Elektromagnetické vlny

→ podmínka $k \perp E_1$, elektromagnetické $\Rightarrow B_1 \neq 0$

Světelné vlny ve vakuu

Setup: $B_0 = E_0 = 0$

vakuum $\Rightarrow$ $n_e = n_i = n_0 = 0$, $\vec{u}_e = \vec{u}_i = 0$

$k \perp E_1$

pouze Maxwell linearizovaně

$$\nabla \times E_1 = -\dot{B}_1 \qquad \Big| \frac{\partial}{\partial t}$$

$$\frac{1}{\mu_0} \nabla \times B_1 = \varepsilon_0 \dot{E}_1 \qquad \nabla \times$$

$$\nabla \cdot B_1 = 0$$

$$\nabla \times \dot{E}_1 = -\ddot{B}_1 \qquad \frac{\partial}{\partial t} \to -i\omega, \; \nabla \to i\vec{k}$$

$$\frac{1}{\mu_0 \varepsilon_0} \nabla \times \nabla \times B_1 = \nabla \times \dot{E}_1 \qquad c \equiv \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$\nabla \cdot B_1 = 0$$

$$-c^2 k \times (k \times B_1) = \omega^2 B_1$$

$$k \cdot B_1 = 0$$

$$-c^2 \left[(k \cdot B_1) k - k^2 B_1 \right] = \omega^2 B_1 \quad ; \quad k \cdot B_1 = 0$$

$$\Rightarrow c^2 k^2 B_1 = \omega^2 B_1 \qquad \boxed{\omega^2 = c^2 k^2}$$

- klasické světelné vlny

Světelné vlny v plazmatu

Setup:
- $B_0 = E_0 = 0$
- pozadí homogenní a stacionární
- ionty tvoří nehybné pozadí ($\omega \gg \Omega_p$)
- ionty i elektrony chladné
- $k \perp E_1$

podobně jako předtím, v Ampérově zákoně zůstával proud

$$\nabla \times E_1 = -\dot{B}_1 \qquad | \nabla \times \qquad \text{a spojit přes } \nabla \times \dot{B}_1$$

$$\frac{1}{\mu_0} \nabla \times B_1 = \varepsilon_0 \dot{E}_1 + j_1 \qquad | \frac{\partial}{\partial t}$$

$$\nabla \times \nabla \times E_1 = -\nabla \times \dot{B}_1 = \mu_0 \varepsilon_0 \ddot{E}_1 - \mu_0 \dot{j}_1 =$$

$$= \frac{1}{c^2} \ddot{E}_1 - \frac{1}{\varepsilon_0 c^2} \dot{j}_1 \qquad \nabla \to i\vec{k}, \quad \frac{\partial}{\partial t} \to -i\omega$$

$$-(k \cdot E_1) k + k^2 E_1 = \frac{i\omega}{\varepsilon_0 c^2} j_1 + \frac{\omega^2}{c^2} E_1$$

podmínka $k \perp E_1 \Rightarrow k \cdot E_1 = 0$

$$\Rightarrow (\omega^2 - c^2 k^2) \vec{E}_1 = -\frac{i\omega}{\varepsilon_0} \vec{j}_1$$

j_1 z tekutinových rovnic pro elektrony

$$\vec{j}_1 = -n_0 e \vec{u}_{e1} \quad \text{– Eulerova rce}$$

$$-i\omega m \vec{u}_{e1} = m \frac{\partial \vec{u}_{e1}}{\partial t} = -e \vec{E}_1 \qquad \partial_t \to -i\omega$$

$$\vec{j}_1 = -n_0 e \frac{e}{i\omega m} \vec{E}_1 = -\frac{n_0 e^2}{i\omega m} \vec{E}_1$$

$$(\omega^2 - c^2 k^2) \vec{E}_1 = \frac{\cancel{i\omega}}{\varepsilon_0} \frac{n_0 e^2}{\cancel{i\omega} m} \vec{E}_1 = \frac{n_0 e^2}{\varepsilon_0 m} \vec{E}_1 = \omega_p^2 \vec{E}_1$$

$$\Rightarrow \boxed{\omega^2 = \omega_p^2 + c^2 k^2}$$

- $v_\varphi^2 = \left(\frac{\omega}{k}\right)^2 = c^2 + \frac{\omega_p^2}{k^2} > c^2$

