frac{\rho_1}{\rho_0}} = \frac{p_0}{\rho_0},$$

where $\frac{p_1}{\rho_0}$ and $\frac{\rho_1}{\rho_0}$ are negligible terms.

[2] A *mode* is defined either as any of the wave-like and/or oscillation-like plasma motions or as one possible configuration of the travelling or steady wave. Both definitions are equivalent.

For simplicity let's define the coordinate system so that the wave propagates in the main direction, thus, wave vector $\boldsymbol{k}$ may be replaced by its scalar equivalent wave number k. Then by solving the equations above one obtains a *dispersion relation*

$$\frac{\omega}{k} = \sqrt{\frac{\gamma p_0}{\rho_0}} \equiv c_\mathrm{s}, \tag{5.16}$$

where c_s is the speed of sound.
A few remarks to what we just obtained:

- The dispersion relation exists, thus the waves propagate. The propagation is fully described by the wave vector $\boldsymbol{k}$. Possible combinations of the wave vector and frequency of the mode are bound by the dispersion relation. The waves propagate in the form of pressure fluctuations.
- The wavenumber of the wave is inversely proportional to its *wavelength* $\lambda = (2\pi)/k$.
- The *phase speed* v_φ is the speed with which places with the same phase propagate through the medium. The phase speed is given by $v_\varphi = \omega/k$. In our case, $v_\varphi = c_\mathrm{s}$.
- On the other hand, the *group speed* v_g is the speed with which the modulation of information propagates. It also relates the wavenumber and the frequency of the mode by $v_\mathrm{g} = \mathrm{d}\omega/\mathrm{d}k$. In our case, again, $v_\mathrm{g} = c_\mathrm{s}$ and $v_\mathrm{g} = v_\varphi$. As we will show later, this does not always have to be the case. It can be illustrated in a very simple example: we have one mode only with a constant amplitude in time. A constant "beep" does not carry any information. Its dispersion relation reads: $\omega = \mathrm{const}$. Thus its phase speed is not defined (it might be viewed as infinite as well), but the group speed is zero. We just showed an additional important property: phase speed does not have to be limited (again, it does not carry any information), while the group speed is limited (from the top) by the speed of light c.
- The term *dispersion relation* indicates that the wave *disperses*, changes its waveform during the propagation. That indeed is the case, when the wave is *dispersive*, i.e., when the propagation phase speed depends on wave frequency or wave number. Then the amplitude spectrum of the wave packet changes as the packet passes through the medium – individual frequencies exhibit various phase shifts. Then the wave-packet propagation is not a simple translation parallel with the direction of propagation, but it is more complex and additionally due to to the movement of the wave packet, the packet itself changes its shape. Dispersive waves usually have group speed different from the phase speed. Sound waves in fluids are non-dispersive. A simplistic illustration of dispersive and non-dispersive waves is given in Fig. 5.1.

- Note that the approach we will use is equivalent to *geometrical optics approximation* a.k.a. *ray approximation*, which ignores the effects of the finite wavelength. Such an approach is justified when the wavelength is much shorter than the typical spatial scale of the changes in the medium. In the opposite case, the effects that are not captured by the ray approximation may be observed. Finite wavelength effects are captured by the *scattering theory*, which are not easy to solve both analytically and numerically. Approximations to the scattering theory, such as the Born approximation, capture some of the most important finite wavelength effects quite well.

Let us see what happens when we relax the assumption of $\boldsymbol{u}_0 = 0$. Then the Fourier image of the continuity equation will read:

$$-\mathrm{i}\omega\rho_1 + \mathrm{i}\rho_1 \boldsymbol{k}\cdot\boldsymbol{u}_0 + \mathrm{i}\rho_0 \boldsymbol{k}\cdot\boldsymbol{u}_1 = 0, \tag{5.17}$$

which may be written as

$$-\mathrm{i}\rho_1(\omega - \boldsymbol{k}\cdot\boldsymbol{u}_0) + \mathrm{i}\rho_0 \boldsymbol{k}\cdot\boldsymbol{u}_1 = 0. \tag{5.18}$$

Hence by introducing $\omega' = \omega - \boldsymbol{k}\cdot\boldsymbol{u}_0$ we change the equation to the previous case. A similar operation may be done to the Euler equation[3]. Hence the final dispersion relation will be

$$\omega - \boldsymbol{k}\cdot\boldsymbol{u}_0 = kc_\mathrm{s}, \tag{5.21}$$

and the motion of the background fluid with respect to the observer's frame introduces a *Doppler shift* of the frequencies.

5.3 Types of plasma waves

To study waves in plasma, we will use the fluid approximation. It is complex enough to capture all essential properties to investigate the possible propagation of waves but it is also simple enough to be able to obtain some analytical solutions. There is a whole spectrum of waves propagating through the plasma, which may be decomposed into several modes with different dispersion relations. We will only study a handful of the possible modes, some of which may have important physical consequences.

Plasma waves are usually excited by means of instabilities (this is not surprising, e.g. sound waves propagating through a convective envelope of solar-like stars are excited by the convective instability below the photosphere of the star) or they may be excited

[3]Due to $\boldsymbol{u}_0 \neq 0$ we must keep the "convective derivative" term in the equation, leading to the linearised form

$$\rho_0\frac{\partial \boldsymbol{u}_1}{\partial t} + \rho_0(\boldsymbol{u}_0\cdot\boldsymbol{\nabla}\boldsymbol{u}_1) = -\gamma\frac{p_0}{\rho_0}\boldsymbol{\nabla}\rho_1, \tag{5.19}$$

which has a Fourier form of

$$-\mathrm{i}\rho_0(\omega - \boldsymbol{k}\cdot\boldsymbol{u}_0)\boldsymbol{u}_1 + \mathrm{i}k\gamma\frac{p_0}{\rho_0}\rho_1 = 0, \tag{5.20}$$

where a possible substitution $\omega' = \omega - \boldsymbol{k}\cdot\boldsymbol{u}_0$ is immediately visible.

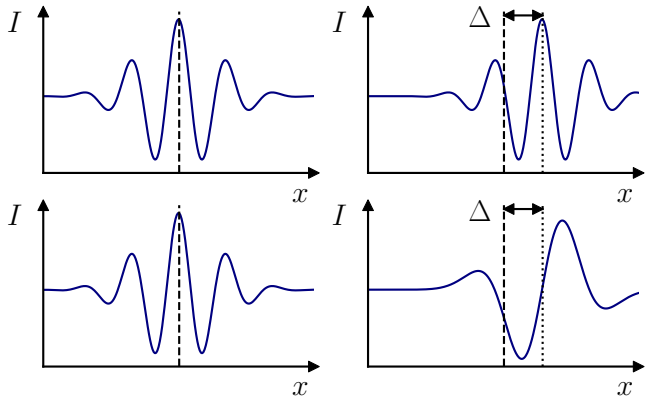

Figure 5.1: Difference between dispersive and non-dispersive waves. In the case of non-dispersive waves, the time evolution of the waveform only consists of a translation in space. In the case of dispersive waves, various modes propagate with different phase speeds, hence the waveform is not only translated but also corrugated.

artificially. The dispersion relation does not hold any information about the excitation process. Basic types of plasma waves can be distinguished by giving one of the two values to the following three properties:

1. Either electrons or ions oscillate in the wave. Thus we have an *electron* or *ion* wave. Usually, when we solve for an electron wave, the ions are approximated as an immovable background, because ions are at least three orders of magnitude heavier than electrons.

2. The wave can be *longitudinal* or *electrostatic* when $\boldsymbol{k} \parallel \boldsymbol{E}_1$ or *tranversal* or *electromagnetic*, when $\boldsymbol{k} \perp \boldsymbol{E}_1$, where $\boldsymbol{E}_1$ is the fluctuating component of the electric field.

3. The wave is *parallel* when $\boldsymbol{k} \parallel \boldsymbol{B}_0$ or *perpendicular* when $\boldsymbol{k} \perp \boldsymbol{B}_0$, where $\boldsymbol{B}_0$ is the background magnetic field.

The plasma waves that we will discuss in the following sections will always be distinguishable by the three above described properties.

The approach we will take is that we will prescribe the expected geometry of the wave (in terms of the type, direction of propagation and which of the components it contains) and search for the solution using the linearised fluid and Maxwell equations. If the solution in the form of a dispersion relation exists, the plasma supports this kind of wave. Our solutions will not give an answer to whether this particular wave truly gets excited under given conditions and what what amplitude it has (for this one would need to state the initial or boundary conditions), not even whether this particular mode is dominant or completely marginal at given conditions.

The linearisation approach explicitly states that we introduce some plasma background, which fulfills the fluid equations and remains unchanged by the wave propagation. It is the perturbations in density, velocity, pressure, electric or magnetic field, which oscillates and possibly propagates from place to place as a wave. We will see that the type of the propagating wave usually depends on the direction with respect to the background magnetic field.

Let's make a simplified example, which we will investigate in detail in Section 5.6.3. Let us consider a stationary and homogeneous plasma cloud in a background magnetic field. Somewhere inside this cloud a light source shortly flashes, a bulb if you want. In ordinary transparent neutral gas, such an impulse would cause the spherical light-wave to propagate from the initial point of the flash of the light bulb with the speed of light in all directions. In plasma, it is more difficult to describe, as the light waves are known to be variations of the electric and magnetic field. These small electric and magnetic fields, oscillating with the wave, interact with the charged particles by displacing them slightly, which causes the modification of the electric and magnetic field in the wave. The motion of particles and the electric and magnetic fields are bound via the set of (linearised) fluid equations, which in general depends on their direction to the background magnetic field. Hence the waveform gets modified depending on the angle of propagation with respect to the background magnetic field. We will see that along the magnetic field, two circularly polarised modes propagating with different speeds appear, accompanied by incoherent local oscillations of plasma. In the direction perpendicular to the magnetic field, two linearly polarised modes, again with different propagation speeds, appear. In a general direction of propagation, we will find a combination of all five modes. Electromagnetic waves do not cause the variations in plasma density, hence only the electric and magnetic fields are affected and inherently also the motions of individual particles. Another kind of impulse, such as a sudden local increase of plasma density, will excite another type of waves, but not the complex electromagnetic ones, only if such an impulse is not accompanied by some emission mechanism.

5.4 Electrostatic waves without a background magnetic field

5.4.1 Plasma oscillations

The very basic oscillatory movement registered in plasmas takes the form of *plasma oscillations*. Let's derive the dispersion relation first. We search for a solution under the following assumptions:

Plasma oscillations

- No background fields, i.e., $\boldsymbol{E}_0 = \boldsymbol{B}_0 = 0$, we do not consider induced perturbation to the magnetic field, i.e., $\boldsymbol{B}_1 = 0$. Thus from the set of Maxwell equations, only the Poisson equation is retained

- No thermal motion, plasma is cold
- Ions are fixed in space and have a uniform distribution
- Background density of ions and electrons is the same, i.e., $n_{e0} = n_{i0} = n_0$
- Background is stationary and homogeneous, $\boldsymbol{u}_{e0} = 0$

Thus we solve the following set of linearised equations:

$$mn_0 \frac{\partial \boldsymbol{u}_{e1}}{\partial t} = -en_0 \boldsymbol{E}_1, \tag{5.22}$$

$$\frac{\partial n_{e1}}{\partial t} + \boldsymbol{\nabla} \cdot [n_0 \boldsymbol{u}_{e1}] = 0, \tag{5.23}$$

$$\boldsymbol{E}_1 = -\nabla \phi_1, \tag{5.24}$$

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}_1 = -en_{e1}. \tag{5.25}$$

Now we express $\boldsymbol{E}_1$ from (5.24) and insert it into appropriate terms in (5.22) and (5.25), and apply a divergence operator to (5.22) to obtain:

$$\varepsilon_0 \triangle \phi_1 = n_{e1} e, \tag{5.26}$$

$$mn_0 \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{u}_{e1}}{\partial t} = en_0 \triangle \phi_1, \tag{5.27}$$

$$\frac{\partial n_{e1}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{e1} = 0. \tag{5.28}$$

Now we express term $\triangle \phi_1$ from (5.26) and insert it into (5.27). We do a simple algebraic adjustment to (5.28) by only leaving $\boldsymbol{\nabla} \cdot \boldsymbol{u}_{e1}$ on the left-hand side of equation:

$$mn_0 \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{u}_{e1}}{\partial t} = \frac{e^2 n_0}{\varepsilon_0} n_{e1}, \tag{5.29}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_{e1} = -\frac{1}{n_0} \frac{\partial n_{e1}}{\partial t}. \tag{5.30}$$

By combining these two equations we finally obtain

$$-m \frac{\partial^2 n_{e1}}{\partial t^2} = \frac{e^2 n_0}{\varepsilon_0} n_{e1}. \tag{5.31}$$

Now we take the Fourier transform of this equation, practically by applying rules (5.11) and (5.12):

$$\omega^2 n_{e1} = \frac{e^2 n_0}{\varepsilon_0 m} n_{e1}. \tag{5.32}$$

Thus we obtained the *dispersion relation for plasma oscillations*

$$\omega_\mathrm{p}^2 = \frac{n_0 e^2}{\varepsilon_0 m}. \tag{5.33}$$

This result requires a little discussion:

- The dispersion relation shows that oscillations exist with this characteristic *plasma frequency* ω_p, but there is no wave propagating since there is no wave vector $\boldsymbol{k}$ in the dispersion relation. The physical interpretation is the following: electrons may be displaced from their equilibrium positions determined by stationary background ions by some random fluctuation. The induced electric field acts as a restoring force to return them back to their original positions. As they move back, they have some inertia and overshoot their equilibrium positions. They are displaced again and the electric field again acts as a restoring force. This random movement is periodic. The motion of individual electrons is independent, their mutual phase shifts are random. In fact, the idea of individual electrons is very simplifying – we should rather speak about metaparticles, "bunch of electrons".

- The phase speed of this motion, $v_\varphi = \omega/k$, is not defined. This is not surprising in the context of what is said above. Since the mutual phase shifts of individual electrons are random, the formal phase speed takes *any* value.

- The group speed, $v_\mathrm{g} = \mathrm{d}\omega/\mathrm{d}k = 0$. No signal propagates.

- The plasma frequency is essentially only a function of the background density.

- Despite the assumption that both the ions and electrons are cold, this condition is not realistic. It is enough to assume that the temperature of electrons is larger than the temperature of ions and we further assume that electrons are isothermal (T_e is uniform). Debye length relates to plasma frequency by $\lambda_\mathrm{D}^2 = \frac{\varepsilon_0 K_\mathrm{B} T_\mathrm{e}}{n_0 e^2} = \frac{K_\mathrm{B} T_\mathrm{e}}{m}\frac{1}{\omega_\mathrm{p}^2} \sim \frac{1}{2} v_\mathrm{T}^2 \frac{1}{\omega_\mathrm{p}^2}$, where we approximated the thermal speed $v_\mathrm{T} \sim \sqrt{\frac{2K_\mathrm{B}T_\mathrm{e}}{m}}$. Obviously, the fluctuations induced by plasma oscillations are shielded on length scales comparable to the Debye length, thus, plasma oscillations do not induce any macroscopic charge.

5.4.2 Electron plasma wave

In the previous section, we showed that plasma supports the oscillatory motion, however, we did not derive the dispersion relation of the wave. We need to keep the additional physical effects. The thermal motion may cause the oscillations to propagate.

Longitudinal electron wave

- No background fields, i.e., $\boldsymbol{E}_0 = \boldsymbol{B}_0 = 0$, we do not consider induced perturbation to the magnetic field, i.e., $\boldsymbol{B}_1 = 0$. Thus from the set of Maxwell equations, only the Poisson equation is retained
- Ions are cold, electrons are subjects to thermal motion
- Ions are fixed in space and have a uniform distribution
- Background density of ions and electrons is the same, i.e., $n_{\mathrm{e}0} = n_{\mathrm{i}0} = n_0$
- Background is stationary and homogeneous, $\boldsymbol{u}_{\mathrm{e}0} = 0$

To include the effect of the thermal motion, we need to keep the pressure term in the equation of motion. The gradient of pressure is coupled with the thermal motion through the equation of state

$$\nabla p_{\mathrm{e}} = \nabla(3K_{\mathrm{B}}T_{\mathrm{e}}n_{\mathrm{e}}) = 3K_{\mathrm{B}}T_{\mathrm{e}}\nabla(n_0 + n_{\mathrm{e}1}), \tag{5.34}$$

where number 3 indicates that there is only one degree of freedom, thus, we (silently) assume the propagation is only in one direction. The modified system of linearised equations to be solved then reads:

$$mn_0\frac{\partial \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = -en_0\boldsymbol{E}_1 - 3K_{\mathrm{B}}T_{\mathrm{e}}\nabla n_{\mathrm{e}1}, \tag{5.35}$$

$$\frac{\partial n_{\mathrm{e}1}}{\partial t} + n_0\boldsymbol{\nabla}\cdot\boldsymbol{u}_{\mathrm{e}1} = 0, \tag{5.36}$$

$$\boldsymbol{E}_1 = -\nabla\phi_1, \tag{5.37}$$

$$\varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}_1 = -en_{\mathrm{e}1}. \tag{5.38}$$

The sequence of algebraic steps performed is identical to those in Section 5.4.1, thus we only state the final step here

$$-m\frac{\partial^2 n_{\mathrm{e}1}}{\partial t^2} = \frac{e^2 n_0}{\varepsilon_0}n_{\mathrm{e}1} - 3K_{\mathrm{B}}T_{\mathrm{e}}\triangle n_{\mathrm{e}1}. \tag{5.39}$$

By taking the Fourier transform we obtain

$$\omega^2 n_{\mathrm{e}1} - \omega_{\mathrm{p}}^2 n_{\mathrm{e}1} - 3k^2\frac{K_{\mathrm{B}}T_{\mathrm{e}}}{m}n_{\mathrm{e}1} = 0 \tag{5.40}$$

and finally a dispersion relation for longitudinal electron waves in plasmas

$$\omega^2 = \omega_{\mathrm{p}}^2 + \frac{3}{2}k^2 v_{\mathrm{T}}^2, \tag{5.41}$$

where $v_\mathrm{T} = \sqrt{\frac{2K_\mathrm{B}T_\mathrm{e}}{m}}$ is the thermal speed of electrons.

The dispersion relation seemingly only slightly differs from the dispersion relation for electron oscillations, however, it implies qualitatively different properties:

- This relation describes a propagating wave (it contains wave vector $\boldsymbol{k}$). It propagates only for frequencies larger than the plasma frequency, for smaller frequencies the resulting wave vector takes imaginary values, thus the spatial part causes the exponential in (5.10) to decay. Excited waves with frequencies smaller than the plasma frequency are rapidly attenuated. The wave only propagates in plasmas, the dispersion relation degenerates to $\omega = 0$ in a vacuum.

- The phase speed is given by $v_\varphi = \frac{\sqrt{\omega_\mathrm{p}^2 + 3/2k^2v_\mathrm{T}^2}}{k}$. It depends on the wavenumber, thus the waves are dispersive (different modes travel with a different phase speed).

- The group speed $v_\mathrm{g} = \frac{3}{2}\frac{v_\mathrm{T}^2 k}{\omega} = \frac{3}{2}\frac{v_\mathrm{T}^2 k}{\sqrt{\omega_\mathrm{p}^2 + 3/2k^2v_\mathrm{T}^2}}$ is not zero, thus the waves are able to carry information. Asymptotically for large k, $\lim\limits_{k\to\infty} v_\mathrm{g} = \frac{3}{2}\lim\limits_{k\to\infty}\frac{v_\mathrm{T}^2 k}{\sqrt{\omega_\mathrm{p}^2 + 3/2k^2v_\mathrm{T}^2}} \sim \frac{3}{2}\frac{v_\mathrm{T}^2 k}{\sqrt{3/2k^2v_\mathrm{T}^2}} = \sqrt{\frac{3}{2}}v_\mathrm{T}$, the waves essentially propagate with the thermal speed.

We solved the dispersion relation for the perturbation in density of electrons n_e1. The definition of the Fourier transform (5.10) allows us to introduce a phase shift, so that the variable fulfills the system of equations. Let's assume that the full solution for n_e1 has the form of

$$n_\mathrm{e1} = \overline{n_\mathrm{e1}} \cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right) \tag{5.42}$$

and we allow for a phase shift δ_E for $\boldsymbol{E}_1$:

$$\boldsymbol{E}_1 = \overline{\boldsymbol{E}_1} \cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta_E\right). \tag{5.43}$$

Both variables $\boldsymbol{E}_1$ and n_e1 are coupled via the Poisson equation $\varepsilon_0 \boldsymbol{\nabla}\cdot\boldsymbol{E}_1 = -en_\mathrm{e1}$. Thus

$$-\varepsilon_0 \boldsymbol{k}\cdot\overline{\boldsymbol{E}_1} \sin\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta_E\right) = -e\overline{n_\mathrm{e1}} \cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right). \tag{5.44}$$

By expressing the left-hand side we obtain

$$\sin\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right)\cos\delta_E + \cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right)\sin\delta_E = \frac{e}{\varepsilon_0}\frac{\overline{n_\mathrm{e1}}}{\overline{\boldsymbol{E}_1}\cdot\boldsymbol{k}} \cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right). \tag{5.45}$$

This equation has a solution when we take for example $\delta_E = \frac{\pi}{2}$.

The perturbation of electron density and the electric field induced by the wave are shifted in phase by a quarter of the period. How about the direction of the vector variables?

The Fourier image of the Lorentz force is

$$-\mathrm{i}\omega m \boldsymbol{u}_\mathrm{e1} = e(\boldsymbol{E}_1 + \boldsymbol{u}_\mathrm{e1}\times\boldsymbol{B}_0). \tag{5.46}$$

Since in our case $\boldsymbol{B}_0 = 0$, the relation simplifies to

$$\boldsymbol{u}_{\mathrm{e}1} = -\frac{e}{\omega}\boldsymbol{E}_1, \tag{5.47}$$

thus $\boldsymbol{u}_{\mathrm{e}1} \parallel \boldsymbol{E}_1$. Similarly Faraday's law written in the Fourier space is

$$\boldsymbol{k} \times \boldsymbol{E}_1 = \omega \boldsymbol{B}_1, \tag{5.48}$$

but $\boldsymbol{B}_1 = 0$, hence $\boldsymbol{k} \parallel \boldsymbol{E}_1$. The waves are longitudinal.

This result may also simply be obtained by application of the curl operator to the Poisson equation $\boldsymbol{E}_1 = -\nabla\phi_1$ and using the vector identity $\boldsymbol{\nabla} \times \nabla\phi_1 \equiv 0$.

5.4.3 Ion waves

When it comes to ion waves, the situation becomes more complicated, since we can't consider ions an immovable background anymore. In section 5.2 we derived the dispersion relation for the acoustic waves in a non-magnetised neutral fluid. Will the solution change when the fluid is composed of charged particles? We search for the wave, assuming:

Ion acoustic wave

- No background fields, i.e., $\boldsymbol{E}_0 = \boldsymbol{B}_0 = 0$, we do not consider an induced perturbation to the magnetic field, i.e., $\boldsymbol{B}_1 = 0$. Thus from the set of Maxwell equations only the Poisson equation is retained
- Both ions and electrons are subjects to thermal motion, their temperatures may be different
- Background density of ions and electrons is the same, i.e., $n_{\mathrm{e}0} = n_{\mathrm{i}0} = n_0$
- Background is stationary and homogeneous, $\boldsymbol{u}_{\mathrm{e}0} = \boldsymbol{u}_{\mathrm{i}0} = 0$

Then we solve a system of linearised equations

$$mn_0\frac{\partial \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = -en_0\boldsymbol{E}_1 - 3K_{\mathrm{B}}T_{\mathrm{e}}\nabla n_{\mathrm{e}1}, \tag{5.49}$$

$$\frac{\partial n_{\mathrm{e}1}}{\partial t} + n_0\boldsymbol{\nabla}\cdot\boldsymbol{u}_{\mathrm{e}1} = 0, \tag{5.50}$$

$$Mn_0\frac{\partial \boldsymbol{u}_{\mathrm{i}1}}{\partial t} = en_0\boldsymbol{E}_1 - 3K_{\mathrm{B}}T_{\mathrm{i}}\nabla n_{\mathrm{i}1}, \tag{5.51}$$

$$\frac{\partial n_{\mathrm{i}1}}{\partial t} + n_0\boldsymbol{\nabla}\cdot\boldsymbol{u}_{\mathrm{i}1} = 0, \tag{5.52}$$

$$\boldsymbol{E}_1 = -\nabla\phi_1, \tag{5.53}$$

$$\varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}_1 = e(n_{\mathrm{i}1} - n_{\mathrm{e}1}). \tag{5.54}$$

Ions are much heavier than electrons, thus we may assume that electrons will simply follow the ions and organize themselves so that they shield possible charge concentrations. Therefore, the perturbation of the electron density $n_{\mathrm{e}1}$ may be estimated by the Boltzmann relation (4.92), thus

$$n_{\mathrm{e}1} = n_{\mathrm{e}} - n_{\mathrm{e}0} = n_0 \left(\exp \frac{e\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}}} - 1 \right) \sim n_0 \frac{e\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}}} \tag{5.55}$$

and we may consider (5.49) and (5.50) solved.

Now we take $\boldsymbol{E}_1$ from (5.53), insert it into (5.51) and use (5.55) in (5.54). Additionally, we apply a divergence operator to (5.51). We have

$$M n_0 \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}1}}{\partial t} = -e n_0 \triangle \phi_1 - 3 K_{\mathrm{B}} T_{\mathrm{i}} \triangle n_{\mathrm{i}1}, \tag{5.56}$$

$$\frac{\partial n_{\mathrm{i}1}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}1} = 0, \tag{5.57}$$

$$-\varepsilon_0 \triangle \phi_1 = e n_{\mathrm{i}1} - e n_0 \frac{e\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}}}. \tag{5.58}$$

Now we express $\boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}1}$ from (5.57), insert into (5.56) and take a Fourier transform of the two remaining equations. We have

$$\omega^2 M n_{\mathrm{i}1} = e n_0 k^2 \phi_1 + 3 K_{\mathrm{B}} T_{\mathrm{i}} k^2 n_{\mathrm{i}1}, \tag{5.59}$$

$$\varepsilon_0 k^2 \phi_1 = e n_{\mathrm{i}1} - e^2 n_0 \frac{\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}}}. \tag{5.60}$$

Finally, we take ϕ_1 from (5.60), insert into (5.59) and solve for $n_{\mathrm{i}1}$ to get

$$\omega^2 = \frac{e^2 n_0 k^2}{M \left(\varepsilon_0 k^2 + \frac{e^2 n_0}{K_{\mathrm{B}} T_{\mathrm{e}}} \right)} + \frac{3 K_{\mathrm{B}} T_{\mathrm{i}} k^2}{M}. \tag{5.61}$$

Let $\Omega_{\mathrm{p}} = \sqrt{\frac{e^2 n_0}{M \varepsilon_0}}$ be an *ion plasma frequency.* If we remind the reader Debye length $\lambda_{\mathrm{D}} = \sqrt{\frac{\varepsilon_0 K_{\mathrm{B}} T_{\mathrm{e}}}{n_0 e^2}}$ and ion thermal speed $v_{\mathrm{Ti}} = \sqrt{\frac{2 K_{\mathrm{B}} T_{\mathrm{i}}}{M}}$, we finally obtain the *dispersion relation of ion acoustic waves*

$$\omega^2 = \frac{\Omega_{\mathrm{p}}^2 \lambda_{\mathrm{D}}^2 k^2}{\lambda_{\mathrm{D}}^2 k^2 + 1} + \frac{3}{2} v_{\mathrm{Ti}}^2 k^2. \tag{5.62}$$

For further discussion, let's take one more approximation, the *plasma approximation.* In practice we focus on large-scale effects, thus we set $k\lambda_{\mathrm{D}} \ll 1$. That is essentially equivalent to setting $n_{\mathrm{i}} = n_{\mathrm{e}}$.[4] Under this approximation, the dispersion relation simplifies to:

$$\omega^2 = k^2 \left(\frac{K_{\mathrm{B}} T_{\mathrm{e}}}{M} + 3 \frac{K_{\mathrm{B}} T_{\mathrm{i}}}{M} \right). \tag{5.63}$$

[4]Note that by assuming the plasma approximation, we consider the bulk densities of ions and electrons to be equal. This does not imply either $n_{\mathrm{i}1} = n_{\mathrm{e}1}$ or $n_{\mathrm{i}0} = n_{\mathrm{e}0}$!

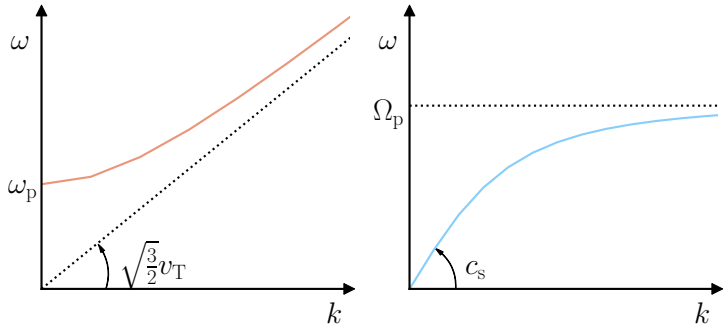

Figure 5.2: Asymptotic solutions for longitudinal electron and ion waves

- Phase speed $v_\varphi = \sqrt{\frac{K_\mathrm{B}T_\mathrm{e}+3K_\mathrm{B}T_\mathrm{i}}{M}} \equiv c_\mathrm{s}$ is the speed of sound in plasma. Under the plasma approximation, these waves are not dispersive. Dispersivity shows up as an effect of Debye shielding, which is mode-dependent.
- Group speed is equal to the phase speed, thus $v_\mathrm{g} = v_\varphi = c_\mathrm{s}$. Ion waves are waves with a constant speed of propagation.
- In laboratory plasmas, $T_\mathrm{i} \ll T_\mathrm{e}$, thus $c_\mathrm{s} \sim \sqrt{\frac{K_\mathrm{B}T_\mathrm{e}}{M}}$. The propagation speed is essentially set by the *mass of ions* and *temperature of electrons*.
- The waves are longitudinal. We leave the proof to the reader.
- The ion acoustic wave exists only if the thermal motion of at least electrons is possible. Electrons are pulled by ions to balance the charge concentrations, hence $v_\mathrm{g} = v_\varphi$. However, the shielding is not perfect, there exist microscopic accumulations of ions, which propagate as the wave through an $\boldsymbol{E}_1$ field.
- By assuming that $T_\mathrm{i} \ll T_\mathrm{e}$, the asymptotic dispersion relations read: $\omega = 0$ for $k \to 0$ and $\omega = \Omega_\mathrm{p}$ for $k \to \infty$. The asymptotic solution for small scales (large k) is fundamentally different from longitudinal *electron* waves, as also illustrated in Fig. 5.2.

5.5 Electrostatic waves in background magnetic field

5.5.1 Electron waves

Electron electrostatic perpendicular waves

- Background magnetic field, no background electric field, i.e., $\boldsymbol{B}_0 \neq 0$, $\boldsymbol{E}_0 = 0$

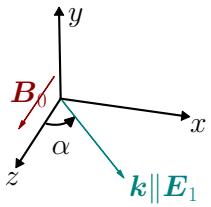

Figure 5.3: The coordinate system used in the derivation of the electrostatic electron waves in the presence of a background magnetic field.

- Ions stationary, thermal motion of electrons is neglected
- Background density of ions and electrons is the same, i.e., $n_{\mathrm{e}0} = n_{\mathrm{i}0} = n_0$
- Background is stationary and homogeneous, $\boldsymbol{u}_{\mathrm{e}0} = 0$
- $\boldsymbol{k} \parallel \boldsymbol{E}_1$

Let us write the Fourier images of fluid equations for electrons:

$$-\mathrm{i}\omega m \boldsymbol{u}_{\mathrm{e}1} = -e(\boldsymbol{E}_1 + \boldsymbol{u}_{\mathrm{e}1} \times \boldsymbol{B}_0), \tag{5.64}$$

$$-\mathrm{i}\omega n_{\mathrm{e}1} + \mathrm{i} n_0 \boldsymbol{k} \cdot \boldsymbol{u}_{\mathrm{e}1} = 0, \tag{5.65}$$

$$\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{E}_1 = -\frac{e}{\varepsilon_0} n_{\mathrm{e}1}, \tag{5.66}$$

We express $\boldsymbol{E}_1$ from (5.64) and combine together (5.65) and (5.66) through $n_{\mathrm{e}1}$.

$$\boldsymbol{E}_1 = \frac{\mathrm{i}\omega m}{e} \boldsymbol{u}_{\mathrm{e}1} - \boldsymbol{u}_{\mathrm{e}1} \times \boldsymbol{B}_0, \tag{5.67}$$

$$\boldsymbol{k} \cdot \boldsymbol{E}_1 = \frac{\mathrm{i}e n_0}{\varepsilon_0 \omega} \boldsymbol{k} \cdot \boldsymbol{u}_{\mathrm{e}1}. \tag{5.68}$$

Since we are considering longitudinal waves, $\boldsymbol{k} \parallel \boldsymbol{E}_1$, hence $\boldsymbol{k} \cdot \boldsymbol{E}_1 = kE_1$. Thus equation (5.68) allows us to express the amplitude $E_1 = |\boldsymbol{E}_1|$ of the electric field:

$$E_1 = \frac{\mathrm{i}e n_0}{\varepsilon_0 \omega k} \boldsymbol{k} \cdot \boldsymbol{u}_{\mathrm{e}1}. \tag{5.69}$$

Let us consider the Cartesian coordinate system (Fig. 5.3), so that $\boldsymbol{e}_z \parallel \boldsymbol{B}_0$ and $\boldsymbol{E}_1$ lies in the $x - z$ plane. Hence $\boldsymbol{B}_0 = (0, 0, B_0)$ and $\boldsymbol{E}_1 = (E_1 \sin\alpha, 0, E_1 \cos\alpha)$, and similarly $\boldsymbol{k} = (k \sin\alpha, 0, k \cos\alpha)$, where α is the angle between $\boldsymbol{B}_0$ and $\boldsymbol{k}$. We may then write (5.67) in components

$$\begin{pmatrix} E_1 \sin\alpha \\ 0 \\ E_1 \cos\alpha \end{pmatrix} = \frac{\mathrm{i}e n_0}{\varepsilon_0 \omega} \left(k u_{\mathrm{e}1,x} \sin\alpha + k u_{\mathrm{e}1,z} \cos\alpha\right) \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix} = \tag{5.70}$$

$$= \frac{\mathrm{i}\omega m}{e} \begin{pmatrix} u_{\mathrm{e}1,x} \\ u_{\mathrm{e}1,y} \\ u_{\mathrm{e}1,z} \end{pmatrix} - B_0 \begin{pmatrix} u_{\mathrm{e}1,y} \\ -u_{\mathrm{e}1,x} \\ 0 \end{pmatrix}. \tag{5.71}$$

Then using (5.69), multiplying the whole equation by $\frac{1}{\mathrm{i}}\frac{e}{\omega m}$ and using $\omega_\mathrm{p}^2 = \frac{e^2 n_0}{\varepsilon_0 m}$ and $\omega_\mathrm{c} = \frac{eB}{m}$, and reorganising everything to the left-hand side of the equation, we have

$$\begin{pmatrix} u_{\mathrm{e}1,x} \\ u_{\mathrm{e}1,y} \\ u_{\mathrm{e}1,z} \end{pmatrix} - \frac{\omega_\mathrm{p}^2}{\omega^2}\left(u_{\mathrm{e}1,x}\sin\alpha + u_{\mathrm{e}1,z}\cos\alpha\right)\begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix} + \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}\begin{pmatrix} u_{\mathrm{e}1,y} \\ -u_{\mathrm{e}1,x} \\ 0 \end{pmatrix} = 0. \quad (5.72)$$

Such an equation may be written as

$$M_u \cdot \boldsymbol{u}_{\mathrm{e}1} = 0, \quad (5.73)$$

with matrix M_u being

$$M_u = \begin{bmatrix} 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\sin^2\alpha & \mathrm{i}\frac{\omega_\mathrm{c}}{\omega} & -\frac{\omega_\mathrm{p}^2}{\omega^2}\sin\alpha\cos\alpha \\ -\mathrm{i}\frac{\omega_\mathrm{c}}{\omega} & 1 & 0 \\ -\frac{\omega_\mathrm{p}^2}{\omega^2}\sin\alpha\cos\alpha & 0 & 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\cos^2\alpha \end{bmatrix} \quad (5.74)$$

A nontrivial solution of equation (5.73) exists only if $\det M_u = 0$. Let's evaluate $\det M_u$ using the Laplace expansion using the third row:

$$\det M_u = -\frac{\omega_\mathrm{p}^4}{\omega^4}\sin^2\alpha\cos^2\alpha + \left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\cos^2\alpha\right)\left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\sin^2\alpha - \frac{\omega_\mathrm{c}^2}{\omega^2}\right) = 0\,, \quad (5.75)$$

which is the dispersion relation for electrostatic waves in the presence of a background magnetic field. Note that according to this dispersion relation no wave propagates, only oscillations appear in plasma.

Let us discuss two extreme cases. For $\alpha = 0$, hence for oscillatory motions along the magnetic field, we have

$$\left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\right)\left(1 - \frac{\omega_\mathrm{c}^2}{\omega^2}\right) = 0, \quad (5.76)$$

which has two physical solutions

$$\omega^2 = \omega_\mathrm{p}^2, \quad \text{and} \quad \omega^2 = \omega_\mathrm{c}^2. \quad (5.77)$$

Therefore, in the direction of the magnetic field, we have ordinary plasma oscillations and the Larmor rotation.

For $\alpha = \pi/2$, i.e., for oscillatory motions perpendicular to the background magnetic field we have

$$1 - \frac{\omega_\mathrm{p}^2}{\omega^2} - \frac{\omega_\mathrm{c}^2}{\omega^2} = 0, \quad (5.78)$$

which may be written as

$$\omega^2 = \omega_\mathrm{p}^2 + \omega_\mathrm{c}^2 \equiv \omega_\mathrm{h}^2. \tag{5.79}$$

The symbol ω_h indicates the upper hybrid frequency, the only possible frequency for electron oscillations perpendicular to $\boldsymbol{B}_0$.

5.5.2 Ion waves

Ion electrostatic perpendicular waves

- Background magnetic field, no background electric field, i.e., $\boldsymbol{B}_0 \neq 0$, $\boldsymbol{E}_0 = 0$
- Electrons are subject to thermal motion if they are allowed to, ions are cold
- Background density of ions and electrons is the same, i.e., $n_{\mathrm{e}0} = n_{\mathrm{i}0} = n_0$
- Background is stationary and homogeneous, $\boldsymbol{u}_{\mathrm{e}0} = \boldsymbol{u}_{\mathrm{i}0} = 0$
- $\boldsymbol{k} \parallel \boldsymbol{E}_1$

In the case of ion waves perpendicular to the background magnetic field, we need to strictly distinguish two cases: when the waves are *nearly* perpendicular, and when they are *strictly* perpendicular (Fig. 5.4). The difference is fundamental. In the case of the nearly perpendicular configuration, there is some, however small, component of the background magnetic field parallel to the wave vector. It is important since in this approximation the electrons follow ions and in such case, they are still able to shield the concentrations of charges via the mechanism of Debye shielding. They are free to move along the magnetic field lines and the shielding is effective. In the case of a strictly perpendicular magnetic field, there is no option for electrons to shield anymore, since they can't move around. Let's investigate the two cases separately.

$\boldsymbol{k}$ nearly perpendicular to $\boldsymbol{B}_0$

In this case we consider the effect of the Debye shielding and thus we can use the Boltzmann relation as a solution to the problem of electrons. We further solve the Euler equation for ions in the form of

$$-\mathrm{i}\omega M \boldsymbol{u}_{\mathrm{i}1} = -e\mathrm{i}\boldsymbol{k}\phi_1 + e\boldsymbol{u}_{\mathrm{i}1} \times \boldsymbol{B}_0 \tag{5.80}$$

in the Fourier space. In components we have (again, assuming the Cartesian coordinate system, where $\boldsymbol{k} = k\boldsymbol{e}_x$ and $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$)

$$-\mathrm{i}\omega M u_{\mathrm{i}1,x} = -e\mathrm{i}k\phi_1 + e u_{\mathrm{i}1,y} B_0, \tag{5.81}$$

$$-\mathrm{i}\omega M u_{\mathrm{i}1,y} = -e u_{\mathrm{i}1,x} B_0, \tag{5.82}$$

which gives the x-component of the perturbed velocity

$$u_{\mathrm{i}1,x} = \frac{ek\phi_1}{\omega M}\left(1 - \frac{\Omega_{\mathrm{c}}^2}{\omega^2}\right)^{-1}. \tag{5.83}$$

The continuity equation as usually gives

$$n_{\mathrm{i}1} = n_0 \frac{k}{\omega} u_{\mathrm{i}1,x}, \tag{5.84}$$

and the electron density perturbation is solved by the Boltzmann relation

$$n_{\mathrm{e}1} = n_0 \frac{e\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}}}. \tag{5.85}$$

By assuming the plasma approximation $n_{\mathrm{i}} = n_{\mathrm{e}}$, we have $n_{\mathrm{i}1} = n_{\mathrm{e}1}$ and hence

$$u_{\mathrm{i}1,x} = \frac{\omega e\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}} k}, \tag{5.86}$$

which may be compared to (5.83) to obtain the dispersion relation

$$\omega^2 - \Omega_{\mathrm{c}}^2 = \frac{K_{\mathrm{B}} T_{\mathrm{e}}}{M} k^2. \tag{5.87}$$

Let us remind the reader that the definition of the speed of sound in plasma reads

$$c_{\mathrm{s}}^2 = \left(\frac{K_{\mathrm{B}} T_{\mathrm{e}}}{M} + \frac{3K_{\mathrm{B}} T_{\mathrm{i}}}{M}\right) \tag{5.88}$$

which has a simpler form due to the assumption of $T_{\mathrm{i}} \to 0$ (or equivalently, $T_{\mathrm{i}} \ll T_{\mathrm{e}}$). Hence the dispersion relation (5.87) may be rewritten in the form

$$\omega^2 = \Omega_{\mathrm{c}}^2 + k^2 c_{\mathrm{s}}^2. \tag{5.89}$$

This case is possible only if the limiting angle $\theta = \frac{\pi}{2} - \alpha$, where α is the angle between $\boldsymbol{k}$ and $\boldsymbol{B}_0$, is larger than the ratio of ion and electron speed in the direction of $\boldsymbol{B}_0$, hence roughly for $\theta > (m/M)^{1/2}$. In this case, the electrons have enough kinetic energy (from the thermal motion) to chase the ions and also move around them to shield the originating concentrations of charges. For angles smaller electrons do not have enough thermal speed to shield the ion, so we need to use a different formalism.

$\boldsymbol{k} \perp \boldsymbol{B}_0$

For $\boldsymbol{k}$ exactly perpendicular to $\boldsymbol{B}_0$ there is no component of $\boldsymbol{k}$ parallel to $\boldsymbol{B}_0$, thus electrons are not allowed to move along the $\boldsymbol{B}_0$ and thus they cannot keep the neutrality. Hence we must solve the fluid equations for both electrons and ions. The system of

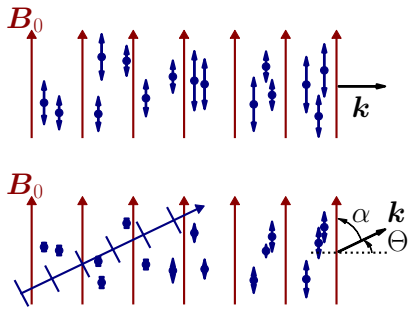

Figure 5.4: Comparison of the cases with $\boldsymbol{k}$ strictly (top) and nearly (bottom) perpendicular to $\boldsymbol{B}_0$. The two-headed arrows indicate the possible motions of the electrons, which are allowed to move only along the magnetic field lines. In the first case of an assumed strictly perpendicular propagation, they are not able to shield, because their allowed motion does not have a component in the direction of the assumed wave propagation.

equations is closed by using the plasma approximation. Hence we have simply used (5.83) for describing the motions of ions in the direction of the wave vector, and equivalently constructed the corresponding equation for electrons just by replacing $e \to -e$ and $M \to m$. Thus

$$u_{\mathrm{e}1,x} = -\frac{ek\phi_1}{\omega m}\left(1 - \frac{\omega_{\mathrm{c}}^2}{\omega^2}\right)^{-1}. \tag{5.90}$$

Similarly for the continuity equation,

$$n_{\mathrm{i}1} = n_0 \frac{k}{\omega} u_{\mathrm{i}1,x}, \tag{5.91}$$

$$n_{\mathrm{e}1} = n_0 \frac{k}{\omega} u_{\mathrm{e}1,x}. \tag{5.92}$$

By applying the plasma approximation approach we have $n_{\mathrm{e}1} = n_{\mathrm{i}1}$ and hence $u_{\mathrm{e}1,x} = u_{\mathrm{i}1,x}$. Thus

$$M\left(1 - \frac{\Omega_{\mathrm{c}}^2}{\omega^2}\right) = -m\left(1 - \frac{\omega_{\mathrm{c}}^2}{\omega^2}\right), \tag{5.93}$$

and further

$$\omega^2(M+m) = m\omega_{\mathrm{c}}^2 + M\Omega_{\mathrm{c}}^2 = e^2B_0^2\left(\frac{1}{m} + \frac{1}{M}\right) = e^2B_0^2\left(\frac{m+M}{mM}\right) \tag{5.94}$$

and thus

$$\omega^2 = \frac{eB_0}{m}\frac{eB_0}{M} = \omega_{\mathrm{c}}\Omega_{\mathrm{c}} \equiv \omega_{\mathrm{d}}^2. \tag{5.95}$$

In this case, the oscillations are allowed to have the only one possible frequency, the *lower hybrid frequency*.

5.6 Electromagnetic waves

When discussing electromagnetic waves, we automatically assume that the resulting solution will fulfill the assumption $\boldsymbol{k} \perp \boldsymbol{E}_1$. This assumption should always be verified at the end. In this text, we will keep the verification to the reader. Since they are electromagnetic, we cannot neglect the magnetic component of the wave, thus $\boldsymbol{B}_1 \neq 0$.

5.6.1 In vacuum: Light waves

Electromagnetic waves in a vacuum degenerate to light waves. Thus we need only the set of Maxwell equations.

Electromagnetic waves in vacuum

- No background magnetic or electric field, i.e., $\boldsymbol{B}_0 = \boldsymbol{E}_0 = 0$
- No plasma, thus $n_\mathrm{e} = n_\mathrm{i} = n_0 = 0$, $\boldsymbol{u}_\mathrm{e} = \boldsymbol{u}_\mathrm{i} = 0$
- $\boldsymbol{k} \perp \boldsymbol{E}_1$

$$\boldsymbol{\nabla} \times \boldsymbol{E}_1 = -\frac{\partial \boldsymbol{B}_1}{\partial t}, \tag{5.96}$$

$$\frac{1}{\mu_0} \boldsymbol{\nabla} \times \boldsymbol{B}_1 = \varepsilon_0 \frac{\partial \boldsymbol{E}_1}{\partial t}, \tag{5.97}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}_1 = 0. \tag{5.98}$$

Now we take the time derivative of (5.96) and apply the curl operator to (5.97). We have

$$\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{E}_1}{\partial t} = -\frac{\partial^2 \boldsymbol{B}_1}{\partial t^2}, \tag{5.99}$$

$$\frac{1}{\mu_0 \varepsilon_0} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{B}_1 = \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{E}_1}{\partial t}, \tag{5.100}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}_1 = 0. \tag{5.101}$$

Both (5.99) and (5.100) contain the term $\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{E}_1}{\partial t}$. Using this term we combine these two equations and take the Fourier transform of the system of the remaining two equations. We have

$$-c^2 \boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{B}_1) = \omega^2 \boldsymbol{B}_1, \tag{5.102}$$

$$\boldsymbol{k} \cdot \boldsymbol{B}_1 = 0, \tag{5.103}$$

where we used the definition of the *speed of light* $c = \sqrt{(\mu_0 \varepsilon_0)^{-1}}$. The double vector multiplication may be written using two terms. We have

$$-c^2 \left[(\boldsymbol{k} \cdot \boldsymbol{B}_1)\boldsymbol{k} - k^2 \boldsymbol{B}_1 \right] = \omega^2 \boldsymbol{B}_1, \tag{5.104}$$

$$\boldsymbol{k} \cdot \boldsymbol{B}_1 = 0. \tag{5.105}$$

Now we finally use (5.105) in (5.104) and have

$$c^2 k^2 \boldsymbol{B}_1 = \omega^2 \boldsymbol{B}_1, \tag{5.106}$$

which simplifies to

$$\omega^2 = c^2 k^2. \tag{5.107}$$

This is the dispersion relation for electromagnetic waves in a vacuum, describing the propagation of light. Obviously, the light waves are non-dispersive and both phase and group speeds are equal to the speed of light in a vacuum.

5.6.2 Electromagnetic waves in plasma

In plasma, the dispersion relation for electromagnetic waves gets modified due to the charged particles.

Electromagnetic waves in plasma

- No background magnetic or electric field, i.e., $\boldsymbol{B}_0 = \boldsymbol{E}_0 = 0$
- Background stationary and homogeneous
- We expect large frequencies ($\omega \gg \Omega_\mathrm{p}$), thus heavy ions remain stationary, only electrons respond
- Both ions and electrons are cold (no thermal motion)
- $\boldsymbol{k} \perp \boldsymbol{E}_1$

We proceed similarly to the case of electromagnetic waves in a vacuum, we only need to retain the electric current $\boldsymbol{j}_1$ in Ampére's law.

$$\boldsymbol{\nabla} \times \boldsymbol{E}_1 = -\frac{\partial \boldsymbol{B}_1}{\partial t}, \tag{5.108}$$

$$\frac{1}{\mu_0} \boldsymbol{\nabla} \times \boldsymbol{B}_1 = \varepsilon_0 \frac{\partial \boldsymbol{E}_1}{\partial t} + \boldsymbol{j}_1. \tag{5.109}$$

Similarly to the previous section, we take the time derivative of (5.109) and apply a curl operator to (5.108) and combine the two using the term $\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{B}_1}{\partial t}$, which is contained in both equations. We have

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E}_1 = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}_1}{\partial t^2} - \frac{1}{\varepsilon_0 c^2}\frac{\partial \boldsymbol{j}_1}{\partial t}. \tag{5.110}$$

Again, we take a Fourier transform of the above given equation and split the double vector multiplication into two terms:

$$-(\boldsymbol{k} \cdot \boldsymbol{E}_1)\boldsymbol{k} + k^2 \boldsymbol{E}_1 = \frac{\mathrm{i}\omega}{\varepsilon_0 c^2}\boldsymbol{j}_1 + \frac{\omega^2}{c^2}\boldsymbol{E}_1. \tag{5.111}$$

From the beginning, we search for the dispersion relation of electromagnetic waves, thus we limit ourselves to the case when $\boldsymbol{k} \perp \boldsymbol{E}_1$, hence $\boldsymbol{k} \cdot \boldsymbol{E}_1 = 0$. Thus (5.111) simplifies to

$$(\omega^2 - c^2 k^2)\boldsymbol{E}_1 = -\frac{\mathrm{i}\omega}{\varepsilon_0}\boldsymbol{j}_1. \tag{5.112}$$

In the solution, we expect frequencies much larger than the ion plasma frequency, thus the perturbed component of the electric current is only composed of electrons, ions are considered fixed in space in this approximation. Thus

$$\boldsymbol{j}_1 = -n_0 e \boldsymbol{u}_{\mathrm{e}1}. \tag{5.113}$$

The fluctuating component of the electron velocity $\boldsymbol{u}_{\mathrm{e}1}$ may be obtained from the Euler equation for electrons, which for cold plasma with no background magnetic field simplifies to

$$m\frac{\partial \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = -e\boldsymbol{E}_1. \tag{5.114}$$

Hence, the current density $\boldsymbol{j}_1$ is

$$\boldsymbol{j}_1 = -\frac{n_0 e^2}{\mathrm{i}\omega m}\boldsymbol{E}_1, \tag{5.115}$$

where we expressed the Euler equation for electrons in the Fourier space. Now we continue to modify (5.112):

$$(\omega^2 - c^2 k^2)\boldsymbol{E}_1 = \frac{\mathrm{i}\omega}{\varepsilon_0}\frac{n_0 e^2}{\mathrm{i}\omega m}\boldsymbol{E}_1 = \frac{n_0 e^2}{\varepsilon_0 m}\boldsymbol{E}_1 = \omega_{\mathrm{p}}^2 \boldsymbol{E}_1. \tag{5.116}$$

Hence the *dispersion relation for electromagnetic waves in plasma* reads

$$\omega^2 = \omega_{\mathrm{p}}^2 + c^2 k^2. \tag{5.117}$$

Let's make a few comments:

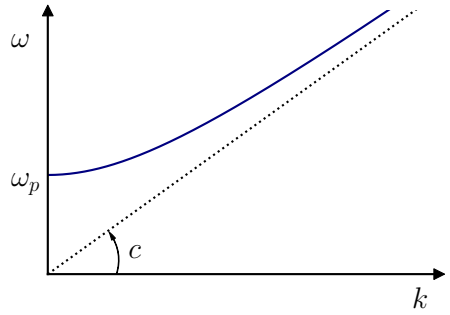

Figure 5.5: Dispersion diagram of electromagnetic waves in plasma.

- The dispersion diagram is displayed in Fig. 5.5.
- Phase velocity: $v_\varphi^2 = \left(\frac{\omega}{k}\right)^2 = c^2 + \frac{\omega_\mathrm{p}^2}{k^2} > c^2$. The phase velocity of these waves is larger than the speed of light. However, since the phase velocity does not carry any signal, the principles of the special theory of relativity are not violated.
- The dispersion relation describes only weakly dispersive waves for large k or in the high-frequency regime.
- Group velocity: $v_\mathrm{g} = \mathrm{d}\omega/\mathrm{d}k = \frac{2c^2k}{2\sqrt{\omega_\mathrm{p}^2+c^2k^2}} = \frac{c^2}{v_\varphi} < c$ for $v_\varphi > c$. The group speed is smaller than the speed of light.
- The waves propagate only for large frequencies, when $\omega > \omega_\mathrm{p}$. The electron plasma frequency ω_p serves as a *cut-off frequency*. For frequencies smaller than ω_p the waves do not propagate (the wave vector $\boldsymbol{k}$ is imaginary). Then we have $ck = (\omega^2 - \omega_\mathrm{p}^2)^{1/2} = \mathrm{i}\sqrt{\omega_\mathrm{p}^2 - \omega^2}$. Taking the spatial part of the Fourier transform (5.10), we find that physical variables vary as $\exp \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} \sim \exp \mathrm{i}kx = \exp\left[-x\frac{\sqrt{\omega_\mathrm{p}^2-\omega^2}}{c}\right] = \exp\left[-\frac{x}{\delta}\right]$, where $\delta = \frac{c}{\sqrt{\omega_\mathrm{p}^2-\omega^2}}$ is the attenuation length.

The electromagnetic waves propagating through plasma may serve as a primitive way to estimate the plasma density. The process is illustrated in Fig. 5.6. From measuring the spectrum we find the cut-off frequency, which only depends on the plasma density n_0. This is used in astrophysics for example in estimating the plasma density of the solar atmosphere. Imagine that we have an ionised plasma cloud hurled during a solar flare by magnetic reconnection, which rises in the atmosphere, and acts as a source for different kinds of plasma waves, including the electromagnetic ones, usually in the band of radio waves. As the cloud rises, the density of ambient plasma decreases, therefore the cut-off frequency also decreases and thus waves at lower frequencies are allowed to escape the region and propagate toward the observer. By measuring the cut-off frequency as a function of time and estimating the rising speed of the plasma blob, we perform a density scan of the solar atmosphere. A spacecraft returning to Earth from

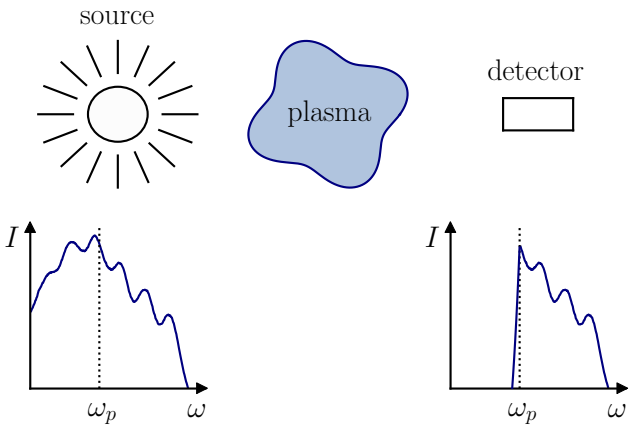

Figure 5.6: A scheme of the influence of a cut-off frequency on the measured spectrum.

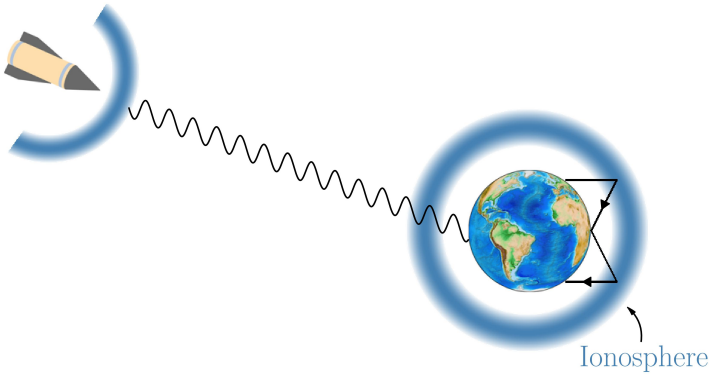

Figure 5.7: The radio trasmission from the spacecraft returning to Earth is blocked by surrounding plasma, while the spacecraft undergoes the atmospheric breaking maneuvre. The breaking heats up and ionises the air in the close vicinity, thereby introducing a plasma-frequency cut-off to the radio communication.

space is unable to communicate directly with control for about three minutes during the atmospheric breaking maneuvre (Fig. 5.7)

(Add a story of measuring the density of the interstellar matter from Voyager 1, must confirm what kind of waves were involved and perform the calculation)

5.6.3 Complex of electromagnetic waves in presence of $\boldsymbol{B}_0$

Electromagnetic waves in plasmas interact mostly with electrons, heavy ions can't adjust to the large frequencies of the electromagnetic waves. Hence we will solve the system of equations

$$-\mathrm{i}\omega m \boldsymbol{u}_{\mathrm{e}1} = -e(\boldsymbol{E}_1 + \boldsymbol{u}_{\mathrm{e}1} \times \boldsymbol{B}_0), \tag{5.118}$$

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{E}_1 = \mathrm{i}\omega \boldsymbol{B}_1, \tag{5.119}$$

$$\frac{\mathrm{i}}{\mu_0}\boldsymbol{k} \times \boldsymbol{B}_1 = -\varepsilon_0 \mathrm{i}\omega \boldsymbol{E}_1 - e n_0 \boldsymbol{u}_{\mathrm{e}1}. \tag{5.120}$$

The equations are already written in the form of Fourier images. Note that we did not consider the continuity equation, which makes the system over-determined. This is

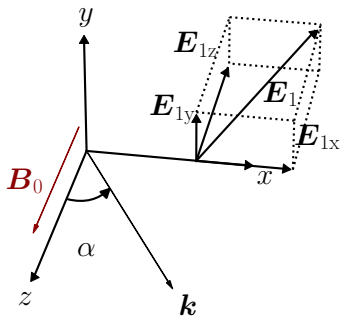

Figure 5.8: Coordinate system: electromagnetic waves in the presence of $\boldsymbol{B}_0$.

because the density perturbation n_{e1} only appears in the continuity equation. Now we express $\boldsymbol{u}_{\mathrm{e1}}$ from (5.118) and combine (5.119) and (5.120) via $\boldsymbol{B}_1$, where in the combined equation we solely keep $\boldsymbol{u}_{\mathrm{e1}}$ on the left-hand side:

$$\boldsymbol{u}_{\mathrm{e1}} = -\mathrm{i}\frac{e}{m\omega}\boldsymbol{E}_1 - \mathrm{i}\frac{eB_0}{m\omega}\boldsymbol{u}_{\mathrm{e1}} \times \boldsymbol{b}_0, \tag{5.121}$$

$$\boldsymbol{u}_{\mathrm{e1}} = -\mathrm{i}\frac{\varepsilon_0\omega}{en_0}\boldsymbol{E}_1 - \mathrm{i}\frac{c^2\varepsilon_0\omega}{\omega^2 en_0}\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_1), \tag{5.122}$$

where we used the definition of the speed of light $c^2 = (\varepsilon_0\mu_0)^{-1}$ and defined the unit vector $\boldsymbol{b}_0$ in the direction of the magnetic field, so that $\boldsymbol{B}_0 = B_0\boldsymbol{b}_0$. Now we insert (5.122) into (5.121). One needs to be careful here, since we find $\boldsymbol{u}_{\mathrm{e1}}$ in (5.121) on both sides of the equation. We further use $\omega_{\mathrm{c}} = eB_0/m$ to obtain

$$\begin{aligned} -\mathrm{i}\frac{\varepsilon_0\omega}{en_0}\boldsymbol{E}_1 - \mathrm{i}\frac{c^2\varepsilon_0}{\omega en_0}\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_1) = \\ - \mathrm{i}\frac{e}{m\omega}\boldsymbol{E}_1 - \mathrm{i}\frac{\omega_{\mathrm{c}}}{\omega}\left[-\mathrm{i}\frac{\varepsilon_0\omega}{en_0}\boldsymbol{E}_1 - \mathrm{i}\frac{c^2\varepsilon_0}{\omega en_0}\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_1)\right] \times \boldsymbol{b}_0. \end{aligned} \tag{5.123}$$

Now we express the double-cross products and multiply the whole equation by $-\frac{en_0\omega}{\mathrm{i}\varepsilon_0}$. We have

$$\begin{aligned} \omega^2\boldsymbol{E}_1 + c^2(\boldsymbol{k}\cdot\boldsymbol{E}_1)\boldsymbol{k} - c^2k^2\boldsymbol{E}_1 = \\ = \frac{e^2n_0}{\varepsilon_0 m}\boldsymbol{E}_1 - \mathrm{i}\omega_{\mathrm{c}}\omega\boldsymbol{E}_1 \times \boldsymbol{b}_0 - \mathrm{i}\frac{\omega_{\mathrm{c}}}{\omega}c^2(\boldsymbol{k}\cdot\boldsymbol{E}_1)(\boldsymbol{k}\times\boldsymbol{b}_0) + \mathrm{i}\frac{\omega_{\mathrm{c}}}{\omega}c^2k^2(\boldsymbol{E}_1\times\boldsymbol{b}_0). \end{aligned} \tag{5.124}$$

By reordering and using $\omega_{\mathrm{p}}^2 = \frac{e^2n_0}{\varepsilon_0 m}$ we have

$$(\omega^2 - \omega_{\mathrm{p}}^2 - c^2k^2)\boldsymbol{E}_1 + \mathrm{i}\frac{\omega_{\mathrm{c}}}{\omega}(\omega^2 - c^2k^2)(\boldsymbol{E}_1\times\boldsymbol{b}_0) + c^2(\boldsymbol{k}\cdot\boldsymbol{E}_1)\boldsymbol{k} + \mathrm{i}\frac{\omega_{\mathrm{c}}}{\omega}c^2(\boldsymbol{k}\cdot\boldsymbol{E}_1)(\boldsymbol{k}\times\boldsymbol{b}_0) = 0. \tag{5.125}$$

To solve this equation, we introduce the Cartesian coordinate system (Fig. 5.8), so that $\boldsymbol{B}_0 \parallel \boldsymbol{e}_z$ and $\boldsymbol{k}$ lies in the x–z plane. Therefore $\boldsymbol{b}_0 = (0, 0, 1)$ and $\boldsymbol{k} = (k \sin\alpha, 0, k \cos\alpha)$, where α is the angle between $\boldsymbol{B}_0$ and $\boldsymbol{k}$. The electric intensity vector has all of its components $\boldsymbol{E}_1 = (E_{1x}, E_{1y}, E_{1z})$. In this particular representation (5.125) turns into

$$\begin{aligned}(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2) \begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \end{pmatrix} + \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2) \begin{pmatrix} E_{1y} \\ -E_{1x} \\ 0 \end{pmatrix} + \\ + c^2(kE_{1x}\sin\alpha + kE_{1z}\cos\alpha) \begin{pmatrix} k\sin\alpha \\ 0 \\ k\cos\alpha \end{pmatrix} + \\ + \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}c^2(E_{1x}k\sin\alpha + kE_{1z}\cos\alpha) \begin{pmatrix} 0 \\ -k\sin\alpha \\ 0 \end{pmatrix} = 0. \end{aligned} \tag{5.126}$$

This is a set of three equations for the three components of perturbation of electric intensity, which may be written in the matrix form as

$$M_{\boldsymbol{E}_1} \cdot \boldsymbol{E}_1 = 0 \tag{5.127}$$

with system matrix

$$M_{\boldsymbol{E}_1} = \begin{pmatrix} \omega^2 - \omega_\mathrm{p}^2 - c^2k^2\cos^2\alpha & \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2) & c^2k^2\sin\alpha\cos\alpha \\ -\mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2\cos^2\alpha) & \omega^2 - \omega_\mathrm{p}^2 - c^2k^2 & -\mathrm{i}\frac{\omega_\mathrm{c}}{\omega}c^2k^2\cos\alpha\sin\alpha \\ c^2k^2\sin\alpha\cos\alpha & 0 & \omega^2 - \omega_\mathrm{p}^2 - c^2k^2\sin^2\alpha \end{pmatrix}. \tag{5.128}$$

Equation (5.127) has a nontrivial solution when $\det M_{\boldsymbol{E}_1} = 0$. We expand $\det M_{\boldsymbol{E}_1}$ around the third row to have

$$\begin{aligned}\det M_{\boldsymbol{E}_1} = c^4k^4\sin^2\alpha\cos^2\alpha \left[\frac{\omega_\mathrm{c}^2}{\omega^2}(\omega^2 - c^2k^2) - (\omega^2 - \omega_\mathrm{p}^2 - c^2k^2)\right] + \\ + (\omega^2 - \omega_\mathrm{p}^2 - c^2k^2\sin^2\alpha)\Bigg[(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2\cos^2\alpha)(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2) - \\ - \frac{\omega_\mathrm{c}^2}{\omega^2}(\omega^2 - c^2k^2)(\omega^2 - c^2k^2\cos^2\alpha)\Bigg] = 0. \end{aligned} \tag{5.129}$$

This is the dispersion relation for a *complex of electromagnetic waves in plasmas*. We will now study two special cases.

Electromagnetic waves parallel to $\boldsymbol{B}_0$

Waves parallel to $\boldsymbol{B}_0$ have $\boldsymbol{k} \parallel \boldsymbol{B}_0$ and thus $\alpha = 0$. The dispersion relation (5.129) simplifies to

$$(\omega^2 - \omega_\mathrm{p}^2)\left[(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2)^2 - \frac{\omega_\mathrm{c}^2}{\omega^2}(\omega^2 - c^2k^2)^2\right] = 0. \tag{5.130}$$

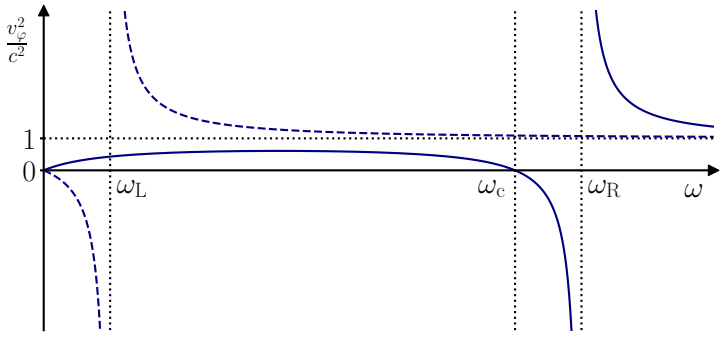

Figure 5.9: The dispersion diagram of the L- (dashed) and R- (solid) waves. The figure illustrates the cut-offs and resonances.

This equation has four roots, one of which is not physical. By setting the first parenthesis equal to zero, one obtains

$$\omega^2 = \omega_p^2, \tag{5.131}$$

therefore, along the magnetic field, the first mode of oscillatory motions is composed of ordinary plasma oscillations.

By setting the second parenthesis equal to zero we obtain two other roots:

$$\omega^2 - \omega_p^2 - c^2k^2 = \pm\frac{\omega_c}{\omega}(\omega^2 - c^2k^2) \tag{5.132}$$

which may be written as

$$\omega^2 - c^2k^2 = \frac{\omega_p^2}{1 - \frac{\omega_c}{\omega}} \tag{5.133}$$

and

$$\omega^2 - c^2k^2 = \frac{\omega_p^2}{1 + \frac{\omega_c}{\omega}}. \tag{5.134}$$

The wave described by (5.133) carries the name *R-wave*, while the other one described by (5.134) is named the *L-wave*. The origin of these names lies in their circular polarisation, which we will study in a moment.

- The dispersion diagrams in two forms are given in Figs. 5.9 and 5.10.
- Both waves are dispersive

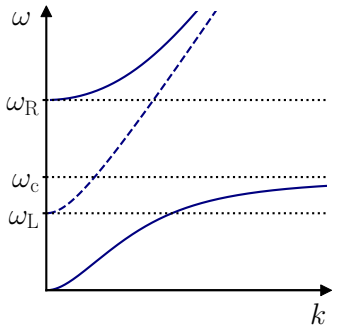

Figure 5.10: The dispersion ($k-\omega$) diagram of L- (dashed) and R- (solid) waves.

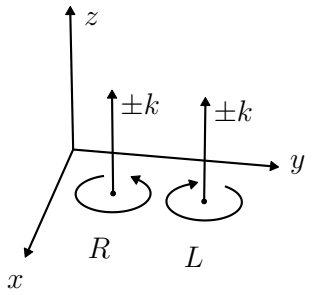

Figure 5.11: The coordinate system to demonstrate the polarisation and propagation of the R- and L-modes.

- We leave the derivation of the phase and group speed to the reader, however already from the dispersion relation we see that at the same frequency, the propagation speed of the R-wave is larger.

- Let's investigate the polarisation of these two waves. We start from (5.127) with $\alpha = 0$. We have

$$\begin{pmatrix} \omega^2 - \omega_\mathrm{p}^2 - c^2k^2 & \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2) & 0 \\ -\mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2) & \omega^2 - \omega_\mathrm{p}^2 - c^2k^2 & 0 \\ 0 & 0 & \omega^2 - \omega_\mathrm{p}^2 \end{pmatrix} \cdot \begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \end{pmatrix} = 0, \tag{5.135}$$

  from which we immediately see that $E_{1z} = 0$. The relation between the two other components is

$$(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2)E_{1x} + \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2)E_{1y} = 0. \tag{5.136}$$

  We further use the dispersion relation in the form $c^2k^2 = \omega^2 - \omega_\mathrm{p}^2\left(1 \mp \frac{\omega_\mathrm{c}}{\omega}\right)^{-1}$ to finally obtain

$$E_{1y} = \pm\mathrm{i}E_{1x}. \tag{5.137}$$

  Thus we have $E_x = -\mathrm{i}E_y$ for the R-wave (the upper sign is valid) and $E_x = \mathrm{i}E_y$ for the L-wave. We immediately see that the waves are circularly polarised (the amplitudes of both components are equal and both components are shifted in phase by a quarter of a circle). We explicitly express the spatio-temporal dependence of the electric intensity,

$$\boldsymbol{E}_1 = \overline{\boldsymbol{E}_1}\exp\left[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\right] = \overline{\boldsymbol{E}_1}\left[\cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right) + \mathrm{i}\sin\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right)\right]. \tag{5.138}$$

  If we introduce a phase shift δ_E between the components and define

$$E_x = \Re(\boldsymbol{E}_1) = \overline{\boldsymbol{E}_1}\cos\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t\right) \tag{5.139}$$

  and

$$E_y = \Im(\boldsymbol{E}_1) = \overline{\boldsymbol{E}_1}\sin\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta_E\right), \tag{5.140}$$

  we see that for the R-wave, $\delta_E = \pi/2$ and thus E_y preceeds E_x, the wave's polarisation is right-handed (see Fig. 5.11). Similarly, in the case of the L-wave, E_y is retarded by $\delta_E = -\pi/2$ and the wave's polarisation is left-handed. Note that the polarisation *does not* depend on the direction of propagation, only on the direction of the background magnetic field.

- The R-wave has a cut-off at $\omega_R = \frac{1}{2}\left[\omega_\mathrm{c} + (\omega_\mathrm{c}^2 + 4\omega_\mathrm{p}^2)^{1/2}\right]$ (hence the designation of this frequency) and a resonance at ω_c. The L-wave has a cut-off at $\omega_L = \frac{1}{2}\left[-\omega_\mathrm{c} + (\omega_\mathrm{c}^2 + 4\omega_\mathrm{p}^2)^{1/2}\right]$ and no resonance.

Application: Faraday rotation. The detection of R- and L-waves in the radio spectrum of astrophysical objects allows us to estimate either the distance of these

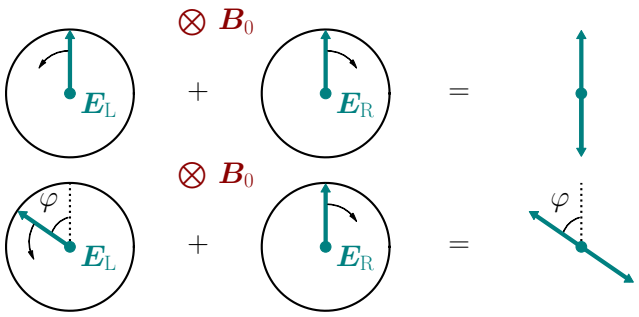

Figure 5.12: The superposition of L- and R- circularly polarised waves with the same amplitude yields a linearly polarised wave. The tilt of the polarisation plane is a consequence of the Faraday rotation effect.

objects or the plasma frequency of the interstellar environment. Assume that we have a plane wave composed of both R- and L-waves, which propagates along interstellar $\boldsymbol{B}_0$. As we pointed out, the R-wave propagates faster.

Let us write the vector of electric fields of both considered R- and L-waves as components with respect to some Cartesian coordinate system

$$\boldsymbol{E}_{\mathrm{R}} = E_0 \exp[\mathrm{i}(\boldsymbol{k}_{\mathrm{R}} \cdot \boldsymbol{r} - \omega t)] \left[\boldsymbol{e}_x + \mathrm{i}\boldsymbol{e}_y\right], \tag{5.141}$$

$$\boldsymbol{E}_{\mathrm{L}} = E_0 \exp[\mathrm{i}(\boldsymbol{k}_{\mathrm{L}} \cdot \boldsymbol{r} - \omega t)] \left[\boldsymbol{e}_x - \mathrm{i}\boldsymbol{e}_y\right]. \tag{5.142}$$

The components are written in agreement with (5.137) and $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are unit vectors in x and y directions.

The two waves superpose to form one detected wave (Fig. 5.12):

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{E}_{\mathrm{R}} + \boldsymbol{E}_{\mathrm{L}}) = \frac{1}{2} E_0 \mathrm{e}^{-\mathrm{i}\omega t} \left[\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\mathrm{R}}\cdot\boldsymbol{r}}(\boldsymbol{e}_x + \mathrm{i}\boldsymbol{e}_y) + \mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\mathrm{L}}\cdot\boldsymbol{r}}(\boldsymbol{e}_x - \mathrm{i}\boldsymbol{e}_y)\right]. \tag{5.143}$$

Should $\boldsymbol{k}_{\mathrm{R}} = \boldsymbol{k}_{\mathrm{L}} = \boldsymbol{k}$ then

$$\boldsymbol{E} = \frac{1}{2} E_0 \mathrm{e}^{-\mathrm{i}\omega t} \left[2\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{e}_x\right] = E_0 \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \boldsymbol{e}_x, \tag{5.144}$$

which is a linearly polarised wave in the x-direction.

In our case, however, $\boldsymbol{k}_{\mathrm{R}} \neq \boldsymbol{k}_{\mathrm{L}}$,

$$k_{\mathrm{R,L}} = \frac{\omega}{c} \sqrt{1 - \frac{\omega_{\mathrm{p}}^2/\omega^2}{1 \mp \omega_{\mathrm{c}}/\omega}}. \tag{5.145}$$

In the high-frequency regime, for $\omega \gg \omega_{\mathrm{c}}, \omega_{\mathrm{p}}$ by using the Taylor series $\sqrt{1-x} \sim 1 - \frac{x}{2}$ and $(1+x)^{-1} \sim 1 - x$ we obtain

$$k_{\mathrm{R,L}} \sim \frac{\omega}{c} \left(1 - \frac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega^2} \frac{1}{1 \mp \omega_{\mathrm{c}}/\omega}\right) \sim \frac{\omega}{c} \left[1 - \frac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\omega^2}\left(1 \pm \frac{\omega_{\mathrm{c}}}{\omega}\right)\right] \tag{5.146}$$

Hence when we use a decomposition of both wavenumbers as

$$k_{\mathrm{R,L}} = k \pm \Delta k, \tag{5.147}$$

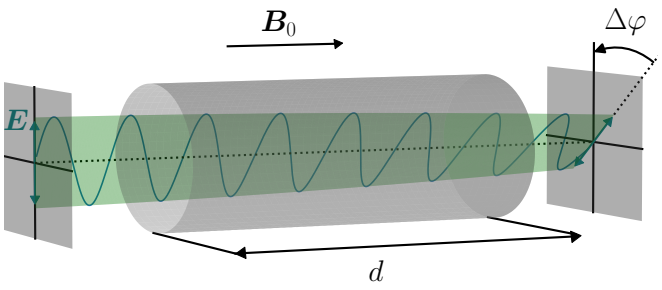

Figure 5.13: A sketch illustrating Faraday's rotation. The combination of the R- and L-modes creates a linearly polarised wave in the plane before entering the plasma cylinder. In the plasma, the L-mode lags behind the R-mode, which causes the rotation of the polarisation axis. In the plane after leaving the plasma cylinder, the linear polarisation depicts the phase shift $\Delta\varphi$.

we have

$$k = \frac{\omega}{c}\left[1 - \frac{1}{2}\frac{\omega_\mathrm{p}^2}{\omega^2}\right] \tag{5.148}$$

and

$$\Delta k = -\frac{1}{2}\frac{\omega_\mathrm{p}^2\omega_\mathrm{c}}{c\omega^2}. \tag{5.149}$$

By inserting (5.147) into (5.143) we have

$$\begin{aligned}\boldsymbol{E} &= \frac{1}{2}E_0\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\left[\mathrm{e}^{\mathrm{i}\Delta\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{e}_x + \mathrm{i}\boldsymbol{e}_y) + \mathrm{e}^{-\mathrm{i}\Delta\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{e}_x - \mathrm{i}\boldsymbol{e}_y)\right] \\ &= E_0\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\left[\cos(-\Delta\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{e}_x + \sin(-\Delta\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{e}_y\right].\end{aligned} \tag{5.150}$$

Obviously, the difference in the phase speeds of the R- and L-wave causes the rotation of the polarisation plane of the resulting linearly polarised wave. When adjusting the coordinate system so that $-\Delta\boldsymbol{k}\cdot\boldsymbol{r} = -\Delta kz$, the angle of the polarisation φ (it is equal to the argument of the goniometric functions) plane fulfills

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z} = -\Delta k. \tag{5.151}$$

To obtain the total rotation of the polarisation plane for waves travelling a distance d (see Fig. 5.13) we must integrate (5.151):

$$\begin{aligned}\varphi &= \varphi_0 + \int_0^d \frac{\mathrm{d}\varphi}{\mathrm{d}z}\mathrm{d}z = \varphi_0 + \int_0^d -\Delta k\mathrm{d}z = \varphi_0 + \frac{1}{2c\omega^2}\int_0^d \omega_\mathrm{p}^2(z)\omega_\mathrm{c}(z)\mathrm{d}z = \\ &= \varphi_0 + \frac{e^3}{2m^2\varepsilon_0 c\omega^2}\int_0^d n_0(z)B_0(z)\mathrm{d}z.\end{aligned} \tag{5.152}$$

When we approximate both the density and magnetic field by constants, we simply obtain

$$\varphi = \varphi_0 + \frac{e^3}{2m^2\varepsilon_0 c}\frac{1}{\omega^2} n_0 B_0 d. \tag{5.153}$$

By measuring the angle of the polarisation plane for various frequencies one may fit this relation and obtain the information about either the properties of the interstellar space or about the distance of the radiating object.

Electromagnetic waves perpendicular to $\boldsymbol{B}_0$

Let us go back to the general dispersion relation of the complex of electromagnetic waves in plasmas (5.129) and discuss the other special case when $\alpha = \pi/2$. The dispersion relation simplifies to

$$(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2)\left[(\omega^2 - \omega_\mathrm{p}^2)(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2) - \omega_\mathrm{c}^2(\omega^2 - c^2k^2)\right] = 0. \tag{5.154}$$

This equation once again has two solutions. By zeroing the first parenthesis we obtain

$$\omega^2 = \omega_\mathrm{p}^2 + c^2k^2, \tag{5.155}$$

which is the dispersion relation identical to electromagnetic waves in non-magnetised plasmas. Due to this correspondence, this wave is called *ordinary* or the *O-mode*. It is linearly polarized with $\boldsymbol{E}_1 \parallel \boldsymbol{B}_0$. It has a cut-off at the plasma frequency.

By zeroing the second parenthesis we have

$$(\omega^2 - \omega_\mathrm{p}^2)(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2) = \omega_\mathrm{c}^2(\omega^2 - c^2k^2), \tag{5.156}$$

which by expressing $\omega^2 - c^2k^2$ from both sides and rearranging becomes

$$\omega^2 - c^2k^2 = \omega_\mathrm{p}^2\frac{\omega^2 - \omega_\mathrm{p}^2}{\omega^2 - \omega_\mathrm{h}^2}, \tag{5.157}$$

where we used the definition of the upper hybrid frequency $\omega_\mathrm{h}^2 = \omega_\mathrm{p}^2 + \omega_\mathrm{c}^2$. Such a wave is termed the *extraordinary wave* or the *X-mode*. It is partly transversal and partly longitudinal, it propagates perpendicularly to $\boldsymbol{B}_0$. As the density is increased, the phase velocity rises from c until the cut-off at ω_R is reached. As the density is further increased, the wave is evanescent until the resonance at the upper hybrid frequency ω_h. Then it can propagate again until the second cut-off at ω_L. The cut-off frequencies are given by: $\omega_R = \frac{1}{2}\left[\omega_\mathrm{c} + (\omega_\mathrm{c}^2 + 4\omega_\mathrm{p}^2)^{1/2}\right]$ and $\omega_L = \frac{1}{2}\left[-\omega_\mathrm{c} + (\omega_\mathrm{c}^2 + 4\omega_\mathrm{p}^2)^{1/2}\right]$.

For the O- and X-modes the plasma acts as a birefringent environment. The dispersion relations in two forms are given in Figs. 5.14, 5.15, and 5.16.

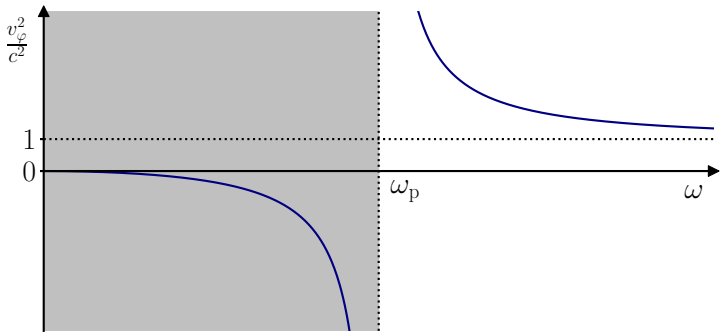

Figure 5.14: Dispersion relation of the O-wave. The shaded region indicates the region of attenuation (square of the phase velocity is negative).

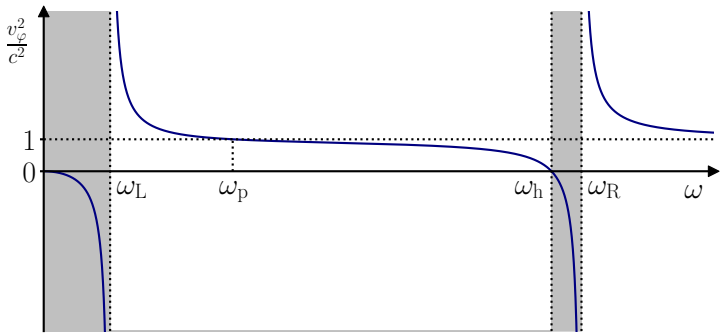

Figure 5.15: (X waves)

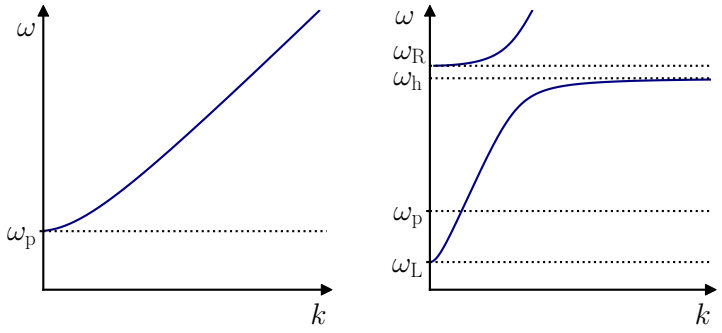

Figure 5.16: The dispersion $k - \omega$ diagram for the O-waves (left) and X-waves (right)

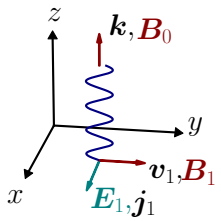

Figure 5.17: Geometry of an Alfvén wave propagating along $\boldsymbol{B}_0$.

5.7 MHD waves

By MHD waves (a.k.a. hydromagnetic waves) we understand low frequency ion oscillations in the presence of the magnetic field. A special case is the Alfvén wave.

5.7.1 Alfvén wave

Alfvén waves

- Background magnetic field, no background electric field, i.e., $\boldsymbol{B}_0 \neq 0$, $\boldsymbol{E}_0 = 0$
- Background stationary and homogeneous
- We solve equations of motion for both ions and electrons
- Both ions and electrons are cold (no thermal motion)

We introduce the Cartesian coordinate system $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$, so that:

- $\boldsymbol{B}_0 = B_0 \boldsymbol{e}_z$,
- $\boldsymbol{E}_1 \parallel \boldsymbol{j}_1 \parallel \boldsymbol{e}_x$, $\boldsymbol{E}_1 \perp \boldsymbol{B}_0$,
- $\boldsymbol{B}_1 \parallel \boldsymbol{u}_1 \parallel \boldsymbol{e}_y$,
- $\boldsymbol{k} \parallel \boldsymbol{B}_0$, $\boldsymbol{B}_1 \perp \boldsymbol{E}_1$.

This system and the wave setup is also indicated graphically in Fig. 5.17.

Again, we use the solution of the system of Maxwell equations (5.108) and (5.109) from the previous section by directly using (5.112):

$$(\omega^2 - c^2 k^2)\boldsymbol{E}_1 = -\frac{\mathrm{i}\omega}{\varepsilon_0}\boldsymbol{j}_1. \tag{5.158}$$

We do not assume that the ions create a stationary background, thus the current density $\boldsymbol{j}_1$ is proportional to the difference between the fluctuation speeds of ions and electrons. Hence

$$\boldsymbol{j}_1 = j_1 \boldsymbol{e}_x = n_0 e(u_{\mathrm{i}1,x} - u_{\mathrm{e}1,x})\boldsymbol{e}_x. \tag{5.159}$$

The expression for both velocity fluctuations must be obtained from the Euler equations. For ions we have

$$M\frac{\partial \boldsymbol{u}_{\mathrm{i}1}}{\partial t} = e(\boldsymbol{E}_1 + \boldsymbol{u}_{\mathrm{i}1} \times \boldsymbol{B}_0), \tag{5.160}$$

which in components and in the Fourier space gives

$$-\mathrm{i}\omega M u_{\mathrm{i}1,x} = eE_1 + eu_{\mathrm{i}1,y}B_0, \tag{5.161}$$

$$-\mathrm{i}\omega M u_{\mathrm{i}1,y} = -eu_{\mathrm{i}1,x}B_0. \tag{5.162}$$

By combining these two equations we have

$$-\mathrm{i}\omega M u_{\mathrm{i}1,x} = eE_1 + eB_0\frac{eB_0}{\mathrm{i}\omega M}u_{\mathrm{i}1,x} \tag{5.163}$$

and after some algebra we obtain

$$u_{\mathrm{i}1,x} = \frac{\mathrm{i}e}{\omega M}\left(1 - \frac{\Omega_{\mathrm{c}}^2}{\omega^2}\right)^{-1} E_1. \tag{5.164}$$

Similarly [e.g. by inserting (5.164) into (5.162)], we obtain for $u_{\mathrm{i}y}$:

$$u_{\mathrm{i}1,y} = \frac{e}{\omega M}\frac{\Omega_{\mathrm{c}}}{\omega}\left(1 - \frac{\Omega_{\mathrm{c}}^2}{\omega^2}\right)^{-1} E_1. \tag{5.165}$$

We solve the Euler equation for electrons by analogy (formally by application of the following substitutions: $M \to m$, $e \to -e$, and $\Omega_{\mathrm{c}} \to -\omega_{\mathrm{c}}$), to obtain:

$$u_{\mathrm{e}1,x} = -\frac{\mathrm{i}e}{\omega m}\left(1 - \frac{\omega_{\mathrm{c}}^2}{\omega^2}\right)^{-1} E_1, \tag{5.166}$$

$$u_{\mathrm{e}1,y} = -\frac{e}{\omega m}\frac{-\omega_{\mathrm{c}}}{\omega}\left(1 - \frac{\omega_{\mathrm{c}}^2}{\omega^2}\right)^{-1} E_1. \tag{5.167}$$

We expect low-frequency waves. Thus we are allowed to use the approximation $\omega_{\mathrm{c}} \gg \omega$. Under this approximation (5.166) and (5.167) becomes:

$$u_{\mathrm{e}1,x} = -\frac{\mathrm{i}e}{\omega m}\left(\frac{\omega^2 - \omega_{\mathrm{c}}^2}{\omega^2}\right)^{-1} E_1 \sim \frac{\mathrm{i}e}{\omega m}\frac{\omega^2}{\omega_{\mathrm{c}}^2}E_1 \to 0 \tag{5.168}$$

and

$$u_{\mathrm{e}1,y} = \frac{e}{\omega m}\frac{\omega_{\mathrm{c}}}{\omega}\left(\frac{\omega^2 - \omega_{\mathrm{c}}^2}{\omega^2}\right)^{-1} E_1 \sim -\frac{e}{\omega m}\frac{\omega_{\mathrm{c}}}{\omega}\frac{\omega^2}{\omega_{\mathrm{c}}^2}E_1 = -\frac{e}{\omega_{\mathrm{c}} m}E_1 = -\frac{E_1}{B_0}. \tag{5.169}$$

Now we use the estimates obtained for the fluctuating speeds (5.164) and (5.168) and use them in (5.159), which we use in (5.158).

$$(\omega^2 - c^2k^2)E_1 = -\frac{\mathrm{i}\omega}{\varepsilon_0} n_0 e \left[\frac{\mathrm{i}e}{\omega M} \left(1 - \frac{\Omega_\mathrm{c}^2}{\omega^2} \right)^{-1} E_1 - 0 \right] =$$
$$= \frac{n_0 e^2}{\varepsilon_0 M} \left(1 - \frac{\Omega_\mathrm{c}^2}{\omega^2} \right)^{-1} E_1 = \Omega_\mathrm{p}^2 \left(1 - \frac{\Omega_\mathrm{c}^2}{\omega^2} \right)^{-1} E_1. \tag{5.170}$$

Hence the dispersion relation reads

$$\omega^2 - c^2k^2 = \Omega_\mathrm{p}^2 \left(1 - \frac{\Omega_\mathrm{c}^2}{\omega^2} \right)^{-1}. \tag{5.171}$$

If we limit ourselves to the case when $\omega \ll \Omega_\mathrm{c}$, which is a very strong assumption, we basically say that we are only interested in very low-frequency waves, the dispersion relation simplifies to

$$\omega^2 - c^2k^2 = \Omega_\mathrm{p}^2 \left(\frac{\omega^2 - \Omega_\mathrm{c}^2}{\omega^2} \right)^{-1} \sim -\omega^2 \frac{\Omega_\mathrm{p}^2}{\Omega_\mathrm{c}^2} = -\omega^2 \frac{n_0 M}{\varepsilon_0 B_0^2} = -\omega^2 \frac{\rho_0}{\varepsilon_0 B_0^2}. \tag{5.172}$$

Hence

$$\frac{\omega^2}{k^2} = \frac{c^2}{1 + \frac{\rho_0}{\varepsilon_0 B_0^2}}. \tag{5.173}$$

In plasma, the expression $1 + \frac{\rho_0}{\varepsilon_0 B_0^2}$ has the meaning of the square-root of relative permitivity ε_r, which in plasmas is usually much larger than unity. Hence $\frac{\rho_0}{\varepsilon_0 B_0^2} \gg 1$ and we have

$$\frac{\omega^2}{k^2} = \frac{c^2}{\frac{\rho \mu_0 c^2}{B_0^2}} = \frac{B_0^2}{\mu_0 \rho_0}. \tag{5.174}$$

Thus the *dispersion relation for Alvén waves* reads

$$\frac{\omega}{k} = \frac{B_0}{\sqrt{\mu_0 \rho_0}} \equiv c_\mathrm{A}. \tag{5.175}$$

c_A is the *Alfvén speed*.
Notes:

- The relations between the oscillating quantities are displayed in Fig. 5.18.
- Magnetic component of the wave, B_y, is in the y direction and looks like sinusoid stringing the field lines of the background magnetic field
- We have a drift $v_y = E_1 \boldsymbol{e}_x \times \boldsymbol{B}_0 / B_0^2$, which is the same for both ions and electrons

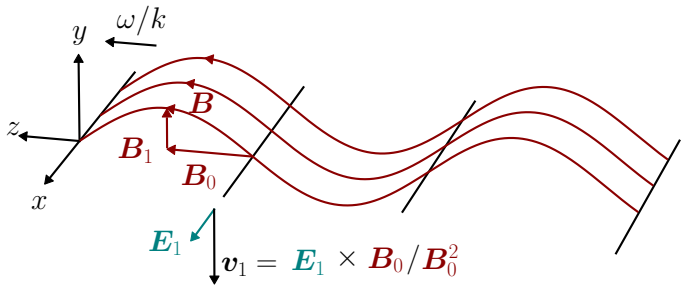

Figure 5.18: Relations between the oscillating quantities in an Alfvén wave and the distortion of the field lines.

- The phase speed of the perturbations to the field lines equals the speed of both ions and electrons. To prove it, let's take the Fourier image of the induction equation without the dissipative term $\omega B_1 = k v_{y,B} B_0$ and Faraday's law $kE_1 = -\omega B_1$. Combining the two we obtain $v_{y,B} = -\frac{\omega}{k}\frac{k}{\omega}\frac{E_1}{B_0} = -\frac{E_1}{B_0}$, which is an expression for the speed of particles. Therefore, both the fluid and the perturbed magnetic field oscillate together. The plasma is frozen in the magnetic field.

- The results we obtained are valid only if there is no background electric field parallel to the magnetic field.

- Alfvén speed is a characteristic speed of propagation of perturbations to the magnetic field. This may yield important estimates. For example, sunspots are believed to be caused by the flux tube, which was perturbed at the base of the solar convection zone and rises up to the photosphere (see the schematic illustration in Fig. 5.19). The characteristic time of rise may be roughly estimated when using the Alfvén speed as the estimate for the rising speed. For the Sun, assuming the depth of the convection zone to be 200 Mm, the average bulk density in the convection zone 200 $\mathrm{kg\,m^{-3}}$, and the initial field strength in the perturbed flux tube to be 10 kG=1 T. Then we have $\tau \sim \frac{d}{c_\mathrm{A}} \sim \frac{200\times10^6}{\frac{1}{\sqrt{4\pi\times10^{-7}\cdot200}}}$ s $\sim$ 36 days. This estimate is not far from the more appropriate modelling, which includes a detailed balance of the rising flux tube and the ambient solar plasma. The characteristic rising time is larger than the solar rotation period, thus one should expect the deflection of the rising flux tube due to the Coriolis force.

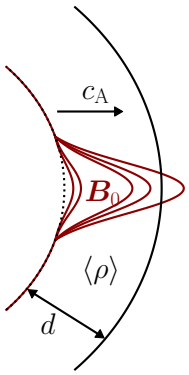

Figure 5.19: Illustration of the emerging magnetic field from the bottom of the convective zone of solar-like stars. A series of snapshots of the perturbed flux tube rising towards solar surface is indicated in the figure.

5.7.2 General MHD waves

As the final case, let us study the general case of magnetohydrodynamic waves. Hence, we will solve the linearised version of the following system of equations:

$$\frac{\partial\rho}{\partial t}+\rho\boldsymbol{\nabla}\cdot\boldsymbol{u}=0, \tag{5.176}$$

$$\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}=-\nabla p+(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}/\mu_0, \tag{5.177}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho^\gamma}\right)=0, \tag{5.178}$$

$$\frac{\partial\boldsymbol{B}}{\partial t}=\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}), \tag{5.179}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0. \tag{5.180}$$

Note that:

- We assume a homogeneous and stationary background and no background motion ($\boldsymbol{u}_0=0$).
- We collapse the set of Maxwell equations into the induction equation with no dissipative terms, to Gauss's law for magnetism and explicitly use Ampére's law in the expression for the Lorentz force; the frozen-field approximation is implied.
- We assume the adiabatic approximation to the equation of state.
- We assume that the flow is non-convecting, that is the terms, where the operator $(\boldsymbol{u}\cdot\boldsymbol{\nabla})$ is applied, are neglected.

Except for the equation of state, the linearisation is trivial and can be done from the top of anyone's head:

$$\frac{\partial \rho_1}{\partial t} + \rho_0(\boldsymbol{\nabla} \cdot \boldsymbol{u}_1) = 0, \tag{5.181}$$

$$\rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} = -\nabla p_1 + (\boldsymbol{\nabla} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0/\mu_0, \tag{5.182}$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u}_1 \times \boldsymbol{B}_0), \tag{5.183}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}_1 = 0. \tag{5.184}$$

The equation of the state needs to be done more carefully. Let us do a slow derivation, in each step we immediately neglect the second and higher-order terms:

$$\begin{aligned} 0 &= \left\{ \frac{\partial}{\partial t} + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla} \right\} \left[\frac{p_0 + p_1}{(\rho_0 + \rho_1)^\gamma} \right] = \\ &= \frac{1}{(\rho_0 + \rho_1)^\gamma} \left[\frac{\partial p_1}{\partial t} + (\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}) p_0 \right] + (p_0 + p_1) \left\{ \frac{-\gamma}{(\rho_0 + \rho_1)^{\gamma+1}} \left[\frac{\partial \rho_1}{\partial t} + (\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}) \rho_0 \right] \right\}. \end{aligned} \tag{5.185}$$

Now we apply the assumption of the homogeneous background, that is we remove the terms where derivatives of the background quantities appear.

$$0 = \frac{1}{(\rho_0 + \rho_1)^\gamma} \left[\frac{\partial p_1}{\partial t} - \gamma \frac{p_0 + p_1}{\rho_0 + \rho_1} \frac{\partial \rho_1}{\partial t} \right] = \frac{1}{(\rho_0 + \rho_1)^\gamma} \left\{ \frac{\partial p_1}{\partial t} - c_\mathrm{s}^2 \frac{\partial \rho_1}{\partial t} \right\}, \tag{5.186}$$

where we used the definition of the speed of sound $c_\mathrm{s}^2 = \gamma p_0/\rho_0$ and we already evaluated term $\frac{p_0+p_1}{\rho_0+\rho_1}$ in Section 5.2. Since the term $\frac{1}{(\rho_0+\rho_1)^\gamma}$ is some non-zero number, the expression in curly parentheses must be zero, which is our linearised equation of state (by further taking into account that the background is homogeneous):

$$\frac{\partial p_1}{\partial t} - c_\mathrm{s}^2 \frac{\partial \rho_1}{\partial t} = 0. \tag{5.187}$$

Thus we combine (5.181), (5.182), (5.183), (5.184), and (5.187) to obtain the dispersion relation. We start by taking the time derivative of (5.182):

$$\frac{\partial^2 \boldsymbol{u}_1}{\partial t^2} = -\frac{1}{\rho_0} \nabla \frac{\partial p_1}{\partial t} + (\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{B}_1}{\partial t}) \times \frac{\boldsymbol{B}_0}{\rho_0 \mu_0}. \tag{5.188}$$

Now we use (5.187) and get

$$\frac{\partial^2 \boldsymbol{u}_1}{\partial t^2} = -\frac{1}{\rho_0} \nabla \left(c_\mathrm{s}^2 \frac{\partial \rho_1}{\partial t} \right) + (\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{B}_1}{\partial t}) \times \frac{\boldsymbol{B}_0}{\rho_0 \mu_0} \tag{5.189}$$

and further by using (5.181) and (5.183) we have

$$\frac{\partial^2 \boldsymbol{u}_1}{\partial t^2} = c_\mathrm{s}^2 \nabla \left(\nabla \cdot \boldsymbol{u}_1 \right) + \{ \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times (\boldsymbol{u}_1 \times \boldsymbol{B}_0)] \} \times \frac{\boldsymbol{B}_0}{\rho_0 \mu_0}. \tag{5.190}$$

The equation is can be solved in the Fourier space again:

$$\begin{aligned}\omega^2\boldsymbol{u}_1 &= c_\mathrm{s}^2\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{u}_1) + \{\boldsymbol{k}\times[\boldsymbol{k}\times(\boldsymbol{u}_1\times\boldsymbol{B}_0)]\}\times\frac{\boldsymbol{B}_0}{\mu_0\rho_0}\\ &= c_\mathrm{s}^2\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{u}_1) + \{\boldsymbol{k}\times[\boldsymbol{k}\times(\boldsymbol{u}_1\times\boldsymbol{b}_0)]\}\times\boldsymbol{b}_0 c_\mathrm{A}^2,\end{aligned} \tag{5.191}$$

where we used $\boldsymbol{b}_0 = \boldsymbol{B}_0/|\boldsymbol{B}_0|$ and $c_\mathrm{A}^2 = B_0^2/(\mu_0\rho_0)$ is the Alfvén speed. The complicated multiplication of the right-hand side is

$$\begin{aligned}\{\boldsymbol{k}\times[\boldsymbol{k}\times(\boldsymbol{u}_1\times\boldsymbol{b}_0)]\}\times\boldsymbol{b}_0 &= \{\boldsymbol{k}\times[\boldsymbol{u}_1(\boldsymbol{k}\cdot\boldsymbol{b}_0) - \boldsymbol{b}_0(\boldsymbol{k}\cdot\boldsymbol{u}_1)]\}\times\boldsymbol{b}_0 =\\ &= (\boldsymbol{k}\cdot\boldsymbol{b}_0)\,[(\boldsymbol{k}\times\boldsymbol{u}_1)\times\boldsymbol{b}_0] - (\boldsymbol{k}\cdot\boldsymbol{u}_1)\,[(\boldsymbol{k}\times\boldsymbol{b}_0)\times\boldsymbol{b}_0] =\\ &= -(\boldsymbol{k}\cdot\boldsymbol{b}_0)\,[\boldsymbol{k}(\boldsymbol{u}_1\cdot\boldsymbol{b}_0) - \boldsymbol{u}_1(\boldsymbol{k}\cdot\boldsymbol{b}_0)] + (\boldsymbol{k}\cdot\boldsymbol{u}_1)\left[\boldsymbol{k}b_0^2 - \boldsymbol{b}_0(\boldsymbol{k}\cdot\boldsymbol{b}_0)\right] =\\ &= (\boldsymbol{k}\cdot\boldsymbol{b}_0)^2\boldsymbol{u}_1 - (\boldsymbol{k}\cdot\boldsymbol{u}_1)(\boldsymbol{k}\cdot\boldsymbol{b}_0)\boldsymbol{b}_0 + \boldsymbol{k}\,[\boldsymbol{k}\cdot\boldsymbol{u}_1 - (\boldsymbol{k}\cdot\boldsymbol{b}_0)(\boldsymbol{u}_1\cdot\boldsymbol{b}_0)]\,.\end{aligned} \tag{5.192}$$

Now we use $\boldsymbol{k}\cdot\boldsymbol{b}_0 = k\cos\alpha$, where α is the field inclination. We continue evaluating (5.191):

$$\omega^2\boldsymbol{u}_1 = c_\mathrm{s}^2\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{u}_1) + c_\mathrm{A}^2\left\{(\boldsymbol{k}\cdot\boldsymbol{b}_0)^2\boldsymbol{u}_1 - (\boldsymbol{k}\cdot\boldsymbol{u}_1)(\boldsymbol{k}\cdot\boldsymbol{b}_0)\boldsymbol{b}_0 + \boldsymbol{k}\,[\boldsymbol{k}\cdot\boldsymbol{u}_1 - (\boldsymbol{k}\cdot\boldsymbol{b}_0)(\boldsymbol{u}_1\cdot\boldsymbol{b}_0)]\right\}, \tag{5.193}$$

thus

$$\frac{\omega^2\boldsymbol{u}_1}{c_\mathrm{A}^2} = k^2\cos^2\alpha\boldsymbol{u}_1 - (\boldsymbol{k}\cdot\boldsymbol{u}_1)k\cos\alpha\boldsymbol{b}_0 + \boldsymbol{k}\left[\left(1+\frac{c_\mathrm{s}^2}{c_\mathrm{A}^2}\right)(\boldsymbol{k}\cdot\boldsymbol{u}_1) - k\cos\alpha(\boldsymbol{u}_1\cdot\boldsymbol{b}_0)\right]. \tag{5.194}$$

We take the projections of (5.194) firstly into the direction of $\boldsymbol{k}$

$$\begin{aligned}\omega^2(\boldsymbol{k}\cdot\boldsymbol{u}_1) &= k^2c_\mathrm{A}^2\cos^2\alpha(\boldsymbol{u}_1\cdot\boldsymbol{k}) - c_\mathrm{A}^2k^2\cos^2\alpha(\boldsymbol{k}\cdot\boldsymbol{u}_1) +\\ &+ c_\mathrm{A}^2(\boldsymbol{k}\cdot\boldsymbol{u}_1)k^2 + c_\mathrm{s}^2(\boldsymbol{k}\cdot\boldsymbol{u}_1)k^2 - c_\mathrm{A}^2k^3\cos\alpha(\boldsymbol{u}_1\cdot\boldsymbol{b}_0).\end{aligned} \tag{5.195}$$

Hence

$$\left[-\omega^2 + c_\mathrm{A}^2k^2 + c_\mathrm{s}^2k^2\right](\boldsymbol{u}_1\cdot\boldsymbol{k}) = c_\mathrm{A}^2k^3\cos\alpha(\boldsymbol{b}_0\cdot\boldsymbol{u}_1). \tag{5.196}$$

Then we project (5.194) into the direction of $\boldsymbol{b}_0$.

$$\begin{aligned}\omega^2(\boldsymbol{b}_0\cdot\boldsymbol{u}_1) &= c_\mathrm{A}^2k^2\cos^2\alpha(\boldsymbol{u}_1\cdot\boldsymbol{b}_0) - c_\mathrm{A}^2(\boldsymbol{k}\cdot\boldsymbol{u}_1)k\cos\alpha +\\ &+ c_\mathrm{A}^2(\boldsymbol{k}\cdot\boldsymbol{u}_1)k\cos\alpha + c_\mathrm{s}^2(\boldsymbol{k}\cdot\boldsymbol{u}_1)k\cos\alpha - c_\mathrm{A}^2k^2\cos^2\alpha(\boldsymbol{u}_1\cdot\boldsymbol{b}_0),\end{aligned} \tag{5.197}$$

hence

$$\omega^2(\boldsymbol{b}_0\cdot\boldsymbol{u}_1) = c_\mathrm{s}^2(\boldsymbol{k}\cdot\boldsymbol{u}_1)k\cos\alpha. \tag{5.198}$$

Equation (5.196) may be further played with:

$$\frac{\boldsymbol{b}_0\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{k}} = \frac{-\omega^2 + c_\mathrm{A}^2k^2 + c_\mathrm{s}^2k^2}{c_\mathrm{A}^2k^3\cos\alpha} \tag{5.199}$$

and similarly (5.198) becomes

$$\frac{\boldsymbol{b}_0 \cdot \boldsymbol{u}_1}{\boldsymbol{u}_1 \cdot \boldsymbol{k}} = \frac{c_\mathrm{s}^2 k \cos\alpha}{\omega^2}. \tag{5.200}$$

These two equations are equal, thus we have

$$\frac{-\omega^2 + c_\mathrm{A}^2 k^2 + c_\mathrm{s}^2 k^2}{c_\mathrm{A}^2 k^3 \cos\alpha} = \frac{c_\mathrm{s}^2 k \cos\alpha}{\omega^2} \tag{5.201}$$

and therefore

$$\omega^4 - \omega^2 k^2 (c_\mathrm{A}^2 + c_\mathrm{s}^2) + c_\mathrm{s}^2 c_\mathrm{A}^2 k^4 \cos^2\alpha = 0. \tag{5.202}$$

By solving the quadratic equation in $(\omega/k)^2$ we have a solution:

$$\left(\frac{\omega}{k}\right)^2 = \left[\frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}\sqrt{c_\mathrm{s}^4 + c_\mathrm{A}^4 - 4c_\mathrm{s}^2 c_\mathrm{A}^2 \cos^2\alpha + 2c_\mathrm{A}^2 c_\mathrm{s}^2}\right]. \tag{5.203}$$

This is the general dispersion relation for the *magnetoacoustic wave*. The one with the plus sign is called the *fast wave*, while the one with the minus sign is called the *slow wave*. Obviously, the speed of propagation depends on the direction of propagation with respect to the vector of the background magnetic field.

For waves travelling *along* the magnetic field $\alpha = 0$ we have:

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}\sqrt{c_\mathrm{s}^4 + c_\mathrm{A}^4 - 2c_\mathrm{A}^2 c_\mathrm{s}^2} = \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}|c_\mathrm{s}^2 - c_\mathrm{A}^2| = \begin{cases} c_\mathrm{s}^2 \\ c_\mathrm{A}^2 \end{cases}, \tag{5.204}$$

for $c_\mathrm{s} > c_\mathrm{A}$ and oppositely for $c_\mathrm{A} > c_\mathrm{s}$ (note that we still keep the notation such that the upper solution represents the fast wave, whereas the lower represents the slow-wave mode). To summarise the two cases in a general solution, we may write that the *fast mode* has the dispersion relation $\omega/k = \max(c_\mathrm{s}, c_\mathrm{A})$, whereas the *slow mode* disperses according to $\omega/k = \min(c_\mathrm{s}, c_\mathrm{A})$. Note that as a special case of waves propagating along the field, we obtained the dispersion relation for the Alfvén wave. Also the *slow wave*, which is purely acoustic, also propagates. For waves travelling across the field for $\alpha = \pi/2$ we have

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}\sqrt{c_\mathrm{s}^4 + c_\mathrm{A}^4 + 2c_\mathrm{A}^2 c_\mathrm{s}^2} = \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) = \begin{cases} c_\mathrm{s}^2 + c_\mathrm{A}^2 \\ 0 \end{cases}, \tag{5.205}$$

therefore, only the *fast wave* propagates. In the special case of no background field, i.e. $\boldsymbol{B}_0 = 0$ we simply have $c_\mathrm{A} = 0$ and thus

$$\left(\frac{\omega}{k}\right)^2 = \begin{cases} 0 \\ c_\mathrm{s}^2 \end{cases}, \tag{5.206}$$

and the wave just degenerates to the ordinary acoustic wave.

5.7.3 Mode conversion

In the stratified medium, we should supply additional terms in our equations (namely the gravitation term) and drop the requirement of the homogeneous background. In some cases, we may find surfaces where the sound speed equals the Alfvén speed, which is the *equipartition surface* (prove that at this surface, the plasma β equals one). Here all the dispersion relations collapse and the waves are resonant. The use of the simple ray approximation (which we were using all the time) is not appropriate in these regions, as finite-wavelength effects play an important role. Using the higher-order theory one may find[5] that a *mode conversion* may occur. Essential in the equipartition layers the waves convert types, purely acoustic waves may become magnetoacoustic fast waves etc. The efficiency of the conversion depends on an incident angle to the magnetic field.

The mode conversion is an effect which essentially prohibits reliable helioseismic tomography of the Sun beneath sunspots. The sunspots are regions of the strong field, thus there definitely is the equipartition layer below the surface of the Sun. The (purely acoustic) seismic waves excited by the vigorous surface convection propagate through the convection zone (in the stratified medium, thus they do not propagate straight but rather bend as the local sound speed varies), go through the equipartition layer and partly convert to the magnetoacoustic modes. They undergo one more conversion after total reflection from the surface. As a consequence, the total power of the acoustic waves, which is measured and analysed by helioseismology, is lower in the regions of the strong field, leading to unknown large systematic errors in analyses.

Similarly, acoustic waves in the solar atmosphere are reflected totally at the temperature minimum region (lower chromosphere, $T \sim 4200$ K), where the acoustic cut-off frequency is around 5.2 mHz (the cut-off frequency is a function of the local plasma state parameters, $\omega_{\mathrm{cut-off}} = \frac{\gamma g}{2c_{\mathrm{s}}}$, where g is the local gravity acceleration). However, in the magnetised regions, the magnetoacoustic waves are allowed to propagate along the magnetic field. Thus when the conversion takes place, the originally acoustic waves may leak to the higher atmosphere as converted magnetoacoustic and Alfvén waves. Numerical simulations revealed that the inclined magnetic field acts as a factor lowering the cut-off frequency, which is then modified to $\omega_{\mathrm{cut-off}} = \frac{\gamma g \cos\alpha}{2c_{\mathrm{s}}}$, where α is the magnetic field inclination to the local vertical. In the magnetised regions with inclined magnetic field the sound waves excited by the solar convection may carry some non-negligible acoustic flux to the chromosphere and thus contributing to its heating[6].

[5] e.g. Cally, P. 2006: *Dispersion relations, rays and ray splitting in magnetohelioseismology*, Royal Society of London Transactions Series A 364, p.333–349

[6] e.g. Sobotka, M., Švanda, M., Jurčák, J. et al. 2013: *Dynamics of the solar atmosphere above a pore with a light bridge*, Astronomy&Astrophysics 560, id.A84, preprint `ArXiv:1309.7790S`

Chapter 6

Diffusion

6.1 Diffusion in weakly ionised plasmas

In Section 4.2 we introduced an additional term to the Euler equation accounting for collisions between particles. We defined the particle flux Γ

$$\boldsymbol{\Gamma} = n\boldsymbol{u}, \tag{6.1}$$

where n was the particle density and $\boldsymbol{u}$ their speed.

The formalism used here is similar. For simplicity, let us assume that the particles flow along the x-axis of the Cartesian coordinate system. Due to collisions occurring in the spatial $\mathrm{d}x$, a fraction of the flux $\mathrm{d}\Gamma$ is lost (see Fig. 6.1). We may write

$$\mathrm{d}\Gamma = -n_\mathrm{n}\sigma\Gamma\mathrm{d}x, \tag{6.2}$$

where n_n is the density of neutral particles because we expect the collisions to be dominated by neutral particles. σ is the effective cross-section of the collisions. From the effective cross-section, we may define useful quantities $\lambda_s = 1/n_\mathrm{n}\sigma$ to be the mean free path between consecutive collisions and similarly the mean collisional frequency

$$\nu = n_\mathrm{n}\langle\sigma v\rangle \approx n_\mathrm{n}\sigma\boldsymbol{u}, \tag{6.3}$$

where the averaging is performed over the velocity space. The averaging is performed because quantum mechanics teaches us that the effective cross-section may be a function of velocity. The mean period between consecutive collisions is then $\tau = 1/\nu$.

We take the time derivative of (6.1) and (6.2), where for simplicity we neglect the possible time dependence of all variables apart from the coordinates and velocity. Then using the definition of Γ we have

$$\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}t} = n\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -n_\mathrm{n}\sigma n\boldsymbol{u}\frac{\mathrm{d}x}{\mathrm{d}t} = -n_\mathrm{n}\sigma nu\boldsymbol{u} = -n\nu\boldsymbol{u}, \tag{6.4}$$

Hence

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\nu\boldsymbol{u}, \tag{6.5}$$

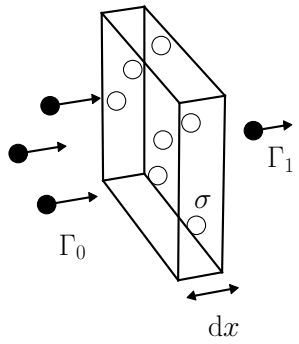

Figure 6.1: Illustration of the model of the change of the particle flux via collisions.

which is the (average) velocity change due to the collisions.

Hence the Euler equation with the magnetic field set to zero and with the collisional term added reads

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] = qn\boldsymbol{E} - \nabla p - mn\nu\boldsymbol{u} \tag{6.6}$$

Let's investigate a system where $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = 0$ (in such a system we only find a constant motion due to the diffusion). Then

$$\boldsymbol{u} = \frac{q\boldsymbol{E}}{m\nu} - \frac{K_\mathrm{B}T\nabla n}{mn\nu} = \frac{\boldsymbol{\Gamma}}{n}, \tag{6.7}$$

where we used the definition of the particle flux. By introducing the mobility $\mu = \frac{q\,\mathrm{sgn}\,q}{m\nu}$ and the diffusion coefficient $D = \frac{K_\mathrm{B}T}{m\nu}$ we have for the particle species j

$$\boldsymbol{\Gamma}_j = \mathrm{sgn}\,q_j\, n\mu_j\boldsymbol{E} - D_j\nabla n. \tag{6.8}$$

For $\boldsymbol{E} = 0$ we have

$$\boldsymbol{\Gamma} = -D\nabla n, \tag{6.9}$$

which is a form of *Fick's law* for diffusion. A simple interpretation of this useful relation is that the particles flow from locations with a higher density to lower density regions (as illustrated in Fig. 6.2), which is to be expected. In plasmas, Fick's law is generalised to (6.8) noting the effect of the electric field. One should also note that the coefficients μ_j and D_j may generally be different for various particle species (most often electrons and ions) and hence the diffusion in plasmas may cause a separation of charges and thus drive an electric field. However, to fulfill the quasineutrality of plasma, $\boldsymbol{\Gamma}_\mathrm{i} = \boldsymbol{\Gamma}_\mathrm{e} = \boldsymbol{\Gamma}$ on scales $L \gg \lambda_\mathrm{D}$, which allows us to derive the necessary electric field that will keep the plasma quasineutral.

By assuming the plasma approximation $n_\mathrm{e} = n_\mathrm{i} = n$ we have

$$n\mu_\mathrm{i}\boldsymbol{E} - D_\mathrm{i}\nabla n = -n\mu_\mathrm{e}\boldsymbol{E} - D_\mathrm{e}\nabla n \tag{6.10}$$

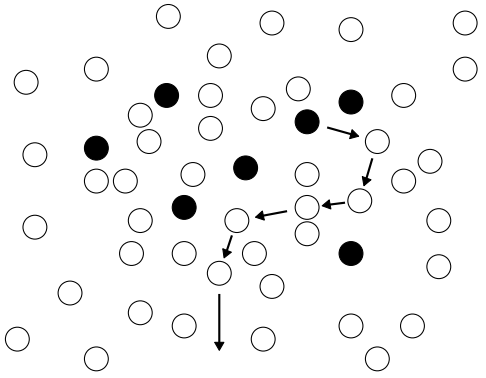

Figure 6.2: Illustrative concept of diffusion. An overabundance of the black "particles" is decreased in time via collisions with white "particles" and their local density thus decreases.

and hence

$$\boldsymbol{E} = \frac{D_\mathrm{i} - D_\mathrm{e}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \frac{\nabla n}{n} \tag{6.11}$$

is the induced electric field triggered by the separation of charges. This is a repulsive electric field, which stops the charges from further separation due to diffusion and hence keeps plasma neutral on large scales. This electric field causes the electrons to slow down and accelerate ions simultaneously, so that the bulk of plasma (with charges separated on scales smaller than λ_D) diffuses according to modified Fick's law. This is the basic picture of *ambipolar diffusion*.

Then

$$\begin{aligned} \boldsymbol{\Gamma} = \boldsymbol{\Gamma}_\mathrm{i} &= n\mu_\mathrm{i} \frac{D_\mathrm{i} - D_\mathrm{e}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \frac{\nabla n}{n} - D_\mathrm{i} \nabla n = \\ &= \frac{\mu_\mathrm{i} D_\mathrm{i} - \mu_\mathrm{i} D_\mathrm{e} - \mu_\mathrm{i} D_\mathrm{i} - \mu_\mathrm{e} D_\mathrm{i}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \nabla n = -\frac{\mu_\mathrm{i} D_\mathrm{e} + \mu_\mathrm{e} D_\mathrm{i}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \nabla n = \\ &\equiv -D_\mathrm{a} \nabla n, \end{aligned} \tag{6.12}$$

where D_a is the coefficient of the ambipolar diffusion. For $\mu_\mathrm{e} \gg \mu_\mathrm{i}$ and using the definition of the mobility coefficient we have

$$D_\mathrm{a} = \frac{\mu_\mathrm{i} D_\mathrm{e} + \mu_\mathrm{e} D_\mathrm{i}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \approx D_\mathrm{i} + \frac{\mu_\mathrm{i}}{\mu_\mathrm{e}} D_\mathrm{e} = \left(1 + \frac{T_\mathrm{e}}{T_\mathrm{i}}\right) D_\mathrm{i}. \tag{6.13}$$

Thus the coefficient of ambipolar diffusion is mostly determined by the diffusion coefficients for ions with some correction.

Typical time evolution by diffusion

As an example, let's take the continuum equation

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot n\boldsymbol{u} = 0, \tag{6.14}$$

and hence

$$\frac{\partial n}{\partial t} = -\boldsymbol{\nabla} \cdot n\boldsymbol{u} = -\boldsymbol{\nabla} \cdot \boldsymbol{\Gamma} = D_\mathrm{a} \triangle n. \tag{6.15}$$

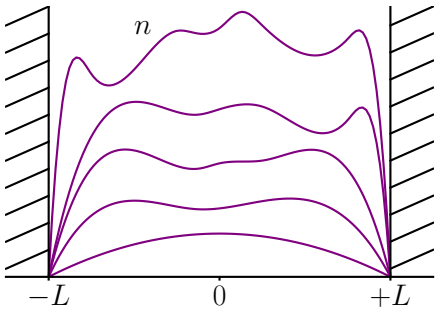

Figure 6.3: A model of plasma diffusion confined between two infinite plane-parallel walls. A series of plasma density snapshots is represented by the purple lines. At the beginning, the density profile is quite complex. Due to diffusion, it simplifies with time, large spatial frequencies (small-scale features) decay the fastest.

This equation is similar to the heat conduction equation. It may be solved by assuming the 1-D solution by separating the variables $n(x,t) = T(t)S(x)$, where T and S are some functions. Then

$$S\frac{\mathrm{d}T}{\mathrm{d}t} = D_\mathrm{a} T \triangle S, \tag{6.16}$$

and hence

$$\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{D_\mathrm{a}}{S}\triangle S = \text{const} = -\frac{1}{\tau}. \tag{6.17}$$

Both sides of the equations must equal the same constant because the left-hand side depends purely on time; thus, the right-hand side can't depend on time and vice versa for the spatial dependency of the right-hand side. For dimension reasons, this constant must have the dimension of reciprocal time. The equation may then be solved by parts

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{T}{\tau} \quad \rightarrow \quad T = T_0 \mathrm{e}^{-\frac{t}{\tau}} \tag{6.18}$$

and

$$\frac{\mathrm{d}^2 S}{\mathrm{d}x^2} = -\frac{1}{D_\mathrm{a}\tau} S, \tag{6.19}$$

which has the solution in the series of harmonic functions,

$$S = S_0 \cos\frac{x}{\sqrt{D_\mathrm{a}\tau}} \quad \text{and} \quad S = S_0 \sin\frac{x}{\sqrt{D_\mathrm{a}\tau}}. \tag{6.20}$$

This solution must be supplied with the boundary conditions. Let's assume a simple model of plasma confined between two infinite plane-parallel walls (see Fig. 6.3), where the density of the plasma vanishes outside interval $x \in (-L, L)$. Then the solution to the problem is

$$n = n_0 \left\{ \sum_{l=0}^{\infty} a_{\mathrm{e},l} \mathrm{e}^{-\frac{t}{\tau_{\mathrm{e},l}}} \cos\left[\frac{(l+\frac{1}{2})\pi x}{L}\right] + \sum_{m=1}^{\infty} a_{\mathrm{o},m} \mathrm{e}^{-\frac{t}{\tau_{\mathrm{o},m}}} \sin\left[\frac{m\pi x}{L}\right] \right\} \tag{6.21}$$

The coefficients $\tau_{\mathrm{e},l}$ and $\tau_{\mathrm{o},m}$ may be determined by inserting the solution above into (6.16) and comparing the terms one by one. For the first term ($l = 0$) in the cosine series we have

$$-a_{\mathrm{e},0}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{e},0}}}\frac{1}{\tau_{\mathrm{e},0}}\cos\frac{\pi x}{2L} = -D_{\mathrm{a}}a_{\mathrm{e},0}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{e},0}}}\left(\frac{\pi}{2L}\right)^2\cos\frac{\pi x}{2L} \tag{6.22}$$

and hence

$$\tau_{\mathrm{e},0} = \frac{4L^2}{\pi^2 D_{\mathrm{a}}}. \tag{6.23}$$

For a general mode l, we would similarly find

$$\tau_{\mathrm{e},l} = \frac{L^2}{(l+\frac{1}{2})^2\pi^2 D_{\mathrm{a}}}. \tag{6.24}$$

For the first term ($m = 1$) in the sine series, we have

$$-a_{\mathrm{o},1}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{o},1}}}\frac{1}{\tau_{\mathrm{o},1}}\sin\frac{\pi x}{L} = -D_{\mathrm{a}}a_{\mathrm{o},1}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{o},1}}}\left(\frac{\pi}{L}\right)^2\sin\frac{\pi x}{L} \tag{6.25}$$

and hence

$$\tau_{\mathrm{o},1} = \frac{L^2}{\pi^2 D_{\mathrm{a}}}. \tag{6.26}$$

For the other modes m we similarly obtain

$$\tau_{\mathrm{o},m} = \frac{L^2}{m^2\pi^2 D_{\mathrm{a}}}. \tag{6.27}$$

The lifetime of various modes $\tau_{\mathrm{e},l}$ and $\tau_{\mathrm{o},m}$ shows that the higher modes are attenuated faster. Therefore, if there is a wild density profile at the beginning (in time $t = 0$), then the profile is simplified with time. A set of snapshots of the evolution of an arbitrary density profile is given in Fig. 6.3.

Other physics-motivated examples for diffusion

The question is whether it is possible to achieve stationarity in the presence of diffusion. It is not if the continuity equation holds. It is possible in the presence of the source term,

$$\boldsymbol{\nabla}\cdot n\boldsymbol{u} = Q. \tag{6.28}$$

In case of collisional ionisation $Q = Zn$, where Z is the ionisation function, we have

$$\boldsymbol{\nabla}\cdot n\boldsymbol{u} = -D_{\mathrm{a}}\triangle n = Zn \quad \rightarrow \quad \triangle n = -\frac{Z}{D_{\mathrm{a}}}n, \tag{6.29}$$

solution to which is again the series of harmonic functions.

The recombination, on the other hand, takes a different form because it does change the number of charged particles. The number density of the ionised particles naturally decays with n^2 in recombination, hence the continuity equation may be written in the form

$$\frac{\partial n}{\partial t} - D_{\mathrm{a}}\nabla n = -\alpha n^2. \tag{6.30}$$

Similarly to the previous cases, we consider the solution separated to the temporal and spatial parts $n = TS$. Further, we will only be interested in the temporal part, which is equivalent to modification of (6.30) to

$$\frac{\partial n}{\partial t} = -\alpha n^2, \tag{6.31}$$

where n^2 dependence is a consequence of the recombination being proportional to both the density of electrons and ions and the use of the plasma approximation. Then

$$\frac{1}{n^2}\mathrm{d}n = -\alpha \mathrm{d}t \quad \rightarrow \quad -\frac{1}{n} = -\frac{1}{n_0} - \alpha t. \tag{6.32}$$

For $t \to \infty$ we have $n \sim \frac{1}{\alpha t}$, which says that due to the recombination in plasmas, the density of the charged particles drops as $1/t$. The spatial part of the continuity equation is not relevant to the problem; thus, we neglect it from the solution.

This solution is important for understanding which process drives the diffusion of charged particles in a cloud of plasma left-alone in time. Obviously, in the initial moment, recombination will be the dominant process, as its time dependence is steeper for small t. On the contrary, when the density of the plasma gets smaller, the diffusion with exponencial time dependence prevails and further drives the diffusion.

6.2 Diffusion in highly ionised plasmas

We will show that in highly ionised plasmas, diffusion resembles the process of recombination rather than ambipolar diffusion. Let us use the fluid approximation to deal with the problem.

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p \tag{6.33}$$

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j} \tag{6.34}$$

Note that generally, the specific resistivity of plasma η is not a scalar value but rather a tensor and thus may depend on direction. In plasma, this usually is the case. Therefore, we split the specific resistivity into the parallel and perpendicular components ($\eta = \eta_\parallel + \eta_\perp$), where in the parallel direction Ohm's law simply is

$$\boldsymbol{E}_\parallel = \eta_\parallel \boldsymbol{j}_\parallel. \tag{6.35}$$

To obtain a similar relation in the direction perpendicular to the magnetic field, when assuming the stationarity, by vector-multiplying (6.34) by $\boldsymbol{B}$ and using (6.33) we have

$$\boldsymbol{E} \times \boldsymbol{B} + (\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B} = \boldsymbol{E} \times \boldsymbol{B} - B^2\boldsymbol{u} + B^2\boldsymbol{u}_\parallel = \eta_\perp \nabla p. \tag{6.36}$$

Since $\boldsymbol{u} = \boldsymbol{u}_\parallel + \boldsymbol{u}_\perp$, then the last two terms on the left-hand side together give $B^2\boldsymbol{u}_\perp$ and thus

$$\boldsymbol{u}_\perp = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} - \frac{\eta_\perp \nabla p}{B^2}. \tag{6.37}$$

The first term on the right-hand side is the same for both (electron and ion) fluids. The second one, on the other hand, may drive the diffusion. The perpendicular particle flux on top of the E-B drift is

$$\boldsymbol{\Gamma}_\perp = n\boldsymbol{u}'_\perp = -\frac{\eta_\perp n(K_\mathrm{B}T_\mathrm{e} + K_\mathrm{B}T_\mathrm{i})}{B^2}\nabla n = -D_\perp \nabla n\,. \tag{6.38}$$

Note that the diffusion coefficient $D_\perp$ is a function of particle density and for simpler manipulation may be rewritten as

$$D_\perp = \frac{\eta_\perp n K_\mathrm{B}\sum\limits_\alpha T_\alpha}{B^2} = 2An. \tag{6.39}$$

The factor of two is introduced so that $\boldsymbol{\nabla}\cdot(2An\nabla n) = 2A\boldsymbol{\nabla}\cdot(n\nabla n) = A\,\triangle n^2$.

The continuity equation

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\boldsymbol{u}) = 0 \tag{6.40}$$

then transforms to

$$\frac{\partial n}{\partial t} = A\,\triangle n^2. \tag{6.41}$$

Such an equation may be solved by separation of variables, hence $n(\boldsymbol{r},t) = T(t)S(\boldsymbol{r})$. Then $S\frac{\mathrm{d}T}{\mathrm{d}t} = AT^2\,\triangle S^2$ and

$$\frac{1}{T^2}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{A}{S}\,\triangle S^2 = \mathrm{const} = -\frac{1}{\tau^2} \tag{6.42}$$

and in the case we solved previously. Then the temporal part has the solution

$$\frac{1}{T} = \frac{t}{\tau^2} + \frac{1}{T_0} \tag{6.43}$$

and the form of τ depends on the spatial part S. For $t \to \infty$

$$T \propto \frac{\tau^2}{t}, \tag{6.44}$$

which reminds us of the the solution of the change of the particle density caused by recombinations. The difference between the ambipolar diffusion and the result we just obtained is in the diffusion coefficient depending on particle density. Note that the result we obtained is not consistent with the assumptions of the derivation of Fick's law, as we allow for non-stationarity here.

6.3 Diffusion in a magnetic field

In presence of a magnetic field, the Euler equation retains an additional term. Let us see what happens in the presence of a magnetic field. We must go back to the double-fluid approximation since collisions are essential in the process of diffusion.

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] = \mathrm{sgn}\,q\,en(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - K_\mathrm{B}T\nabla n - mn\nu\boldsymbol{u} \tag{6.45}$$

Let's assume that the collisions are much faster than the overall evolution of the system in time, or in other words $\frac{\mathrm{d}}{\mathrm{d}t} \ll \omega_\mathrm{c}, \nu$, where ν is the collision frequency with neutrals and $\tau \equiv 1/\nu$ is the corresponding time between collisions. Then the left-hand side may be neglected. Thus in components of the Cartesian system with the z axis oriented along the magnetic field ($\boldsymbol{B} = B\boldsymbol{e}_z$) we have

$$mn\nu u_x = \operatorname{sgn} q\, enE_x - K_\mathrm{B}T\frac{\partial n}{\partial x} + \operatorname{sgn} q\, enu_y B, \tag{6.46}$$

$$mn\nu u_y = \operatorname{sgn} q\, enE_y - K_\mathrm{B}T\frac{\partial n}{\partial y} - \operatorname{sgn} q\, enu_x B. \tag{6.47}$$

Along the magnetic field, a classic diffusion as derived above occurs.

By introducing $\mu = \frac{e}{m\nu}$, $D = \frac{K_\mathrm{B}T}{m\nu} = \frac{\omega_\mathrm{c}}{\nu}\frac{K_\mathrm{B}T}{eB}$ we have

$$u_x = \operatorname{sgn} q\, \mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \operatorname{sgn} q\, \frac{\omega_\mathrm{c}}{\nu}u_y, \tag{6.48}$$

$$u_y = \operatorname{sgn} q\, \mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \operatorname{sgn} q\, \frac{\omega_\mathrm{c}}{\nu}u_x, \tag{6.49}$$

and (using $\tau \equiv 1/\nu$)

$$u_x(1+\omega_\mathrm{c}^2\tau^2) = \operatorname{sgn} q\, \mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \omega_\mathrm{c}^2\tau^2\frac{E_y}{B} - \operatorname{sgn} q\, \omega_\mathrm{c}^2\tau^2\frac{K_\mathrm{B}T}{neB}\frac{\partial n}{\partial y}, \tag{6.50}$$

$$u_y(1+\omega_\mathrm{c}^2\tau^2) = \operatorname{sgn} q\, \mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \omega_\mathrm{c}^2\tau^2\frac{E_x}{B} + \operatorname{sgn} q\, \omega_\mathrm{c}^2\tau^2\frac{K_\mathrm{B}T}{neB}\frac{\partial n}{\partial x}. \tag{6.51}$$

The last two terms on the right-hand side obviously indicate the already known E-B and diamagnetic drifts. Hence

$$\boldsymbol{u}_\perp = \frac{\boldsymbol{v}_E + \boldsymbol{v}_\mathrm{D}}{1 + 1/(\omega_\mathrm{c}^2\tau^2)} + \operatorname{sgn} q\frac{\mu}{1+\omega_\mathrm{c}^2\tau^2}\boldsymbol{E} - \frac{D}{1+\omega_\mathrm{c}^2\tau^2}\frac{\nabla n}{n}, \tag{6.52}$$

where $\boldsymbol{v}_E = \boldsymbol{E}\times\boldsymbol{B}/B^2$ and $\boldsymbol{v}_\mathrm{D} = -\nabla p\times\boldsymbol{B}/(qnB^2)$. Let us further identify

$$\mu_\perp \equiv \frac{\mu}{1+\omega_\mathrm{c}^2\tau^2} \quad \text{and} \quad D_\perp \equiv \frac{D}{1+\omega_\mathrm{c}^2\tau^2}. \tag{6.53}$$

Hence generalised Fick's law transforms into

$$\boldsymbol{u}_\perp = \operatorname{sgn} q\, \mu_\perp\boldsymbol{E} - D_\perp\frac{\nabla n}{n} + \frac{\boldsymbol{v}_E + \boldsymbol{v}_\mathrm{D}}{1 + 1/(\omega_\mathrm{c}^2\tau^2)}. \tag{6.54}$$

The last term represents the already familiar drifts in the plasma fluid, only with the correction term for the diffusion. Obviously, the correcting term is important for $\nu > \omega_\mathrm{c}$. Hence in the weak field regime, drifts are suppressed by collisions.

The correction term for the mobility and the diffusion coeficient act in opposite ways. In the strong field, i.e., when $\nu < \omega_\mathrm{c}$, diffusion is suppressed. In particular, we have

$$D_\perp \propto \frac{K_\mathrm{B}T\nu}{m\omega_\mathrm{c}^2} \quad \text{and} \quad D_\parallel = \frac{K_\mathrm{B}T}{m\nu}, \tag{6.55}$$

where $D_\parallel$ is the non-modified diffusion coefficient which appears in the component of the Euler equation parallel to the magnetic field. Hence the decay caused by diffusion in plasmas has different rates in different directions. In case the collisional frequency is proportional[1] to $m^{-1/2}$ then the diffusion depends also on the species of the particles. Then

$$D_\perp \propto m^{1/2} \quad \text{and} \quad D_\parallel \propto m^{-1/2}. \tag{6.56}$$

In the perpendicular direction, the diffusion is faster for ions than electrons, while in the parallel direction the electrons diffuse faster. It has to be noted that generally speaking, collisions are important for particles to be able to penetrate across the magnetic field.

[1] As we discussed at the beginning of this chapter, $\nu = n_\mathrm{n}\langle\sigma v\rangle$. Leaving out a possible dependence of the effective cross-section on velocity, we have $\nu \propto \langle v\rangle \propto \sqrt{\langle v^2\rangle}$. Since $T \propto m\langle v^2\rangle$, for a fixed temperature T we have $\langle v^2\rangle \propto m^{-1}$ and finally $\nu \propto m^{-1/2}$.

Chapter 7

Equilibrium and stability

7.1 Equilibrium

The state of equilibrium indicates the forces acting on plasma are in balance. In the investigation of the possible equilibrium of plasma, we will use the fluid approximation. However, it should be noted that this approximation might be too rough. In plasmas, drifting motions are usually the sources of instabilities driving the plasma from equilibrium, and the drifting motions are largely not captured by the fluid approximation.

In investigating the stability, we assume that the forces are in balance, hence the solution is stationary. Hence the necessary condition of equilibrium is $\frac{\partial \bullet}{\partial t} \to 0$.

We recognise a *stable equilibrium*, where small perturbations deviating the system from an exact balance are attenuated, and an *unstable equilibrium*, when even a small perturbation of the system grows and brings the system out of equilibrium. Finding the equilibrium is generally a non-linear problem (see Fig. 7.1). We will limit ourselves to discussing linear problems by only considering small perturbations. To describe the equilibrium, the MHD description suffices, however it is usually not sufficient to discuss the stability of such an equilibrium. Let us remind the reader that *mechanical stability* is equal to the work that is needed to bring the object from a stable equilibrium position to another (investigated) equilibrium position. Should this work be negative, the position at which it is located is not stable.

Let us describe a general concept of investigation of equilibrium on an example of *hydromagnetic equilibrium*. Let's consider a plasma cylinder, where the magnetic field is aligned with the cylinder axis (see Fig. 7.2). The simplified equation of motion reads

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p. \tag{7.1}$$

We search for a stationary state, hence we set $\frac{\partial \boldsymbol{u}}{\partial t} = 0$ and the Euler equation degenerates to

$$\nabla p = \boldsymbol{j} \times \boldsymbol{B} \tag{7.2}$$

It is clear now that there is an obvious balance between the Lorentz force which supports the pressure gradient. Such a balance may be achieved when an azimuthal current flows

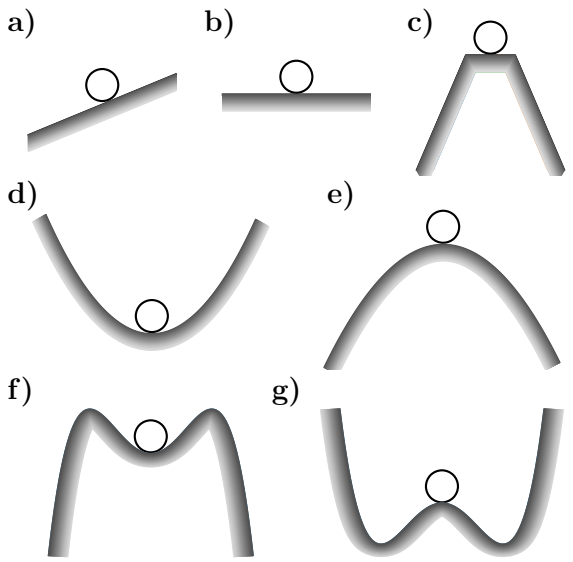

Figure 7.1: Illustrations of various kinds of equilibria. a) inequilibrium, b) neutrally stable, c) metastable, d) stable, e) unstable, f) linearly stable, non-linearly unstable, g) linearly unstable, non-linearly stable.

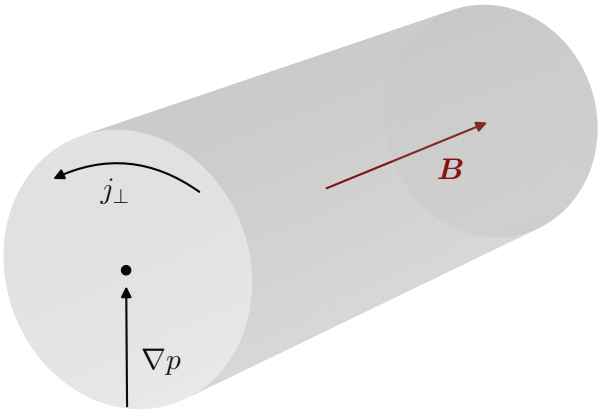

Figure 7.2: Illustration of hydromagnetic equilibrium.

through the plasma cylinder. By taking the vector multiplication by $\boldsymbol{B}$ we have

$$\boldsymbol{B} \times \nabla p = \boldsymbol{B} \times (\boldsymbol{j} \times \boldsymbol{B}) = B^2 \boldsymbol{j} - \boldsymbol{B} \cdot \boldsymbol{j} \boldsymbol{B} = B^2 (\boldsymbol{j} - \boldsymbol{j}_{\parallel}). \tag{7.3}$$

By assuming that $\boldsymbol{j} = \boldsymbol{j}_{\parallel} + \boldsymbol{j}_{\perp}$ we obtain the expression for a required azimuthal current

$$\boldsymbol{j}_{\perp} = \frac{\boldsymbol{B} \times \nabla p}{B^2} = (K_{\mathrm{B}} T_{\mathrm{i}} + K_{\mathrm{B}} T_{\mathrm{e}}) \frac{\boldsymbol{B} \times \nabla n}{B^2}, \tag{7.4}$$

which is an expression for the diamagnetic current. Such a situation may be viewed as if a plasma cylinder is to be in equilibrium, there must be a current induced, which takes the form of a diamagnetic current. One should bear in mind that this expression was derived when neglecting all other terms in the Euler equation.

One should also note that both $\boldsymbol{j}$ and $\boldsymbol{B}$ are perpendicular to ∇p, hence both $\boldsymbol{j}$ and $\boldsymbol{B}$ lie on the isosurface of constant pressure. The field lines and the current lines may be curved on this surface.

7.2 Plasma β

By considering Ampére's law

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} \tag{7.5}$$

for the equilibrium situation, we have

$$\begin{aligned} \nabla p = \boldsymbol{j} \times \boldsymbol{B} = \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = \\ = \frac{1}{\mu_0} \left[-(\nabla \boldsymbol{B}) \cdot \boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B} \right] = \frac{1}{\mu_0} \left[(\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B} - \frac{1}{2} \nabla B^2 \right] \end{aligned} \tag{7.6}$$

By rearranging we have

$$\nabla \left(p + \frac{1}{\mu_0} B^2 / 2 \right) = \frac{1}{\mu_0} (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B}. \tag{7.7}$$

The right-hand side indicates the action of forces along the field lines, the *magnetic tension*. If we assume that there are negligible changes of the magnetic field along the field lines, the right-hand side vanishes. Then

$$p + \frac{B^2}{2\mu_0} = \text{const.} \tag{7.8}$$

The second term has the physical meaning of *magnetic field pressure*. By comparing these two terms we define the *plasma beta*,

$$\beta \equiv \frac{p}{p_{\mathrm{mag}}} = \frac{n \sum\limits_{\alpha} K_{\mathrm{B}} T_{\alpha}}{\frac{B^2}{2\mu_0}}, \tag{7.9}$$

where α identificates the particles by type.

For $\beta > 1$ the gas pressure prevails. This high-beta regime is common to the interior of stars and most of laboratory experiments. For $\beta < 1$ the magnetic field pressure prevails. This low-beta regime is common to solar/stellar atmosphere(s) and also to interstellar matter.

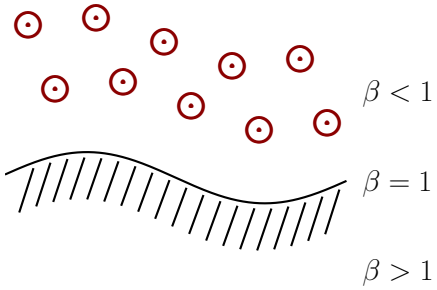

Figure 7.3: Plasma and magnetic field boundary and definition of plasma β,

7.2.1 Diffusion of the magnetic field into plasmas

Let us discuss the system of two limited half-spaces, one of them filled with plasma only, and the other one containing the magnetic field in a vacuum (Fig. 7.3). In the case where there is no resistivity in plasmas, the two regions will remain disconnected. The electromotive force induced by any possible movement of the field would drive an infinite electric current on the boundary separating these two regions, but the field would never penetrate the plasma. However, the shape of the boundary may change significantly. If a pressure balance (7.8) is to be held, then it must be that if we somehow *force* the magnetic field to penetrate the plasma region, the gas must leave such a region.

Since (7.8) holds, automatically $\beta = 1$ on the boundary. For the "plasma half-space" the second term vanishes and vice versa for the "magnetic field half-space". However, by considering the finite resistivity of the plasma we practically allow the two distinct "fluids" to mix. In the following, we want to estimate the time scale of this process. Let us start from Faraday's law of induction, Ampére's law, and generalised Ohm's law:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{7.10}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} \tag{7.11}$$

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j} \tag{7.12}$$

Let us assume that the plasma is stationary, hence $\boldsymbol{u} = 0$ and thus the second term on the left-hand side of Ohm's law vanishes. Note that even when we assume no bulk motion of the plasma, electric currents still might flow through, as we have two distinct charged species in the plasma. We remind the reader that the velocity of the bulk motion is defined as

$$\boldsymbol{u} = \frac{1}{\rho}(n_\mathrm{i} M \boldsymbol{u}_\mathrm{i} + n_\mathrm{e} m \boldsymbol{u}_\mathrm{e}) \sim \frac{M\boldsymbol{u}_\mathrm{i} + m\boldsymbol{u}_\mathrm{e}}{M+m}, \tag{7.13}$$

hence as a weighted mean of the velocities of the two species, while the current is defined as a *difference* of the velocities of the two species:

$$\boldsymbol{j} = e(n_\mathrm{i}\boldsymbol{u}_\mathrm{i} - n_\mathrm{e}\boldsymbol{u}_\mathrm{e}) \sim ne(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}). \tag{7.14}$$

By combining equations (7.10), (7.11), and (7.12) we have

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \eta \boldsymbol{j} = -\frac{\eta}{\mu_0} \left[\nabla(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \triangle \boldsymbol{B}\right] = \frac{\eta}{\mu_0} \triangle \boldsymbol{B}, \tag{7.15}$$

where we used $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. The above equation is a type of "heat-conduction equation" and may be understood using the following model: the equation describes the temporal evolution of the magnetic field when it is distorted from the equilibrium. We seek the solution and study whether it is possible to recover the equilibrium and at what cost.

Such an equation may be solved using the separation of variables. For simplicity, we approximate the ∇ operator to represent division over the *length-scale* L, which may be understood as the penetration depth of the magnetic field. Then we solve the equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \frac{\eta}{\mu_0 L^2} \boldsymbol{B}, \tag{7.16}$$

which has a solution in a form of an exponential

$$\boldsymbol{B} = \boldsymbol{B}_0 \exp\left[\pm \frac{t}{\tau}\right], \tag{7.17}$$

from which we only take the negative solution as physical since we solve for the dissipation of the magnetic field into plasma.

$$\tau = \frac{\mu_0 L^2}{\eta} \tag{7.18}$$

is the characteristic time-scale of the penetration of the magnetic field into plasma.

The solution might also be understood as the magnetic field dissipating due to the induction of electric currents in plasma, which dissipate via Ohmic heating. Let us estimate how much of the energy of the magnetic field dissipates due to Ohmic heating processes.

The loss output of Ohmic heating is given by

$$P = \eta j^2. \tag{7.19}$$

Hence the energy drained from the magnetic field over the time τ is

$$\Delta W = \eta j^2 \tau. \tag{7.20}$$

The electric current is given by Ampére's law (7.11). Hence the energy loss (again, when approximating the ∇ operator) we have

$$\Delta W = \eta j^2 \tau = \eta \left(\frac{B}{\mu_0 L}\right)^2 \frac{\mu_0 L^2}{\eta} = \frac{B^2}{\mu_0}. \tag{7.21}$$

It turned out that all the energy of the magnetic field dissipates and heats up the plasma. Actually it is larger by a factor of two, which is due to our rough estimates.

7.3 Instabilities

Systems out of thermodynamical equilibrium contain some free energy, which may drive the rise of random fluctuations. In this case, for instance, the self-excitation of plasma waves is possible. There are no other means to self-excite plasma waves. Of course, they might be driven from the outside. In this case, we speak about the unstable equilibrium.

We recognise four basic types of plasma instabilities and in further sections, we will study some of the examples.

Stream instabilities A stream of plasma or an electric current propagates through plasma. A spectrum of waves is excited that leads to an instability of the system.

Rayleigh-Taylor instability Should there be a sharp density gradient in plasmas or the plasma be confined *and* there is an external force (for instance, gravity), the external force drives the growth of instabilities. A classical example of a non-plasmatic Rayleigh-Taylor instability is a glass of water turned upside down when the atmospheric pressure supports the water in the glass. The system is allowed to be in an unstable equilibrium, however once perturbed, the instability grows.

Universal instability is called universal because it is always present in confined plasmas (and the plasmas are always confined somehow). The confined plasma is not in total thermodynamical balance even when there are no important external forces. The pressure (expanding the plasma) drives the instabilities.

Kinetic instability The kinetic instability is driven by the non-equilibrium distribution function (usually non-Maxwellian). An anisotropy in the distribution function may drive the instabilities.

7.3.1 Two-stream instability

Let us study an example of stream instabilities, the two-stream instability. Let us assume a homogeneous plasma in the coordinate system, where ions are stationary. Let us allow electrons to have a systematic bulk speed $\boldsymbol{u}_0$ with respect to the ion background. We assume $\nabla \cdot \boldsymbol{u}_0 = 0$. The plasma is cold ($T_\mathrm{i} = T_\mathrm{e} = 0$) and there is no magnetic field (hence $\boldsymbol{B}_0 = 0$, $\boldsymbol{B}_1 = 0$).

Then the linearised Euler equations read:

$$Mn_0 \frac{\partial \boldsymbol{u}_{\mathrm{i}1}}{\partial t} = en_0 \boldsymbol{E}_1, \tag{7.22}$$

$$mn_0 \left[\frac{\partial \boldsymbol{u}_{\mathrm{e}1}}{\partial t} + (\boldsymbol{u}_0 \cdot \boldsymbol{\nabla}) \boldsymbol{u}_{\mathrm{e}1} \right] = -en_0 \boldsymbol{E}_1. \tag{7.23}$$

We search for electrostatic waves propagating in the x direction, where $\boldsymbol{E}_1 \parallel \boldsymbol{e}_x \parallel \boldsymbol{k} \parallel \boldsymbol{u}_0$. Let us use the approach of solving the equations in the Fourier space (using the Fourier transform (5.10)) similarly to the problem of deriving the dispersion relation of

plasma waves. Hence

$$-\mathrm{i}\omega M u_{\mathrm{i}1} = eE_1, \tag{7.24}$$

$$m(-\mathrm{i}\omega + \mathrm{i}ku_0)u_{\mathrm{e}1} = -eE_1. \tag{7.25}$$

The continuity equation for ions reads (with assumptions applied: $\nabla n_0 = 0$ and $\boldsymbol{u}_{\mathrm{i}0} \equiv 0$)

$$\frac{\partial n_{\mathrm{i}1}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}1} = 0, \tag{7.26}$$

where we again consider only the x-component of the differential operator. We express $n_{\mathrm{i}1}$ and further use (7.24) to get

$$n_{\mathrm{i}1} = \frac{k}{\omega} n_0 u_{\mathrm{i}1} = \frac{\mathrm{i}en_0 k}{M\omega^2} E_1. \tag{7.27}$$

Similarly, the continuity equation for electrons reads

$$\frac{\partial n_{\mathrm{e}1}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e}1} + (\boldsymbol{u}_0 \cdot \boldsymbol{\nabla}) n_{\mathrm{e}1} = 0, \tag{7.28}$$

hence the parallel component in the Fourier space is

$$(-\mathrm{i}\omega + \mathrm{i}ku_0)n_{\mathrm{e}1} + \mathrm{i}kn_0 u_{\mathrm{e}1} = 0, \tag{7.29}$$

from which we express $n_{\mathrm{e}1}$ by using (7.25):

$$n_{\mathrm{e}1} = \frac{kn_0 u_{\mathrm{e}1}}{\omega - ku_0} = -\frac{\mathrm{i}ekn_0}{m(\omega - ku_0)^2} E_1. \tag{7.30}$$

We do not assume the plasma approximation to be valid in this case, thus we further use Poisson equation to finally derive the dispersion relation

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}_1 = \frac{e}{\varepsilon_0}(n_{\mathrm{i}1} - n_{\mathrm{e}1}), \tag{7.31}$$

where the density perturbations of the two species are given by (7.27) and (7.30). Then

$$\mathrm{i}kE_1 = \frac{e}{\varepsilon_0} \mathrm{i}ekn_0 \left[\frac{1}{M\omega^2} + \frac{1}{m(\omega - ku_0)^2} \right] E_1 \tag{7.32}$$

and thus

$$1 = \frac{e^2 n_0}{\varepsilon_0 m} \left[\frac{m/M}{\omega^2} + \frac{1}{(\omega - ku_0)^2} \right] = \omega_{\mathrm{p}}^2 \left[\frac{m/M}{\omega^2} + \frac{1}{(\omega - ku_0)^2} \right]. \tag{7.33}$$

This is the dispersion relation of the two-stream instability. Please note that for $u_0 = 0$ and $m/M \to 0$ this relation degenerates to $\omega^2 = \omega_{\mathrm{p}}^2$, hence to normal plasma oscillations.

This is an equation of the fourth-order in ω. The equation may either have four real roots, each of them corresponding to an individual oscillatory mode, hence it will not describe an instability. The other option is that the equation will have two real roots

and the remaining two will be complex, one conjugated to the other. Let us discuss these two. They might be symbolically written as

$$\omega_j = \alpha_j + \mathrm{i}\gamma_j, \quad \text{where} \quad \alpha_j = \Re[\omega_j] \quad \text{and} \quad \gamma_j = \Im[\omega_j]. \tag{7.34}$$

In the case of the complex roots, the solution for e.g. the electric field then reads

$$E_1 = E\boldsymbol{e}_x \exp\left[\mathrm{i}(kx - \omega t)\right] \mathrm{e}^{\gamma_j t}. \tag{7.35}$$

Since the two complex roots are complexly conjugated, one of them will have $\gamma_j > 0$ and at least one of the modes will depict an exponential growth of the perturbation.

To study the dispersion relation at least qualitatively, we introduce new variables $\xi = \omega/\omega_\mathrm{p}$ and $\xi_0 = ku_0/\omega_\mathrm{p}$. Then the dispersion relation reads

$$1 = \frac{m/M}{\xi^2} + \frac{1}{(\xi - \xi_0)^2} = F(\xi; \xi_0). \tag{7.36}$$

We draw $F(\xi; \xi_0)$ as a function of ξ for a given ξ_0. Two possible plots are displayed in Fig. 7.4.

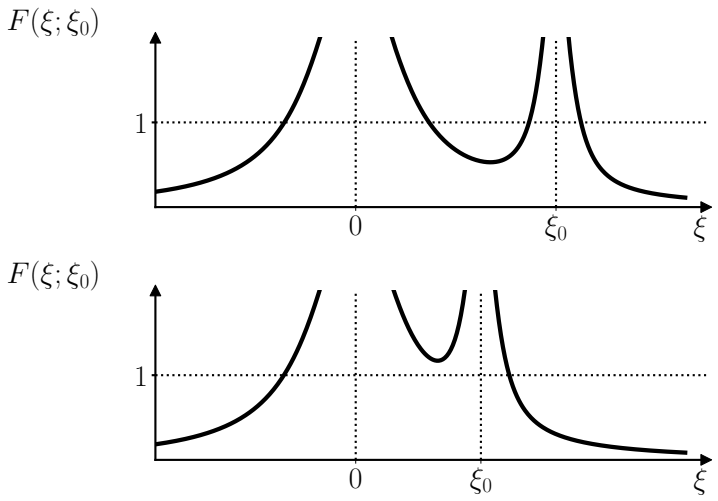

Figure 7.4: A graphical solution of the dispersion relation of the two-stream instability. In the case displayed above, the solution has four real roots. No instability takes place. In the case below, only two real roots solve the dispersion relation. The remaining root must necessarily have an imaginary component responsible for the instability.

It turns out that for small ξ_0 the plasma is unstable (function $F(\xi; \xi_0)$ has two conjugated complex roots). In other words, for a given u_0, the plasma is unstable with respect to perturbations with small k (and hence large wavelength). In this case, an internal inconsistency appears. The linearisation approach to solve the system of initial equations will no longer be valid, the assumptions of small perturbations will break. Our solution is therefore no longer correct. We just derived that such a solution is

qualitatively unstable (that is a correct result), however quantitatively (deriving the exponential growth) is not correct.[1]

7.3.2 Gravitational instability

Let us consider another example which belongs in the family of Rayleigh-Taylor instabilities: the space is divided into two half-spaces, one empty and one occupied by plasma. The Cartesian coordinate system is set so that the $\boldsymbol{e}_x$ is normal to the boundary between the two half-spaces and $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ lie within the boundary, perpendicular to each other. The background magnetic field is parallel to z-axis. The density gradient of the plasma is parallel to the x-axis. The plasma is cold and the magnetic field is homogeneous. We allow for the plasma bulk velocity to be parallel to y-axis. Then in the state of equilibrium, the Euler equation reads

$$Mn_{\mathrm{i}0}[(\boldsymbol{u}_{\mathrm{i}0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{i}0}] = en_{\mathrm{i}0}\boldsymbol{u}_{\mathrm{i}0}\times\boldsymbol{B}_0 + Mn_{\mathrm{i}0}\boldsymbol{g}. \tag{7.37}$$

Note that we added an *ad hoc* term of the gravity force. For $\boldsymbol{g}$ constant, $\boldsymbol{u}_{\mathrm{i}0}$ is also constant and hence $(\boldsymbol{u}_{\mathrm{i}0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{i}0}\to 0$. Then

$$0 = en_{\mathrm{i}0}\boldsymbol{u}_{\mathrm{i}0}\times\boldsymbol{B}_0 + Mn_{\mathrm{i}0}\boldsymbol{g} \tag{7.38}$$

We multiply (7.38) by $\boldsymbol{B}_0$ to obtain

$$0 = e\boldsymbol{B}_0\times(\boldsymbol{u}_{\mathrm{i}0}\times\boldsymbol{B}_0) + M\boldsymbol{B}_0\times\boldsymbol{g} = eB_0^2\boldsymbol{u}_{\mathrm{i}0} - e\boldsymbol{B}_0\cdot\boldsymbol{u}_{\mathrm{i}0}\boldsymbol{B}_0 + M\boldsymbol{B}_0\times\boldsymbol{g}, \tag{7.39}$$

where $\boldsymbol{B}_0\cdot\boldsymbol{u}_{\mathrm{i}0}$ due to the orthogonality of these vectors and hence

$$\boldsymbol{u}_{\mathrm{i}0} = -\frac{M}{e}\frac{\boldsymbol{B}_0\times\boldsymbol{g}}{B_0^2} = -\frac{g}{\Omega_{\mathrm{c}}}\boldsymbol{e}_y, \tag{7.40}$$

where $\boldsymbol{u}_{\mathrm{i}0}$ is the velocity of the ion drift in the field of gravity. The drift of electrons is negligible since they have an m/M-times smaller effect. Regarding the other drifts, there is no E-B drift, since $\boldsymbol{E}_0 = 0$, there is also no diamagnetic drift because we assumed cold plasma. However, the conclusions might be different when the boundary is perturbed and corrugated (see Fig. 7.5). Then the following effects occur:

1. The ions drift and will cause an accumulation of charge. The electrons react with some delay.
2. The accumulated charges cause the additional electric field $\boldsymbol{E}_1$.
3. Now the E-B drift appears with drift velocity pointing parallel to the normal direction to the boundary.

[1] The solution has the form of growth in time. The growth factor gives an approximate idea for the timescale of the growth of the instability. It is also important to compare this timescale with the timescale of the study we are performing. Should, for instance, the time span of the experiment be much shorter than the instability timescale, the instability will simply not have enough time to fully develop and may thus be neglected.

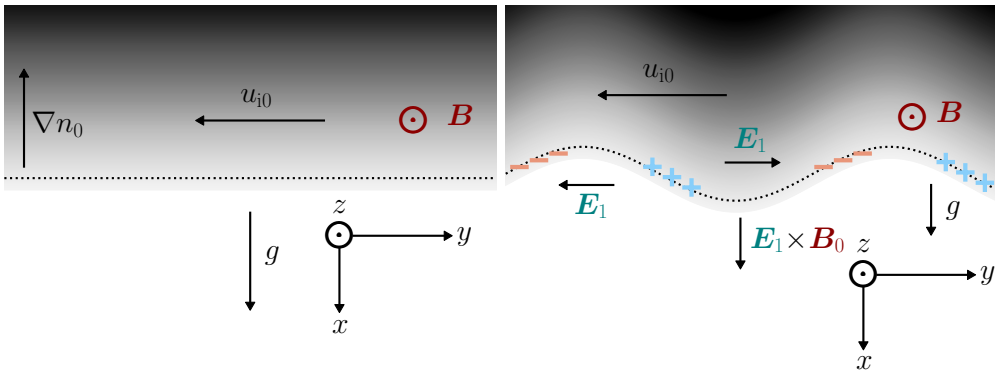

Figure 7.5: Left: The coordinate setup for the gravitational instability. Note that the bulk velocity $\boldsymbol{u}$ is a mass-weighted combination of background electron $\boldsymbol{u}_{e0}$ and ion $\boldsymbol{u}_{i0}$ velocities. However, in the drift of the gravity field, $\boldsymbol{u}_{e0} \ll \boldsymbol{u}_{i0}$. Right: The E-B drift in the evolved gravitational instability.

Let us search for the increment of the growth. Let us assume that there is a wave propagating parallel to $\boldsymbol{e}_y$. For ions, we introduce perturbations to the density and velocity, similarly as we did in the case of the plasma waves. The Euler equation reads:

$$M(n_{i0}+n_{i1})\left\{\frac{\partial}{\partial t}(\boldsymbol{u}_{i0}+\boldsymbol{u}_{i1})+[(\boldsymbol{u}_{i0}+\boldsymbol{u}_{i1})\cdot\boldsymbol{\nabla}](\boldsymbol{u}_{i0}+\boldsymbol{u}_{i1})\right\}= \\ e(n_{i0}+n_{i1})\left[\boldsymbol{E}_1+(\boldsymbol{u}_{i0}+\boldsymbol{u}_{i1})\times\boldsymbol{B}_0\right]+M(n_{i0}+n_{i1})\boldsymbol{g} \tag{7.41}$$

We further proceed slightly differently from the procedure we took when studying the waves. We append (7.41) with (7.37) multiplied by $1+n_{i1}/n_{i0}$ to obtain the system of two equations:

$$M(n_{i0}+n_{i1})\left\{\frac{\partial}{\partial t}(\boldsymbol{u}_{i0}+\boldsymbol{u}_{i1})+(\boldsymbol{u}_{i0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i0}+(\boldsymbol{u}_{i0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i1}+(\boldsymbol{u}_{i1}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i0}\right\}= \\ e(n_{i0}+n_{i1})\left[\boldsymbol{E}_1+(\boldsymbol{u}_{i0}+\boldsymbol{u}_{i1})\times\boldsymbol{B}_0\right]+M(n_{i0}+n_{i1})\boldsymbol{g} \tag{7.42}$$

$$M(n_{i0}+n_{i1})(\boldsymbol{u}_{i0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i0}=(n_{i0}+n_{i1})e\boldsymbol{u}_{i0}\times\boldsymbol{B}_0+M(n_{i0}+n_{i1})\boldsymbol{g}. \tag{7.43}$$

By subtracting the two equations, some of the terms cancel each other out, and when we further use $\frac{\partial \boldsymbol{u}_{i0}}{\partial t}=0$ and $(\boldsymbol{u}_{i1}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i0}=0$ we have to the first order

$$Mn_{i0}\left[\frac{\partial \boldsymbol{u}_{i1}}{\partial t}+(\boldsymbol{u}_{i0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i1}\right]=en_{i0}(\boldsymbol{E}_1+\boldsymbol{u}_{i1}\times\boldsymbol{B}_0). \tag{7.44}$$

It seems that apparently the term with the gravity acceleration disappeared. An explicit term containing $\boldsymbol{g}$ indeed disappeared, however we have to keep in mind that the gravity acceleration is still hidden in $\boldsymbol{u}_{i0}$!

Equation (7.44) in the Fourier space transforms to

$$M(\omega-\boldsymbol{k}\cdot\boldsymbol{u}_{i0})\boldsymbol{u}_{i1}=\mathrm{i}e(\boldsymbol{E}_1+\boldsymbol{u}_{i1}\times\boldsymbol{B}_0). \tag{7.45}$$

Let us write this equation in components by assuming $E_{1x} = 0$, since we are searching for a wave propagating in the y-direction. We further use $\boldsymbol{k} = k\boldsymbol{e}_y$ and $\boldsymbol{u}_{\mathrm{i}0} = u_{\mathrm{i}0}\boldsymbol{e}_y$. For ions we have

$$u_{\mathrm{i}1x} = \frac{\mathrm{i}e}{M(\omega - ku_{\mathrm{i}0})} u_{\mathrm{i}1y} B_0, \tag{7.46}$$

$$u_{\mathrm{i}1y} = \frac{-\mathrm{i}e}{M(\omega - ku_{\mathrm{i}0})} (E_{1y} - u_{\mathrm{i}1x} B_0). \tag{7.47}$$

By combining the two we obtain for $u_{\mathrm{i}y}$

$$u_{\mathrm{i}1y} = \frac{-\mathrm{i}e}{M(\omega - ku_{\mathrm{i}0})} \left(E_{1y} + \frac{-\mathrm{i}eB_0^2}{M(\omega - ku_{\mathrm{i}0})} u_{\mathrm{i}1y} \right) \tag{7.48}$$

and hence

$$u_{\mathrm{i}1y} = \frac{-\mathrm{i}eE_{1y}}{M(\omega - ku_{\mathrm{i}0})} \left[1 - \frac{e^2 B_0^2 / M^2}{(\omega - ku_{\mathrm{i}0})^2} \right]^{-1} = \frac{-\mathrm{i}eE_{1y}}{M(\omega - ku_{\mathrm{i}0})} \left[1 - \frac{\Omega_{\mathrm{c}}^2}{(\omega - ku_{\mathrm{i}0})^2} \right]^{-1}. \tag{7.49}$$

To obtain a relation for $u_{\mathrm{i}1x}$ we simply insert (7.49) into (7.46) and have

$$u_{\mathrm{i}1x} = -\frac{e\Omega_{\mathrm{c}} E_{1y}}{M(\omega - ku_{\mathrm{i}0})^2} \left[1 - \frac{\Omega_{\mathrm{c}}^2}{(\omega - ku_{\mathrm{i}0})^2} \right]^{-1}. \tag{7.50}$$

By further assuming low-frequency waves, hence using approximation $\Omega_{\mathrm{c}}^2 \gg (\omega - ku_{\mathrm{i}0})^2$, we have

$$u_{\mathrm{i}1x} = \frac{E_{1y}}{B_0}, \quad u_{\mathrm{i}1y} = -\mathrm{i}\frac{\omega - ku_{\mathrm{i}0}}{\Omega_{\mathrm{c}}} \frac{E_{1y}}{B_0} \tag{7.51}$$

and analogously for electrons

$$u_{\mathrm{e}1x} = \frac{E_{1y}}{B_0}, \quad u_{\mathrm{e}1y} = -\mathrm{i}\frac{\omega - ku_{\mathrm{e}0}}{\omega_{\mathrm{c}}} \frac{E_{1y}}{B_0}, \tag{7.52}$$

where $\omega_{\mathrm{c}} \gg \Omega_{\mathrm{c}}$ and thus $u_{\mathrm{e}1y} \to 0$.

The continuity equation for ions is

$$\frac{\partial n_{\mathrm{i}1}}{\partial t} + \boldsymbol{\nabla} \cdot (n_{\mathrm{i}0} \boldsymbol{u}_{\mathrm{i}0}) + (\boldsymbol{u}_{\mathrm{i}0} \cdot \boldsymbol{\nabla}) n_{\mathrm{i}1} + n_{\mathrm{i}1} \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}0} + (\boldsymbol{u}_{\mathrm{i}1} \cdot \boldsymbol{\nabla}) n_{\mathrm{i}0} + n_{\mathrm{i}0} \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}1} + \boldsymbol{\nabla} \cdot (n_{\mathrm{i}1} \boldsymbol{u}_{\mathrm{i}1}) = 0. \tag{7.53}$$

The second term vanishes, because by assumptions $\nabla n_{\mathrm{i}0} \perp \boldsymbol{u}_{\mathrm{i}0}$, the fourth term vanishes because we assumed that $\boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}0} = 0$ and the last term on the left-hand side also vanishes because it is the second-order term. Hence in the Fourier space, we have

$$-\mathrm{i}\omega n_{\mathrm{i}1} + \mathrm{i}k u_{\mathrm{i}0} n_{\mathrm{i}1} + u_{\mathrm{i}1x} \frac{\partial n_{\mathrm{i}0}}{\partial x} + \mathrm{i}k n_{\mathrm{i}0} u_{\mathrm{i}1y} = 0. \tag{7.54}$$

Similarly, we construct the continuity equation for electrons, however it is simpler, because we assumed that $u_{\mathrm{e}0} \ll u_{\mathrm{i}0}$ and we showed that $u_{\mathrm{e}1y} \to 0$. We will also expect low-frequency waves, hence the use of plasma approximation $n_1 = n_{\mathrm{e}1} = n_{\mathrm{i}1}$ and

$n_0 = n_{e0} = n_{i0}$ is justified. In the end, we show that the results are consistent with the assumptions. Thus for electrons, we have

$$-\mathrm{i}\omega n_1 + u_{e1x}\frac{\partial n_0}{\partial x} = 0. \tag{7.55}$$

By combining (7.54) and (7.51) we have

$$-\mathrm{i}\omega n_1 + \mathrm{i}k u_{i0} n_1 + \frac{E_{1y}}{B_0}\frac{\partial n_0}{\partial x} - \mathrm{i}^2 k n_0 \frac{\omega - k u_{i0}}{\Omega_c}\frac{E_{1y}}{B_0} = 0, \tag{7.56}$$

which trivially turns into

$$(\omega - k u_{i0}) n_1 + \mathrm{i}\frac{E_{1y}}{B_0}\frac{\partial n_0}{\partial x} + \mathrm{i}k n_0 \frac{\omega - k u_{i0}}{\Omega_c}\frac{E_{1y}}{B_0} = 0. \tag{7.57}$$

Similarly we combine (7.55) and (7.52) to get the equivalent of (7.57):

$$-\mathrm{i}\omega n_1 + \frac{E_{1y}}{B_0}\frac{\partial n_0}{\partial x} = 0, \tag{7.58}$$

which allows us to express the term E_{1y}/B_0 and insert it into (7.57). Then

$$(\omega - k u_{i0}) n_1 - \frac{\omega n_1}{\frac{\partial n_0}{\partial x}}\frac{\partial n_0}{\partial x} - k n_0 \frac{\omega - k u_{i0}}{\Omega_c}\frac{\omega n_1}{\frac{\partial n_0}{\partial x}} = 0. \tag{7.59}$$

After some algebra we obtain

$$\omega(\omega - k u_{i0}) = -\frac{u_{i0}\Omega_c}{n_0}\frac{\partial n_0}{\partial x}. \tag{7.60}$$

Now we use (7.40) and have

$$\omega(\omega - k u_{i0}) = \frac{g}{n_0}\frac{\partial n_0}{\partial x}, \tag{7.61}$$

or

$$\omega^2 - k u_{i0}\omega - g\left(\frac{\frac{\partial n_0}{\partial x}}{n_0}\right) = 0. \tag{7.62}$$

This is the dispersion relation of the wave we searched for. The solution is

$$\omega = \frac{1}{2}k u_{i0} \pm \left[\frac{1}{4}k^2 u_{i0}^2 + g\left(\frac{\frac{\partial n_0}{\partial x}}{n_0}\right)\right]^{\frac{1}{2}}. \tag{7.63}$$

We will have an instability (and its exponential growth) in the case that ω is complex. That will be fulfilled for

$$-g\left(\frac{\frac{\partial n_0}{\partial x}}{n_0}\right) > \frac{1}{4}k^2 u_{i0}^2, \tag{7.64}$$

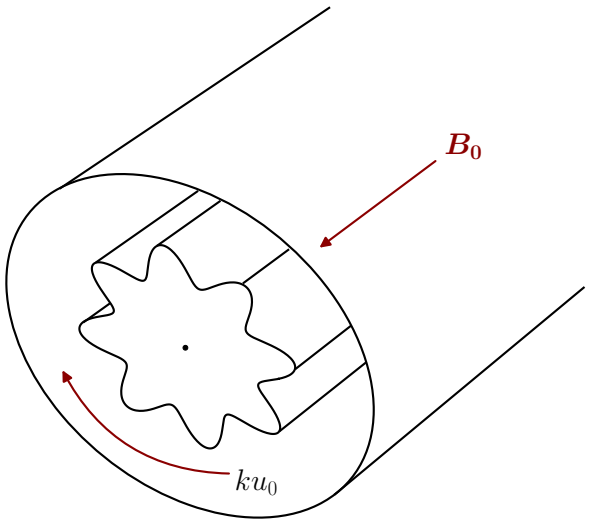

Figure 7.6: A flute instability.

thus $\boldsymbol{g}$ and ∇n_0 must have opposite signs. By waving our hands, "the lighter fluid supports the heavier one". We had a similar situation in the case of a glass of water turned upside down.

The interpretation is seen in Fig. 7.5 on the right. The corrugated boundary together with the flow along the boundary causes the concentration of charges on the opposite sides of the "wave". The perturbed E_1 electric field is induced by this charge separation and the resulting $\boldsymbol{v}_\mathrm{E}$ drift causes the initial perturbation to grow.

A similar instability appears in the column of plasma when the gravity force is replaced by the centrifugal force (Fig. 7.6). The conclusions are the same. The boundary of the plasma column is then corrugated and the instability is termed the *flute instability*. The onset of this instability is observed in experiments of laser-induced fusion. The idea of the laser-induced fusion is that the fusion begins faster than the flute instability sets on and disintegrates the plasma confinement (Fig. 7.7).

7.3.3 Universal instability

We will proceed similarly to the previous section, however, the $\boldsymbol{g} \times \boldsymbol{B}$ drift will be replaced by the diamagnetic drift. Since in this drift there is no scaling by mass of the particles, we have to solve equations for both ions and electrons.

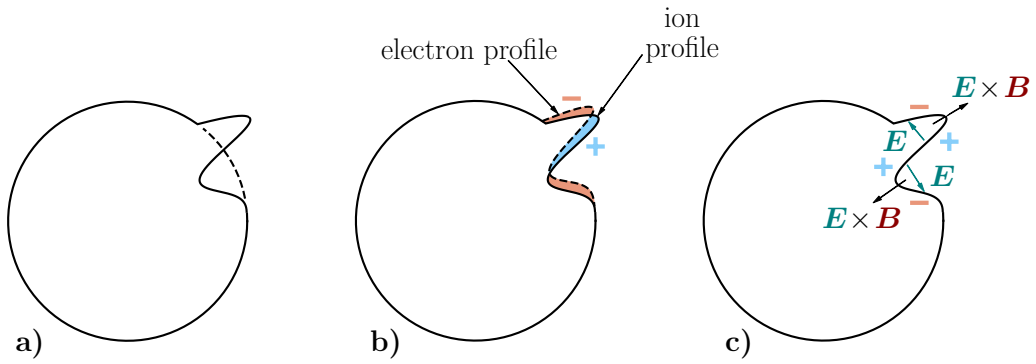

Figure 7.7: Development of the flute instability. a) Initial disturbance. b) Effect of ion and electron azimuthal drifts. c) Resulting $\boldsymbol{E} \times \boldsymbol{B}$ drifts increase the amplitude.

For the background velocities we have the relations:

$$\boldsymbol{u}_{\mathrm{i}0} = \boldsymbol{u}_{\mathrm{Di}} = \frac{K_{\mathrm{B}} T_{\mathrm{i}}}{eB_0} \frac{1}{n_0} \frac{\partial n_0}{\partial x} \boldsymbol{e}_y, \tag{7.65}$$

$$\boldsymbol{u}_{\mathrm{e}0} = \boldsymbol{u}_{\mathrm{De}} = -\frac{K_{\mathrm{B}} T_{\mathrm{e}}}{eB_0} \frac{1}{n_0} \frac{\partial n_0}{\partial x} \boldsymbol{e}_y, . \tag{7.66}$$

We expect that the component k_z of the wavevector is non-zero, as the motion of electrons along the background magnetic field is important to shield the electrostatic field induced by ions. We will further assume that the Boltzmann relation (4.92) fully describes the density perturbation of the electrons,

$$\frac{n_{\mathrm{e}1}}{n_0} = \frac{e\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}}}. \tag{7.67}$$

The geometry of the problem is sketched in Fig. 7.8.

At point A, the density is larger than in equilibrium, which is denoted by the solid line, hence $n_{\mathrm{e}1} > 0$ and thus $\phi_1 > 0$. At point B, the density is smaller than in equilibrium, hence $n_{\mathrm{e}1} < 0$ and $\phi_1 < 0$. Thus there must be an electric field E_1 between points A and B, which drives the E-B drift with the drift velocity

$$\boldsymbol{u}_1 = \frac{\boldsymbol{E}_1 \times \boldsymbol{B}_0}{B_0^2}. \tag{7.68}$$

There will be a wave in direction $\boldsymbol{e}_y$, hence both $n_{\mathrm{e}1}$ and ϕ_1 oscillate in time. Therefore $\boldsymbol{u}_{\mathrm{e}1}$ also oscillates in time and in fact the oscillations in the drift velocity are the cause of the oscillations of density. As a result, there will be a motion of the plasma fluid in the x direction. Let us support this physical scenario with equations. The Fourier image of (7.68) is

$$u_{1x} = \frac{E_y}{B_0} = -\frac{\mathrm{i}k_y \phi_1}{B_0}, \tag{7.69}$$

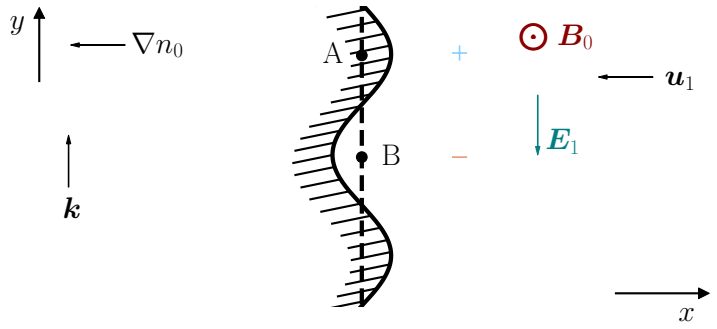

Figure 7.8: An illustration of the universal instability. The corrugated boundary induces the charge separation, which drives the perturbation of the electric field. This together with the background magnetic field yields the $E - B$ drift, which causes the instability to grow.

where we used the Poisson equation $\boldsymbol{E}_1 = -\nabla\phi_1 = -\mathrm{i}\boldsymbol{k}\phi_1$. The speed of the drift is the same for both ions and electrons. Let us further assume for simplicity, that the fluid of plasmas is incompressible and only oscillates in the x direction. Mathematically speaking: $u_{1x} \neq u_{1x}(x)$ and $k_z \ll k_y$. Then the continuity equation for divergence centres reads:

$$\frac{\partial n_1}{\partial t} = -u_{1x}\frac{\partial n_0}{\partial x}, \tag{7.70}$$

other components are being neglected. We express (7.70) in the Fourier space and use the Boltzmann relation to handle n_1 for electrons and (7.69) to handle u_{1x}. We have:

$$-\mathrm{i}\omega n_{\mathrm{e}1} = -u_{\mathrm{e}1x}\frac{\partial n_0}{\partial x} = \frac{\mathrm{i}k_y\phi_1}{B_0}\frac{\partial n_0}{\partial x} = -\mathrm{i}\omega n_0\frac{e\phi_1}{K_{\mathrm{B}}T_{\mathrm{e}}}, \tag{7.71}$$

which is solved as

$$\frac{\omega}{k} = -\frac{K_{\mathrm{B}}T_{\mathrm{e}}}{eB_0}\frac{1}{n_0}\frac{\partial n_0}{\partial x}, \tag{7.72}$$

which is exactly the expression (7.66). We confirmed the physical picture sketched in words a few paragraphs earlier. The perturbation waves propagate with the speed of the diamagnetic drift, hence termed *drift waves*.

However, the dispersion relation shows that no instability occurs, it does not have any term assuring the growth. We did not prove that this configuration is unstable. Let us only comment that the approach we took is not correct enough and that we neglected two important ingredients: the polarisation drift and the drift in the inhomogeneous electric field. The polarisation drift shows up in the case that we have a variable electric field. Then an additional drift with a speed of $v_{\mathrm{p}} = \operatorname{sgn} q\frac{1}{\omega_{\mathrm{c}}B}\frac{\mathrm{d}E}{\mathrm{d}t}$ needs to be taken into account. We will further only draw a physical picture in words: the corrections by the additional two drifts causes the phase shift between ϕ_1, which depicts a delay between u_1 and n_1. Hence, in places, where the plasma is already skewed ($n_1 > 0$), u_1 directs out of the plasma and vice versa. The perturbation on the boundary grows.

7.3.4 Landau damping

The highlight of these lecture notes consists of an example of the kinetic instability. We started this text with an introduction to statistical physics to describe important properties of an ensemble of particles. We will end the text by investigating oscillatory motions in plasmas, which occur in the case of the particle distribution function not being in equilibrium. We need to leave the fluid approximation and need to go one step deeper, to the single-particle distribution function. Let us remind the reader that for the single-particle distribution function f the Boltzmann equation may be derived,

$$\frac{\partial f}{\partial t} + \frac{\partial h}{\partial p_i}\frac{\partial f}{\partial r_i} - \frac{\partial h}{\partial r_i}\frac{\partial f}{\partial p_i} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}, \tag{7.73}$$

where $h = \frac{p^2}{2m} + V(r)$ is the Hamiltonian of one particle. We showed that this equation may be rewritten as

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f + \frac{F}{m}\cdot\frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}, \tag{7.74}$$

where F are the external forces coming from the external potential $V(r)$. By playing explicitly with the term on the right-hand side to extract the term of the self-potential, which has a similar shape as the action of the external potential, we got

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f + \frac{\langle F\rangle}{m}\cdot\frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)'_{\text{coll}}, \tag{7.75}$$

where $\langle F\rangle$ is the averaged action of both external forces and the self potential combined. Then the Vlasov approximation consists of neglecting the right-hand side, which is equivalent to neglecting the mutual correlation of distribution functions of different particles. We can go a step further, where we substitute $\langle F\rangle$ by the Lorentz force (as an effect of the sum of both external and internal electric and magnetic fields). Hence finally we have a Vlasov equation to solve:

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0, \tag{7.76}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ represent the joint action of both external and internal electric and magnetic fields respectively. In the following, we will neglect the external fields and in the case of the magnetic field, we will also neglect the internal one. Altogether, we use the following approximations:

1. $\boldsymbol{B} = 0$,
2. $\boldsymbol{E}_0 = 0$, but $\boldsymbol{E}_1 \neq 0$,
3. $f = f_0 + f_1$, $f_0 = f_0(v)$, $f_1 = f_1(t, \boldsymbol{r}, \boldsymbol{v})$,
4. all perturbations are expressed by means of the Fourier series as $A_1 = \bar{A}_1 \exp\left[\mathrm{i}(kx - \omega t)\right]$,

5. $A_1 \ll A_0$ and $\frac{\partial A_1}{\partial a} \ll \frac{\partial A_0}{\partial a}$. The perturbations are much smaller than the background values and also the derivatives of the perturbations are much smaller than the derivatives of the background.

6. We will further assume that the electric field only has a component along the $\hat{x}$-axis.

We solve the problem for electrons. Thus the Vlasov equation (7.76), under the assumptions listed above, reduces to

$$-\mathrm{i}\omega f_1 + \mathrm{i}k f_1 v_x = \frac{e}{m} E_{1x} \frac{\partial f_0}{\partial v_x}, \tag{7.77}$$

hence

$$f_1 = \frac{\mathrm{i}eE_{1x}}{m} \frac{\partial f_0/\partial v_x}{\omega - k v_x}. \tag{7.78}$$

we add the Poisson equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}_1 = \mathrm{i}kE_{1x} = -\frac{en_1}{\varepsilon_0} = -\frac{e}{\varepsilon_0} \int\limits_{-\infty}^{+\infty} f_1 \mathrm{d}^3\boldsymbol{v}. \tag{7.79}$$

Let us rescale the distribution function so that $f_0 = n_0 \hat{f}_0(\boldsymbol{v})$, where the normalisation of the newly introduced $\hat{f}_0$ is $\int \hat{f}_0 \mathrm{d}^3\boldsymbol{v} = 1$. This definition has the advantage of including n_0, which may be put outside of the integrals, because it is not a function of $\boldsymbol{v}$. Let us integrate (7.78) over the whole velocity space

$$\int\limits_{-\infty}^{+\infty} f_1 \mathrm{d}^3\boldsymbol{v} = \frac{\mathrm{i}eE_{1x}}{m} n_0 \int\limits_{-\infty}^{+\infty} \frac{\partial \hat{f}_0/\partial v_x}{\omega - k v_x} \mathrm{d}^3\boldsymbol{v} = -\frac{\mathrm{i}kE_{1x}\varepsilon_0}{e}, \tag{7.80}$$

where we also used (7.79). From that we obtain

$$1 = -\frac{e^2 n_0}{\varepsilon_0 m k} \int\limits_{-\infty}^{+\infty} \frac{\partial \hat{f}_0/\partial v_x}{\omega - k v_x} \mathrm{d}^3\boldsymbol{v} = +\frac{\omega_\mathrm{p}^2}{k^2} \int\limits_{-\infty}^{+\infty} \frac{\partial \hat{f}_0/\partial v_x}{v_x - \frac{\omega}{k}} \mathrm{d}^3\boldsymbol{v}. \tag{7.81}$$

Let us assume that $\hat{f}_0$ is a background equilibrium distribution, essentially having the shape of the Maxwell-Boltzmann distribution. Then we may decompose $\hat{f}_0$ as a product of

$$\hat{f}_0(\boldsymbol{v}) = \tilde{f}_0(v_x)\tilde{f}_0(v_y)\tilde{f}_0(v_z), \tag{7.82}$$

with the normalisation $\int \tilde{f}_0(v)\mathrm{d}v = 1$. Then (7.81) simplifies to

$$1 = \frac{\omega_\mathrm{p}^2}{k^2} \int\limits_{-\infty}^{+\infty} \mathrm{d}v_x \frac{\partial \tilde{f}_0(v_x)/\partial v_x}{v_x - \frac{\omega}{k}} \int\limits_{-\infty}^{+\infty} \mathrm{d}v_y \tilde{f}_0(v_y) \int\limits_{-\infty}^{+\infty} \mathrm{d}v_z \tilde{f}_0(v_z) = \frac{\omega_\mathrm{p}^2}{k^2} \int\limits_{-\infty}^{+\infty} \frac{\partial \tilde{f}_0/\partial v_x}{v_x - \frac{\omega}{k}} \mathrm{d}v_x. \tag{7.83}$$

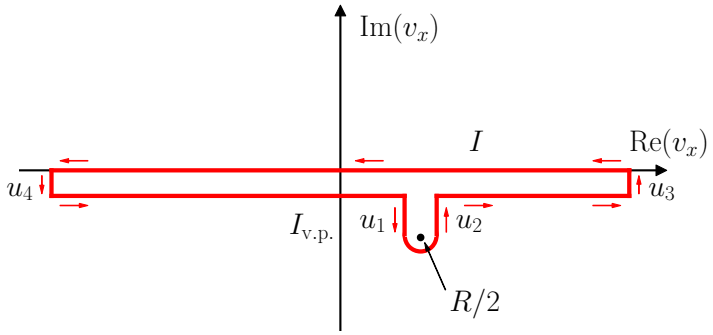

Figure 7.9: A sketch of the integration path including the pole in ω/k.

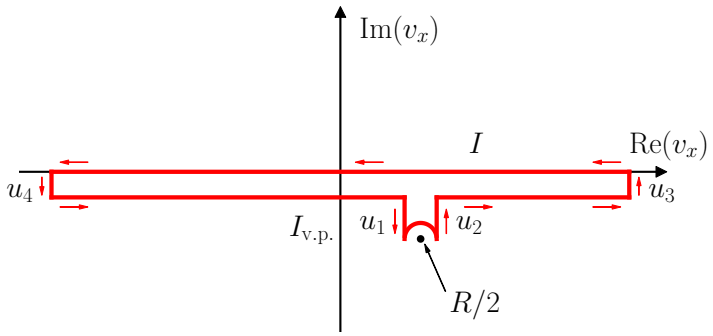

Figure 7.10: A sketch of the integration path bypassing the pole in ω/k.

We will let ω be complex. The function (7.83) has a pole in ω/k. By using the residuum theorem for the integration of complex functions, we must integrate along a closed curve, but bypass a pole, which results in an additional contribution of the residuum of this pole. According to the integration path displayed in Fig. 7.9 we have

$$-I + I_{\text{v.p.}} + u_1 + u_2 + \frac{R}{2} + u_3 + u_4 = 0, \tag{7.84}$$

where I is the integral we want to evaluate (note the negative sign as we integrate against the direction of the x axis), $I_{\text{v.p.}}$ is the principal value of this integral, $R = \left[2\pi \mathrm{i} \frac{\partial \tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}$ is a residuum in the pole and the segments u_3 and u_4 may be considered infinitesimally small. The segments u_1 and u_2 are in the close vicinity of the pole and we may assume that their length will be similar. Their orientation is opposite on the integration path, hence we may safely assume that their total contribution, effectively the *difference* of their lengths, will be negligible.

UNVERIFIED: Why did we take only one half of the residuum in the pole? We assume that the damping is small and therefore the pole is very close to the real axis, effectively lying on it. The principal value integral may then be computed as an average

of the integrals above and below the real axis. There is no pole (and no residuum) for the integral above the axis, for the integral below the axis one needs to account for the whole residuum, therefore the total contribution contains only one half of the residuum. The case of non-negligible damping is discussed elsewhere[2].

ALTERNATIVELY: We may also draw an integration path that bypasses the pole, as in Fig. 7.10. In that case, we use the integration path that avoids the pole with a semicircular detour towards the upper half of the complex plane. The theorem from complex analysis in mathematics says that if we bypass the pole of a complex differentiable (meromorphic) function using a trajectory in a form of an arc, we only need to consider a proportional fraction of the corresponding residuum in that pole. The fraction is then given as a ratio of the length of the arc and the length of the corresponding full circle. In our case, since we bypass the residuum using a half-circle, we only consider one half of the value of the residuum.

(7.83) may thus be written as

$$1 = \frac{\omega_\mathrm{p}^2}{k^2}\left\{\mathrm{v.p.}\int\limits_{-\infty}^{+\infty}\frac{\partial\tilde{f}_0/\partial v_x}{v_x-\frac{\omega}{k}}\mathrm{d}v_x + \left[\mathrm{i}\pi\frac{\partial\tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.85}$$

The integral may be solved by parts

$$\int\limits_{-\infty}^{+\infty}\frac{\partial\tilde{f}_0}{\partial v_x}\frac{\mathrm{d}v_x}{v_x-\omega/k} = \left[\frac{\tilde{f}_0}{v_x-\omega/k}\right]_{+\infty}^{-\infty} + \int\limits_{-\infty}^{+\infty}\frac{\tilde{f}_0\mathrm{d}v_x}{(v_x-\omega/k)^2}. \tag{7.86}$$

The first term vanishes due to the fast decay of the distribution function. Note that the phase speed $v_\varphi = \omega/k$ and let v_φ be positive, i.e., we will deal with the positive velocities only. Let's approximate using the Taylor expansion

$$(v_x - v_\varphi)^{-2} \sim v_\varphi^{-2}\left(1 + \frac{2v_x}{v_\varphi} + \frac{3v_x^2}{v_\varphi^2}\right). \tag{7.87}$$

Then (7.86) turns into

$$\begin{aligned}\int\limits_{-\infty}^{+\infty}\frac{\tilde{f}_0\mathrm{d}v_x}{(v_x-v_\varphi)^2} &\sim v_\varphi^{-2}\left[\int\limits_{-\infty}^{+\infty}\tilde{f}_0\mathrm{d}v_x + \int\limits_{-\infty}^{+\infty}\frac{2v_x}{v_\varphi}\tilde{f}_0\mathrm{d}v_x + \int\limits_{-\infty}^{+\infty}\frac{3v_x^2}{v_\varphi^2}\tilde{f}_0\mathrm{d}v_x\right] = \\ &= v_\varphi^{-2}\left[1 + \frac{2}{v_\varphi}\langle v_x\rangle + \frac{3\langle v_x^2\rangle}{v_\varphi^2}\right] = v_\varphi^{-2}\left(1 + \frac{3\langle v_x^2\rangle}{v_\varphi^2}\right),\end{aligned} \tag{7.88}$$

where we used $\langle v_x\rangle = 0$, which certainly is true for symmetrical velocity distributions, such as the Maxwell-Boltzmann one.

Let's continue to deal with (7.85):

$$1 = \frac{\omega_\mathrm{p}^2}{k^2}\frac{k^2}{\omega^2}\left\{1 + \frac{3\langle v_x^2\rangle}{v_\varphi^2} + \mathrm{i}\pi\left[\frac{\omega^2}{k^2}\frac{\partial\tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.89}$$

[2] T. H. Stix: Waves in Plasmas, 1992, AIP-Press, ISBN 978-0-88318-859-0

Let us take a step to the side. If we were to neglect the residuum in the pole, we would neglect the last term. By further using $\frac{1}{2}m\langle v_x^2\rangle = \frac{1}{2}K_\mathrm{B}T_\mathrm{e}$ we have

$$1 = \frac{\omega_\mathrm{p}^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}\frac{K_\mathrm{B}T_\mathrm{e}}{m}\right). \tag{7.90}$$

If we further assume that $\omega \to \omega_\mathrm{p}$ in the second-order (in other words $\omega^4 \sim \omega^2\omega_\mathrm{p}^2$) we have

$$\omega^2 = \omega_\mathrm{p}^2 + 3k^2\frac{K_\mathrm{B}T_\mathrm{e}}{m}, \tag{7.91}$$

which is the dispersion relation for electron electrostatic waves. By neglecting the contribution in the pole we obtained the classical result.

The new physics shows up when we properly treat the residuum in the pole. For simplicity, let's omit the term with the thermal motion, which (as we just showed) results in the electrostatic electron waves. The dispersion relation then reads

$$\omega^2 = \omega_\mathrm{p}^2\left\{1 + \mathrm{i}\pi\left[\frac{\omega^2}{k^2}\frac{\partial\tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.92}$$

Using the expansion to the Taylor series to the first order and assuming $\omega_\mathrm{p} > 0$ we have

$$\omega \sim \omega_\mathrm{p}\left\{1 + \frac{1}{2}\mathrm{i}\pi\left[\frac{\omega^2}{k^2}\frac{\partial\tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.93}$$

The Taylor expansion is justified because we expect ω to be very close to ω_p. In that case (7.92) may be written as

$$\frac{\omega^2}{\omega_\mathrm{p}^2} = \left\{1 + \mathrm{i}\pi\left[\frac{\omega^2}{k^2}\frac{\partial\tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\} \sim 1 \tag{7.94}$$

and thus the second term on the right-hand side must be small compared to unity. For $\omega \to \omega_\mathrm{p}$ the equation (7.93) then gives

$$\omega \sim \omega_\mathrm{p}\left\{1 + \frac{1}{2}\mathrm{i}\pi\left[\frac{\omega_\mathrm{p}^2}{k^2}\frac{\partial\tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.95}$$

This relation has a suitable form of a complex number with purely real and purely imaginary parts separated.

By going back to the definition of the Fourier series $A_1 = \bar{A}_1 \exp\left[\mathrm{i}(kx - \omega t)\right]$, the new results will come across as obvious. For $\frac{\partial\tilde{f}_0}{\partial v_x} > 0$ $\Im(\omega) > 0$ and thus we have an instability. For $\frac{\partial\tilde{f}_0}{\partial v_x} < 0$ $\Im(\omega) < 0$ the external perturbations (such as waves) are damped.

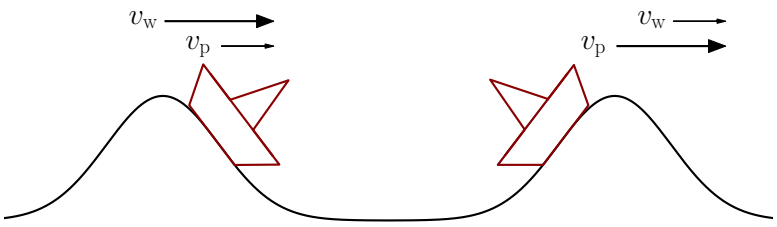

Figure 7.11: A system in the frame moving with phase speed v_φ. At point A the speed of the particle is a little smaller than v_φ, hence the particle falls behind to the point x', where the electric potential is higher and thus pushes the particle forward. Particles with $v < v_\varphi$ are being accelerated. On the contrary, particles with $v > v_\varphi$ are decelerated. (Not a language note but the points A, x' and speed v_{phi} are not included in the diagram)

So if we plot the distribution function and the perturbation appears at the given phase speed, the derivative of the distribution function decides whether this perturbation will be damped or will grow. Note that for the Maxwell-Boltzmann distribution, $\frac{\partial \tilde{f}_0}{\partial v_x} < 0$ for all speeds, thus for plasma in a perfect equilibrium the perturbations are always damped. This effect is termed the *Landau damping*.

For a Maxwell-Boltzmann distribution, we may obtain an explicit term of the imaginary part, which is

$$\Im\left(\frac{\omega}{\omega_\mathrm{p}}\right) \sim 0.22\sqrt{\pi}\left(\frac{\omega_\mathrm{p}}{k v_\mathrm{T}}\right)^3 \exp\left[-\frac{1}{2k^2\lambda_\mathrm{D}}\right], \tag{7.96}$$

which shows that the waves with wavenumbers comparable to the Debye shielding length are the ones most effectively damped.

A different situation arises when a stread of particles with a different typical speed perturbs the distribution function and creates a bump. Part of this bump will have $\frac{\partial \tilde{f}_0}{\partial v_x} > 0$. If the system is perturbed e.g. by the waves with the phase speeds in the susceptible interval, the instability will occur. An energy exchange between the wave and the particles will occur (see Fig. 7.11). A mixture of two Doppler-shifted Maxwellian distributions is unstable. An alternative interpretation can be seen in Fig. 7.12.

The Landau damping effect plays an important role in the heating of the solar corona (and also coronae of Sun-like stars), where the outer layers of the atmosphere (coronae) are much hotter (millions of degrees) than the photospheres located below (thousands of degrees). The heating is therefore due to a non-thermodynamical effect, such as the dissipation of the coronal currents flowing along with the loops of the magnetic fields, thermal energy from the all-scale reconnection processes and the heating due to the MHD waves propagating to the coronae. The waves exchange energy with the coronal plasma by means of Landau damping.

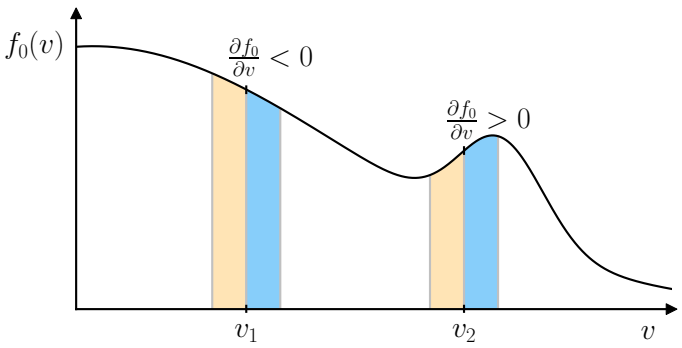

Figure 7.12: An interpretation of Landau damping: If in total there are fewer faster-than-the-wave particles than slower-than-the-wave particles, the energy streams from the wave to the particles and the perturbation is damped. This is the case of Landau damping when $\frac{\partial \tilde{f}_0}{\partial v_x} < 0$. The other case is obvious.

Appendices

Appendix A

Basics of statistical physics

A.1 Energy balance equation as the second moment of the Vlasov equation

The energy balance equation will be obtained by multiplying eq. (1.48) with $\frac{1}{2}m|\boldsymbol{v}|^2$ and integrating over the velocity space.

It is once again convenient to use the Einstein summation rule and perform some of the algebra in the components.

We remind the reader that

$$|\boldsymbol{v}|^2 = (u_i + w_i)(u_i + w_i) \tag{A.1}$$

for $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$, where $\boldsymbol{u}$ is the bulk velocity and $\boldsymbol{w}$ the random (thermal) component. We also remind the reader that

$$\rho\langle\mathcal{X}\rangle \equiv \int m\mathcal{X} f \,\mathrm{d}^3\boldsymbol{v}, \tag{A.2}$$

defines the mean value of quantity $\mathcal{X}$.

Multiplying the Boltzmann equation (only repeating Eq. 1.48)

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{r}} - \frac{1}{m}\boldsymbol{\nabla}\left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right]\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0\,. \tag{A.3}$$

by $\frac{1}{2}m|\boldsymbol{v}|^2$ and integrating over the velocity space we get

$$\int \frac{\partial f}{\partial t}\frac{1}{2}m|\boldsymbol{v}|^2\,\mathrm{d}^3\boldsymbol{v} + \int v_k\frac{\partial f}{\partial r_k}\frac{1}{2}m|\boldsymbol{v}|^2\,\mathrm{d}^3\boldsymbol{v} - \int \frac{1}{m}\frac{\partial}{\partial r_k}\left[V+\Phi\right]\frac{\partial f}{\partial v_k}\frac{1}{2}m|\boldsymbol{v}|^2\,\mathrm{d}^3\boldsymbol{v} = 0\,. \tag{A.4}$$

We will further deal with the terms of this equation one by one. In the first term, we swap the order of the time derivative and integration over velocity space, assuming that

it is mathematically possible. We further utilise the definition of the mean value (A.2) and decomposition of the velocity to bulk and random components.

$$\frac{\partial}{\partial t}\int \frac{1}{2}m|\boldsymbol{v}|^2 f\,\mathrm{d}^3\boldsymbol{v} = \frac{\partial}{\partial t}\langle\frac{\rho}{2}|\boldsymbol{v}|^2\rangle = \frac{\partial}{\partial t}\frac{\rho}{2}\langle u^2+w^2\rangle = \frac{\partial}{\partial t}\left[\frac{\rho}{2}u^2+\frac{\rho}{2}\langle w^2\rangle\right] = \\ = \frac{\partial}{\partial t}\left[\frac{\rho}{2}u^2+\rho\mathcal{E}\right], \tag{A.5}$$

where we used the definition of the internal energy as $\rho\mathcal{E} \equiv \rho\langle 1/2w^2\rangle$.

When dealing with the second term of (A.4), we will utilise the following relation:

$$\begin{aligned}&\langle(v_iv_i)v_k\rangle = \langle(u_i+w_i)^2(u_k+w_k)\rangle = \langle(u_i^2+2u_iw_i+w_i^2)(u_k+w_k)\rangle = \\ &= \langle u_i^2u_k+2u_iw_iu_k+w_i^2u_k+u_i^2w_k+2u_iw_iw_k+w_i^2w_k\rangle = \\ &= u^2u_k+u_k\langle w^2\rangle+2u_i\langle w_iw_k\rangle+\langle w_i^2w_k\rangle,\end{aligned} \tag{A.6}$$

where we used the fact that the mean value is a linear operator and $\langle w_i\rangle = 0$. We will further define the following quantities:

$$\text{heat conduction} \qquad F_k \equiv \rho\langle w_k\frac{1}{2}w^2\rangle, \tag{A.7}$$

$$\text{pressure} \qquad P \equiv \frac{1}{3}\rho\langle w^2\rangle, \tag{A.8}$$

$$\text{stress tensor} \qquad \pi_{ij} \equiv \rho\langle\frac{1}{3}|w^2|\delta_{ij}-w_iw_j\rangle = P\delta_{ij}-\rho\langle w_iw_j\rangle. \tag{A.9}$$

Note that from the last line it follows that $\langle w_iw_k\rangle = 1/\rho(P\delta_{ik}-\pi_{ik})$.

Using the identity (A.6) we may evaluate the second term of (A.4):

$$\int v_k\frac{\partial f}{\partial r_k}\frac{1}{2}m|\boldsymbol{v}|^2\,\mathrm{d}^3\boldsymbol{v} = \frac{\partial}{\partial r_k}\int v_kf\frac{1}{2}m|\boldsymbol{v}|^2\,\mathrm{d}^3\boldsymbol{v} = \\ = \frac{\partial}{\partial r_k}\frac{\rho}{2}\langle v^2v_k\rangle = \frac{\partial}{\partial r_k}\frac{\rho}{2}\langle(v_iv_i)v_k\rangle = \\ = \frac{\partial}{\partial r_k}\left[\frac{\rho}{2}u^2u_k+F_k+\rho\mathcal{E}u_k+u_i(P\delta_{ik}-\pi_{ik})\right]. \tag{A.10}$$

The third term of (A.4) gives:

$$\int\frac{1}{m}\frac{\partial\left[V+\Phi\right]}{\partial r_k}\frac{\partial f}{\partial v_k}\frac{1}{2}m|\boldsymbol{v}|^2\,\mathrm{d}^3\boldsymbol{v} = \frac{\partial\left[V+\Phi\right]}{\partial r_k}\int\frac{1}{2}\frac{m}{m}v_iv_i\frac{\partial f}{\partial v_k}\,\mathrm{d}^3\boldsymbol{v} = -\frac{\rho}{m}u_k\frac{\partial[V+\Phi]}{\partial r_k}, \tag{A.11}$$

where we used integration by parts and also the normalisation condition of the distribution function

$$\int v_iv_i\frac{\partial f}{\partial r_k}\,\mathrm{d}^3\boldsymbol{v} = [v_iv_if]_{-\infty}^{+\infty}-\int\frac{\partial(v_iv_i)}{\partial v_k}f\,\mathrm{d}^3\boldsymbol{v} = -2\int v_i\frac{\partial v_i}{\partial v_k}f\,\mathrm{d}^3\boldsymbol{v} = -2\int v_i\delta_{ik}f\,\mathrm{d}^3\boldsymbol{v} = \\ = -2\int v_kf\,\mathrm{d}^3\boldsymbol{v} = -2\frac{\rho}{m}u_k. \tag{A.12}$$

So in total, (A.4) may be written as

$$\frac{\partial}{\partial t}\left[\frac{\rho}{2}u^2+\rho\mathcal{E}\right]+\frac{\partial}{\partial r_k}\left[\frac{\rho}{2}u^2u_k+F_k+\rho\mathcal{E}u_k+u_i(P\delta_{ik}-\pi_{ik})\right]=-\frac{\rho}{m}u_k\frac{\partial[V+\Phi]}{\partial r_k}. \tag{A.13}$$

Now we take the Euler equation (1.60)

$$\rho\frac{\partial u_k}{\partial t}+\rho u_j\frac{\partial u_k}{\partial r_j}=-\frac{\rho}{m}\frac{\partial[V+\Phi]}{\partial r_k}-\frac{\partial P}{\partial r_k}+\frac{\partial\pi_{kj}}{\partial r_j} \tag{A.14}$$

and multiply it by u_k:

$$u_k\rho\frac{\partial u_k}{\partial t}+\rho u_ku_j\frac{\partial u_k}{\partial r_j}=-\frac{\rho}{m}u_k\frac{\partial[V+\Phi]}{\partial r_k}-u_k\frac{\partial P}{\partial r_k}+u_k\frac{\partial\pi_{kj}}{\partial r_j}. \tag{A.15}$$

The left-hand side becomes:

$$\begin{aligned}u_k\rho\frac{\partial u_k}{\partial t}+\rho u_ku_j\frac{\partial u_k}{\partial r_j}&=\frac{\rho}{2}\frac{\partial u_k^2}{\partial t}+\frac{\rho}{2}u_j\frac{\partial u_k^2}{\partial r_j}+\frac{u_k^2}{2}\frac{\partial\rho}{\partial t}+\frac{u_k^2}{2}\frac{\partial\rho u_j}{\partial r_j}=\\&=\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_k^2\right)+\frac{\partial}{\partial r_j}\left(\frac{1}{2}\rho u_ju_k^2\right)=\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}\right)+\frac{\partial}{\partial r_j}\left(\frac{\rho u^2}{2}u_j\right),\end{aligned} \tag{A.16}$$

where as a trick we added a zero term in a form of the $u^2/2$-multiple of the continuity equation. Hence the u_k-multiple of the Euler equation reads:

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}\right)+\frac{\partial}{\partial r_k}\left(\frac{\rho u^2}{2}u_k\right)=-\frac{\rho}{m}u_k\frac{\partial[V+\Phi]}{\partial r_k}-u_k\frac{\partial P}{\partial r_k}+u_k\frac{\partial\pi_{kj}}{\partial r_j}. \tag{A.17}$$

Now we subtract the previous from (A.13) and get

$$\frac{\partial\rho\mathcal{E}}{\partial t}=-\frac{\partial F_k}{\partial r_k}-\frac{\partial\rho\mathcal{E}u_k}{\partial r_k}-\frac{\partial}{\partial r_k}u_iP\delta_{ik}+u_k\frac{\partial P}{\partial r_k}+\frac{\partial u_i\pi_{ik}}{\partial r_k}-u_k\frac{\partial\pi_{kj}}{\partial r_j}. \tag{A.18}$$

The term on the left-hand side together with the second term on the right-hand side give

$$\frac{\partial\rho\mathcal{E}}{\partial t}+\frac{\partial\rho\mathcal{E}u_k}{\partial r_k}=\mathcal{E}\frac{\partial\rho}{\partial t}+\rho\frac{\partial\mathcal{E}}{\partial t}+\mathcal{E}\frac{\partial u_k\rho}{\partial r_k}+\rho u_k\frac{\partial\mathcal{E}}{\partial r_k}=\rho\frac{\partial\mathcal{E}}{\partial t}+\rho u_k\frac{\partial\mathcal{E}}{\partial r_k}=\rho\frac{\partial\mathcal{E}}{\partial t}+\rho\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\mathcal{E}=\rho\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}, \tag{A.19}$$

where in the first step we identified the $\mathcal{E}$-multiple of the continuity equation, hence two terms cancelled out.

The third and forth terms on the right-hand side of (A.18) give

$$-\frac{\partial}{\partial r_k}u_iP\delta_{ik}+u_k\frac{\partial P}{\partial r_k}=-\frac{\partial u_kP}{\partial r_k}+u_k\frac{\partial P}{\partial r_k}=-P\frac{\partial u_k}{\partial r_k}-u_k\frac{\partial P}{\partial r_k}+u_k\frac{\partial P}{\partial r_k}=-P\frac{\partial u_k}{\partial r_k}. \tag{A.20}$$

And finally, the fifth and sixth terms on the right-hand side of (A.18) give

$$\frac{\partial u_i\pi_{ik}}{\partial r_k}-u_k\frac{\partial\pi_{kj}}{\partial r_j}=\frac{\partial u_i\pi_{ik}}{\partial r_k}-u_i\frac{\partial\pi_{ik}}{\partial r_k}=\pi_{ik}\frac{\partial u_i}{\partial r_k}+u_i\frac{\partial\pi_{ik}}{\partial r_k}-u_i\frac{\partial\pi_{ik}}{\partial r_k}=\pi_{ik}\frac{\partial u_i}{\partial r_k}. \tag{A.21}$$

The resulting expression is usually termed the local heat dissipation rate, $\Psi \equiv \pi_{ik}\frac{\partial u_i}{\partial r_k}$.

Putting the intermediate results together we finally have

$$\rho\frac{\partial \mathcal{E}}{\partial t} + \rho u_k \frac{\partial \mathcal{E}}{\partial r_k} = -\frac{\partial F_k}{\partial r_k} - P\frac{\partial u_k}{\partial r_k} + \pi_{ik}\frac{\partial u_i}{\partial r_k} \tag{A.22}$$

or in a vector form

$$\rho\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -\boldsymbol{\nabla}\cdot\boldsymbol{F} - P\boldsymbol{\nabla}\cdot\boldsymbol{u} + \boldsymbol{\pi} : (\boldsymbol{\nabla}\boldsymbol{u}), \tag{A.23}$$

where we need to stress out that the object $(\boldsymbol{\nabla}\boldsymbol{u})$ is a dyadic (a second-order tensor) and the colon (:) operator indicates the double-scalar product between the tensors.

A.2 Properties of the stress tensor and heat flow with the isotropic distribution function

Let us properly derive the properties of the stress tensor π_{ij} and (conductive) heat flow F_i in a special case of the distribution function, which is isotropic in the velocity space, hence $f(\boldsymbol{r},\boldsymbol{v}) = f(\boldsymbol{r},|\boldsymbol{v}|)$.

Note that the isotropic distribution function is a special case of the symmetric distribution function, for which we have $f(\boldsymbol{r},\boldsymbol{v}) = f(\boldsymbol{r},-\boldsymbol{v})$. In such a case, when splitting the particle velocity component into its bulk and fluctuating components, $v_i = u_i + w_i$ we have for the mean values

$$\langle w_i\rangle = 0,\ \langle v_i\rangle = u_i = 0,\ \forall i, \tag{A.24}$$

where we followed the description for the mean value using (A.2). This is due to the symmetry of the distribution function. Such useful properties will also be valid for the isotropic distribution function (which again is a special case of the symmetric one) and furthermore, the velocity dispersions in all directions will be the same, symbolically

$$\langle v_i^2\rangle = \langle w_i^2\rangle \equiv \langle w^2\rangle,\ \forall i. \tag{A.25}$$

Note that $\langle w^2\rangle$ indicates the velocity dispersion in any direction, which is a different quantity than the square of the length of the fluctuating velocity vector $\boldsymbol{w}$. We would indicate the latter as $|\boldsymbol{w}|^2$. For the isotropic distribution function, these two quantities differ by a factor of 3, which we will use immediately in what is to follow. For the stress tensor we can thus write:

$$\begin{aligned}\pi_{ij} &= \rho\left\langle\frac{1}{3}|\boldsymbol{w}|^2\delta_{ij} - w_i w_j\right\rangle = \rho\left[\frac{1}{3}\langle|\boldsymbol{w}|^2\rangle\delta_{ij} - \langle w_i w_j\rangle\right] = \\ &= \rho\left[\frac{1}{3}\left\langle\sum_{i=1}^{3} w_i^2\right\rangle\delta_{ij} - \langle w_i w_j\rangle\right] = \rho\left[\frac{1}{3}3\langle w^2\rangle\delta_{ij} - \langle w^2\rangle\delta_{ij}\right] = 0,\end{aligned} \tag{A.26}$$

where we applied the properties of the velocity dispersions for an isotropic system in the velocity space.

Similarly, for conductive heat flow we have

$$F_k = \frac{\rho}{2}\left\langle w_k|\boldsymbol{w}|^2\right\rangle = \frac{\rho}{2}\left\langle w\sum_{i=1}^{3} w^2\right\rangle = \frac{1}{2}m\int w\,3w^2 f\mathrm{d}^3\boldsymbol{v} = \frac{3}{2}m\int v^3 f\mathrm{d}^3\boldsymbol{v} = 0,\ \forall k \tag{A.27}$$

once again due to the symmetries (the odd powers of velocity component average to zero).

In the case of the symmetric distribution function ($f(\boldsymbol{r},\boldsymbol{v}) = f(\boldsymbol{r},-\boldsymbol{v})$), the system symmetry also simplifies the two considered quantites significantly. For the heat flow we have

$$\begin{aligned} F_k &= \frac{\rho}{2}\left\langle w_k|\boldsymbol{w}|^2\right\rangle = \frac{\rho}{2}\left\langle w_k\sum_{i=1}^{3} w_i^2\right\rangle = \frac{m}{2}\int v_k\sum_{i=1}^{3} v_i^2 f\mathrm{d}^3\boldsymbol{v} = \\ &= \frac{m}{2}\sum_{i=1}^{3}\int v_k v_i^2 f\mathrm{d}^3\boldsymbol{v} = \frac{m}{2}\left[\int v_k^3 f\mathrm{d}^3\boldsymbol{v} + \sum_{i\neq k}\int v_i^2\left(\int v_k f\mathrm{d}v_k\right)\mathrm{d}^2\boldsymbol{v}\right] = 0, \end{aligned} \tag{A.28}$$

where we symbolically split the integration element to $\mathrm{d}^3\boldsymbol{v} = \mathrm{d}v_k\mathrm{d}^2\boldsymbol{v}$. In the last step and the second term, note that the integral in the round parenthesis (integration of v_k) could in principle still depend on the two remaining velocity components. The symmetry of f in the velocity space implies the symmetry in the individual components, hence the averaging of the velocity component vanishes and the whole term becomes zero.

When assessing the properties of the stress tensor for the symmetric distribution function, let us go back to the velocity correlations, from which the stress-tensor components are defined. The elements of the correlation tensor $\langle v_i v_j\rangle = \int v_i v_j f\mathrm{d}^3\boldsymbol{v}$ substantially differ for the diagonal and non-diagonal elements. For the diagonal elements we trivially have

$$\langle v_i v_i\rangle = \langle v_i^2\rangle = \langle w_i^2\rangle, \tag{A.29}$$

hence on the diagonal, we have velocity dispersions in each direction. Note that the values of the diagonal terms need not to be the same. They are for the isotropic distribution function. For the non-diagonal terms we have

$$\langle v_i v_j\rangle = \frac{m}{\rho}\int v_i v_j f\mathrm{d}^3\boldsymbol{v} = \frac{m}{\rho}\int\left(\int v_i f\mathrm{d}v_i\right) v_j\mathrm{d}v_j\mathrm{d}v = 0, \tag{A.30}$$

where similarly to the previous case we split the integration element into $\mathrm{d}^3\boldsymbol{v} = \mathrm{d}v_i\mathrm{d}v_j\mathrm{d}v$ and did the averaging of the components with caution.

Since the velocity-correlation tensor has a diagonal form, the stress tensor π_{ij} must also have a diagonal form. However, only two of them are independent, as the sum of the diagonal elements is bound to the value of pressure P,

$$\pi_{ii} = \rho\left\langle\frac{1}{3}|\boldsymbol{w}|^2 - w_i^2\right\rangle \text{ and } \pi_{ij} = 0 \text{ for } i\neq j. \tag{A.31}$$

Appendix B

Guiding centre motion

B.1 Relativistic E-B drift

In Section 3.2 we derived an expression for the classical E-B drift. It is interesting to show that the E-B drift naturally shows up as a consequence of the Lorentz transform of the electromagnetic field tensor. That is we transform $F_{\mu\nu}$,

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \tag{B.1}$$

using the tensor of Lorentz transform Λ^{μ}_{ν}

$$\Lambda^{\mu}_{\nu} = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}, \tag{B.2}$$

where $\beta = v/c$ and $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. Note that the velocity v is not known yet and we will obtain it as a solution to the problem. The transformed $F'_{\xi\zeta}$ is

$$\begin{aligned} F'_{\xi\zeta} &= \Lambda^{\mu}_{\xi} F_{\mu\nu} \Lambda^{\nu}_{\zeta} = \\ &= \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}. \end{aligned} \tag{B.3}$$

Now we need to properly multiply these three matrices to obtain

$$F'_{\xi\zeta} = \begin{bmatrix} 0 & \gamma(E_x/c - \beta B_y) & \gamma(E_y/c + \beta B_x) & E_z/c \\ \gamma(\beta B_y - E_x/c) & 0 & B_z & \gamma(\beta E_x/c - B_y) \\ -\gamma(E_y/c + \beta B_x) & -B_z & 0 & \gamma(\beta E_y/c + B_x) \\ -E_z/c & \gamma(B_y - \beta E_x/c) & -\gamma(\beta E_y/c + B_x) & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & E'_x/c & E'_y/c & E'_z/c \\ -E'_x/c & 0 & B'_z & -B'_y \\ -E'_y/c & -B'_z & 0 & B'_x \\ -E'_z/c & B'_y & -B'_x & 0 \end{bmatrix}. \tag{B.4}$$

Hence

$$E'_x/c = \gamma(E_x/c - \beta B_y), \tag{B.5}$$
$$E'_y/c = \gamma(E_y/c + \beta B_x), \tag{B.6}$$
$$E'_z = E_z, \tag{B.7}$$
$$B'_x = \gamma(\beta E_y/c + B_x), \tag{B.8}$$
$$B'_y = \gamma(B_y - \beta E_x/c), \tag{B.9}$$
$$B'_z = B_z \tag{B.10}$$

are the transformation equations for all the components of the electromagnetic tensor. The target mutual velocity of both coordinate systems is the one when the effect of the magnetic and electric fields will be separable, thus when $\boldsymbol{E} \parallel \boldsymbol{B}$, which is an equivalent of $\boldsymbol{E} \times \boldsymbol{B} = 0$. For reasons which will show up later, let's compute the square of the cross product amplitude, which will certainly be zero for parallel vectors. Then by using the rules for a mixed product we have

$$|\boldsymbol{E} \times \boldsymbol{B}|^2 = (\boldsymbol{E} \times \boldsymbol{B}) \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \boldsymbol{E} \cdot [\boldsymbol{B} \times (\boldsymbol{E} \times \boldsymbol{B})] = E^2 B^2 - (\boldsymbol{E} \cdot \boldsymbol{B})^2 = 0 \tag{B.11}$$

Let's compute this relation for transformed vectors $\frac{\boldsymbol{E}'}{c}$ and $\boldsymbol{B}'$ and let's only deal with perpendicular components, because the parallel components are already separated. Doing this algebra we have

$$\left|\frac{\boldsymbol{E}'_\perp}{c}\right|^2 |\boldsymbol{B}'_\perp|^2 = \left[\gamma^2(E_x/c - \beta B_y)^2 + \gamma^2(E_y/c + \beta B_x)^2\right] \left[\gamma^2(\beta E_y/c + B_x)^2 + \gamma^2(B_y - \beta E_x/c)^2\right] =$$
$$= \gamma^4(E_x/c - \beta B_y)^2(\beta E_y/c + B_x)^2 + \gamma^4(E_x/c - \beta B_y)^2(B_y - \beta E_x/c)^2 +$$
$$+ \gamma^4(E_y/c + \beta B_x)^2(\beta E_y/c + B_x)^2 + \gamma^4(E_y/c + \beta B_x)^2(B_y - \beta E_x/c)^2 \tag{B.12}$$

and

$$\left|\frac{\boldsymbol{E}'_\perp}{c} \cdot \boldsymbol{B}'_\perp\right|^2 = \gamma^4(E_x/c - \beta B_y)^2(\beta E_y/c + B_x)^2 + \gamma^4(E_y/c + \beta B_x)^2(B_y - \beta E_x/c)^2 +$$
$$+ 2\gamma^4(E_x/c - \beta B_y)(E_y/c + \beta B_x)(\beta E_y/c + B_x)(B_y - \beta E_x/c). \tag{B.13}$$

Hence [we immediately see that when subtracting (B.13) from (B.12), the first terms cancel each other out, as do also the fourth term from (B.12) and second term from (B.13)]:

$$\begin{aligned}\left|\frac{\boldsymbol{E}'_\perp}{c}\cdot\boldsymbol{B}'\right|^2-\left|\frac{\boldsymbol{E}'_\perp}{c}\right|^2|\boldsymbol{B}'|^2 &= \gamma^4(E_x/c-\beta B_y)^2(B_y-\beta E_x/c)^2+\\ &+\gamma^4(E_y/c+\beta B_x)^2(\beta E_y/c+B_x)^2-\\ &-2\gamma^4(E_x/c-\beta B_y)(E_y/c+\beta B_y)(\beta E_y/c+B_x)(B_y-\beta E_x/c)=\\ &=\gamma^4\left[\left(\frac{E_x}{c}-\beta B_y\right)\left(B_y-\beta\frac{E_x}{c}\right)-\left(\frac{E_y}{c}+\beta B_x\right)\left(\beta\frac{E_y}{c}+B_x\right)\right]^2=\\ &=\gamma^4\left[-\beta\left(\frac{E_x^2}{c^2}+\frac{E_y^2}{c^2}+B_x^2+B_y^2\right)+(1+\beta^2)\left(\frac{E_y}{c}B_x-\frac{E_x}{c}B_y\right)\right]^2=\\ &=\gamma^4\left[-\beta\left(\frac{E^2}{c^2}+B^2\right)+(1+\beta^2)\left(\boldsymbol{B}\times\frac{\boldsymbol{E}}{c}\right)_z\right]=0,\end{aligned}\tag{B.14}$$

where we used

$$(1+\beta^2)\left(\boldsymbol{B}\times\frac{\boldsymbol{E}}{c}\right)_z=(1+\beta^2)\left(B_x\frac{E_y}{c}-B_y\frac{E_x}{c}\right).\tag{B.15}$$

Equation (B.14) may be true if and only if we have

$$\frac{\beta}{1+\beta^2}=\frac{\frac{\boldsymbol{E}}{c}\times\boldsymbol{B}}{\frac{E^2}{c^2}+B^2},\tag{B.16}$$

which implicitly defines the target speed v_E of the coordinate system, in which the effects of magnetic and electric fields are separable. This speed is hidden in $\beta=v_E/c$. We will not solve this equation properly but rather use an approximation of $\beta<1$, when we may use β^2 as a correction term and obtain

$$\boldsymbol{v}_E=(1+\beta^2)\frac{\boldsymbol{E}\times\boldsymbol{B}}{\frac{E^2}{c^2}+B^2}.\tag{B.17}$$

This is the relativistic version of (3.29). When assuming a non-relativistic case with $\beta\ll1$ and further $|\boldsymbol{E}|\ll c$, the two expressions equal exactly. The latter condition applies to most real-world systems.

October 1, 2026

B.2 Alternative derivation of (3.118)

$$
\begin{aligned}
\langle (\boldsymbol{e}_\perp \times \boldsymbol{b})(\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla})\boldsymbol{B}\rangle &= \left\langle \left[\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} \right] \left[\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \nabla_1 B \\ \nabla_2 B \\ \nabla_3 B \end{pmatrix} \right] \right\rangle = \\
&= \left\langle \begin{pmatrix} b\sin\phi \\ -b\cos\phi \\ 0 \end{pmatrix} (\cos\phi \nabla_1 B + \sin\phi \nabla_2 B) \right\rangle = \\
&= \left\langle \begin{pmatrix} b\sin\phi\cos\phi\nabla_1 B + b\sin^2\phi\nabla_2 B \\ -b\cos^2\phi\nabla_1 B - b\sin\phi\cos\phi\nabla_2 B \\ 0 \end{pmatrix} \right\rangle = \\
&= -\frac{1}{2} \begin{pmatrix} -b\nabla_2 B \\ b\nabla_1 B \\ 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} \times \begin{pmatrix} \nabla_1 B \\ \nabla_2 B \\ \nabla_3 B \end{pmatrix} = \\
&= -\frac{1}{2}\boldsymbol{b} \times \nabla B
\end{aligned} \tag{B.18}
$$

$$
\begin{aligned}
\langle (\boldsymbol{e}_\perp \times [\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla}]\, \boldsymbol{b}\rangle &= \left\langle \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \times \left[\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \nabla_1 \boldsymbol{b} \\ \nabla_2 \boldsymbol{b} \\ \nabla_3 \boldsymbol{b} \end{pmatrix} \right] \right\rangle = \\
&= \left\langle \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \times \left[(\cos\phi, \sin\phi, 0) \begin{pmatrix} \nabla_1 b_1 & \nabla_1 b_2 & \nabla_1 b_3 \\ \nabla_2 b_1 & \nabla_2 b_2 & \nabla_2 b_3 \\ \nabla_3 b_1 & \nabla_3 b_2 & \nabla_3 b_3 \end{pmatrix} \right] \right\rangle = \\
&= \left\langle \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\phi\nabla_1 b_1 + \sin\phi\nabla_2 b_1 \\ \cos\phi\nabla_1 b_2 + \sin\phi\nabla_2 b_2 \\ \cos\phi\nabla_1 b_3 + \sin\phi\nabla_2 b_3 \end{pmatrix} \right\rangle = \\
&= \left\langle \begin{pmatrix} \sin\phi\cos\phi\nabla_1 b_3 + \sin^2\phi\nabla_2 b_3 \\ -\cos^2\phi\nabla_1 b_3 - \cos\phi\sin\phi\nabla_2 b_3 \\ -\sin\phi\cos\phi\nabla_1 b_1 - \sin^2\phi\nabla_2 b_1 + \cos^2\phi\nabla_1 b_2 + \cos\phi\sin\phi\nabla_2 b_2 \end{pmatrix} \right\rangle = \\
&= \begin{pmatrix} \frac{1}{2}\nabla_2 b_3 \\ -\frac{1}{2}\nabla_1 b_3 \\ -\frac{1}{2}\nabla_2 b_1 + \frac{1}{2}\nabla_1 b_2 \end{pmatrix} = \frac{1}{2}(\boldsymbol{e}_1\nabla_2 - \boldsymbol{e}_2\nabla_1)b_3 + \frac{1}{2}\boldsymbol{b}(\nabla_1 b_2 - \nabla_2 b_1) = \\
&= \frac{1}{2}\boldsymbol{b}(\boldsymbol{\nabla} \times \boldsymbol{b})_3 + \mathcal{O}\left[(\nabla b_{[1,2]})^2\right] = \frac{1}{2}\boldsymbol{b}[\boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b})].
\end{aligned} \tag{B.19}
$$

Note that in the second step we expressed the dot product by means of matrix multiplication respecting the rule $\boldsymbol{a} \cdot \boldsymbol{b} = a^{\mathrm{T}} b = a b^{\mathrm{T}}$, where a is the matrix representation of the vector $\boldsymbol{a}$.

October 1, 2026

B.3 Derivation of equations (3.122)–(3.124)

Derive equations for variables in the perpendicular plane

$$\left\langle \frac{\mathrm{d}v_\parallel}{\mathrm{d}t} \right\rangle = \frac{E_\parallel}{\epsilon} - \frac{u_\perp^2}{2B}\nabla_\parallel B + \boldsymbol{v}_\mathrm{E} \cdot D_t \boldsymbol{b} \tag{B.20}$$

$$\left\langle \frac{\mathrm{d}u_\perp}{\mathrm{d}t} \right\rangle = \frac{v_\parallel u_\perp}{2B}\nabla_\parallel B - \frac{u_\perp}{2}(\boldsymbol{\nabla} \cdot \boldsymbol{v}_\mathrm{E} - \boldsymbol{b} \cdot \nabla_\parallel \boldsymbol{v}_\mathrm{E}) \tag{B.21}$$

$$\left\langle \frac{\mathrm{d}\phi}{\mathrm{d}t} \right\rangle = -\frac{B}{\epsilon} - \boldsymbol{e}_2 \cdot D_t \boldsymbol{e}_1 - \frac{v_\parallel}{2}\boldsymbol{b} \cdot \boldsymbol{\nabla} \times (v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E}) \tag{B.22}$$

Appendix C

Magnetic dipole

C.1 Magnetic dipole expressed in various coordinate systems

The magnetic induction $\boldsymbol{B}$ of a dipole with a magnetic moment $\mathcal{M}$ is given[1] by

$$\boldsymbol{B} = (B_r, B_\vartheta, B_\varphi) = \frac{\mathcal{M}}{r^3}\left[2\cos\vartheta \boldsymbol{e}_r + \sin\vartheta \boldsymbol{e}_\vartheta\right] \tag{C.1}$$

in spherical coordinates (r, ϑ, φ). For the sake of an example studied in Section 3.9, it is convenient to express the same in cylindrical coordinates (R, φ, z). To complete this, we consider the following transformation equations

$$R = r\sin\vartheta, \tag{C.2}$$
$$\varphi = \varphi, \tag{C.3}$$
$$z = r\cos\vartheta \tag{C.4}$$

which can easily be derived from the definition of each using the usual Cartesian system. Consequently

$$r = \sqrt{R^2 + z^2}, \tag{C.5}$$
$$\sin\vartheta = \frac{R}{r} = \frac{R}{\sqrt{R^2+z^2}}, \tag{C.6}$$
$$\cos\vartheta = \frac{z}{r} = \frac{z}{\sqrt{R^2+z^2}}. \tag{C.7}$$

To transform (C.1) to cylindrical coordinates, we first express the unit vectors from (C.1) in cylindrical coordinates. For the positional vector of a point we have

$$\boldsymbol{R} + \boldsymbol{z} = R\boldsymbol{e}_r + z\boldsymbol{e}_z = \boldsymbol{r} = r\boldsymbol{e}_r, \tag{C.8}$$

[1] e.g. `https://en.wikipedia.org/wiki/Magnetic_dipole`

where vectors $\boldsymbol{e}$ are unit vectors in the appropriate coordinate direction indicated by their subscript. Hence

$$\boldsymbol{e}_r = \frac{R}{r}\boldsymbol{e}_R + \frac{z}{r}\boldsymbol{e}_z = \sin\vartheta\boldsymbol{e}_R + \cos\vartheta\boldsymbol{e}_z. \tag{C.9}$$

To express $\boldsymbol{e}_\vartheta$ we use the orthonormality of both the spherical and cylindrical coordinate systems:

$$\boldsymbol{e}_\vartheta = \boldsymbol{e}_\varphi \times \boldsymbol{e}_r \quad \text{and} \quad \boldsymbol{e}_\varphi = \boldsymbol{e}_z \times \boldsymbol{e}_R. \tag{C.10}$$

Then

$$\boldsymbol{e}_\vartheta = (\boldsymbol{e}_z \times \boldsymbol{e}_R) \times \boldsymbol{e}_{\boldsymbol{r}} = -\boldsymbol{e}_z\boldsymbol{e}_R \cdot \boldsymbol{e}_r + \boldsymbol{e}_R\boldsymbol{e}_r \cdot \boldsymbol{e}_z = -\sin\vartheta\boldsymbol{e}_z + \cos\vartheta\boldsymbol{e}_R. \tag{C.11}$$

We use these expression in (C.1) to get

$$\boldsymbol{B} = \frac{\mathcal{M}}{r^3}\left[3\cos\vartheta\sin\vartheta\boldsymbol{e}_R + (2 - 3\sin^2\vartheta)\boldsymbol{e}_z\right]. \tag{C.12}$$

By expressing r in cylindrical coordinates we obtain components of the magnetic induction:

$$B_R = \frac{3\mathcal{M}}{r^3}\cos\vartheta\sin\vartheta = \frac{3\mathcal{M}Rz}{(R^2+z^2)^{5/2}} \tag{C.13}$$

and

$$B_z = \frac{\mathcal{M}}{r^3}\left(2 - 3\sin^2\vartheta\right) = \frac{\mathcal{M}}{(R^2+z^2)^{3/2}}\left[2 - \frac{3R^2}{R^2+z^2}\right] = \frac{\mathcal{M}}{(R^2+z^2)^{5/2}}\left[2z^2 - R^2\right]. \tag{C.14}$$

In Section 3.9 we claimed that for an axisymmetric magnetic field, we may utilise the flux function

$$F = \mathcal{M}\frac{R^2}{(R^2+z^2)^{3/2}} \tag{C.15}$$

to derive the magnetic induction components as

$$B_z = \frac{1}{R}\frac{\partial F}{\partial R} \quad \text{and} \quad B_R = -\frac{1}{R}\frac{\partial F}{\partial z}, \tag{C.16}$$

which is consistent with the approach described in Section 4.4.2. Let us verify that we indeed get the dipole components (C.13) and (C.14).

For the polar component, we have

$$B_R = -\frac{\mathcal{M}R^2}{R}\frac{\partial}{\partial z}\frac{1}{(R^2+z^2)^{3/2}} = -\mathcal{M}R\left[-\frac{3}{2}\frac{2z}{(R^2+z^2)^{5/2}}\right] = \frac{3\mathcal{M}Rz}{(R^2+z^2)^{5/2}}. \tag{C.17}$$

For the vertical component, we have

$$B_z = \frac{\mathcal{M}}{R}\frac{\partial}{\partial R}\frac{R^2}{(R^2+z^2)^{3/2}} = \frac{\mathcal{M}}{R}\left[\frac{2R}{(R^2+z^2)^{3/2}} - \frac{3}{2}\frac{R^2 2R}{(R^2+z^2)^{5/2}}\right] = \tag{C.18}$$

$$= \frac{\mathcal{M}}{(R^2+z^2)^{5/2}}\left[2(R^2+z^2) - 3R^2\right] = \frac{\mathcal{M}}{(R^2+z^2)^{5/2}}\left[2z^2 - R^2\right]. \tag{C.19}$$

Both components were therefore verified.

C.2 Flux function in spherical coordinates

In Fig. 3.12 we plotted the flux function F conveniently in spherical coordinates, whereas its expression used was derived in cylindrical coordinates. Following the transformations (C.2) and (C.4) we have

$$F = \mathcal{M}\frac{R^2}{(R^2+z^2)^{3/2}} = \frac{\mathcal{M}r^2\sin^2\vartheta}{(r^2\sin^2\vartheta + r^2\cos^2\vartheta)^{3/2}} = \tag{C.20}$$

$$\frac{\mathcal{M}r^2\sin^2\vartheta}{(r^2)^{3/2}} = \frac{\mathcal{M}}{r}\sin^2\vartheta = \frac{\mathcal{M}}{R}\sin\vartheta, \tag{C.21}$$

which constitutes the explicit expression for F in spherical coordinates.

It may be instructive to show that the approach of the use of the flux function to derive the components of the magnetic induction is applicable regardless of the coordinate system. The requirement is that the Gauss law $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ holds. That leads to different expressions to derive the components, namely:

$$B_r = \frac{1}{r^2\sin\vartheta}\frac{\partial F}{\partial\vartheta} = \frac{1}{r^2\sin\vartheta}\frac{\mathcal{M}}{r}\frac{\partial\sin^2\vartheta}{\partial\vartheta} = \frac{2\mathcal{M}}{r^3}\cos\vartheta \tag{C.22}$$

and

$$B_\vartheta = -\frac{1}{r\sin\vartheta}\frac{\partial F}{\partial r} = -\frac{\mathcal{M}\sin^2\vartheta}{r\sin\vartheta}\frac{\partial}{\partial r}\frac{1}{r} = \frac{\mathcal{M}\sin\vartheta}{r}\frac{1}{r^2} = \frac{\mathcal{M}}{r^3}\sin\vartheta. \tag{C.23}$$

These components are consistent with (C.1).