  → fázová rychlost > c ! ale nenese signál

- v_φ závisí na k ⇒ v plazmatu disperzivní

  pro $k \to \infty$ $\omega^2 \sim c^2 k^2$, pak nedisperzivní

- $v_g = \left(\frac{d\omega}{dk}\right) = \frac{2c^2 k}{2\sqrt{\omega_p^2 + c^2 k^2}} = \frac{c^2}{v_\varphi} < c$ pro $v_\varphi > 0$

  → grupová rychlost < c ← nese signál

• propagace jen pro $\omega^2 > \omega_p^2$

↖ hraniční (cut-off) frekvence

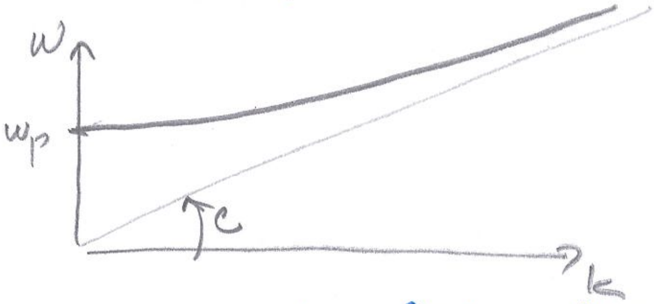
pro $\omega^2 < \omega_p^2$:

$$ck = (\omega^2 - \omega_p^2)^{1/2} = i\sqrt{\omega_p^2 - \omega^2}$$

$$\text{pro } E_1 = \bar{E}_1 \exp[i(k \cdot r - \omega t)] \propto \exp[ikx] =$$

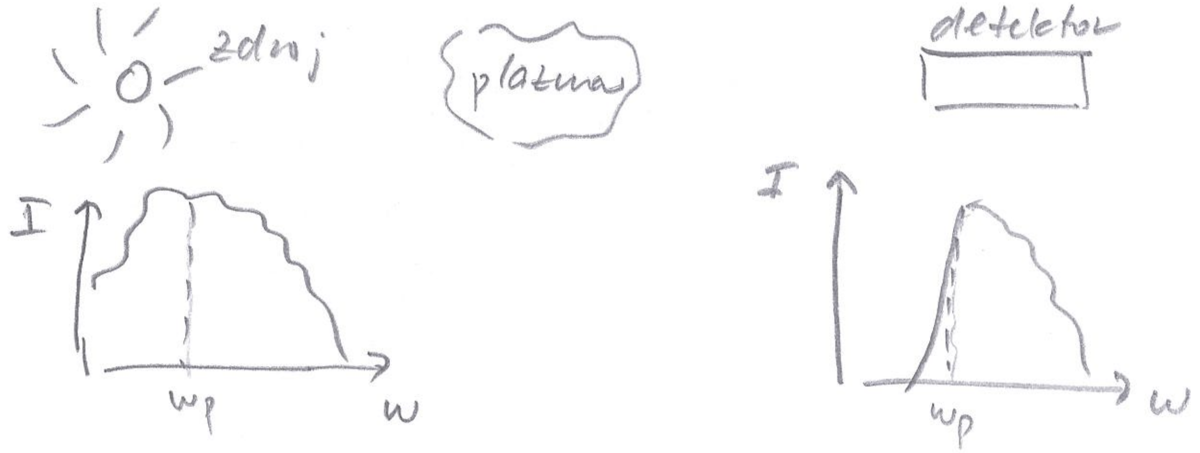
$$= \exp\left[-x\frac{\sqrt{\omega_p^2-\omega^2}}{c}\right] = \exp\left[-\frac{x}{\delta}\right]$$

$$\delta \equiv \frac{c}{\sqrt{\omega_p^2-\omega^2}}$$ ... útlumová délka

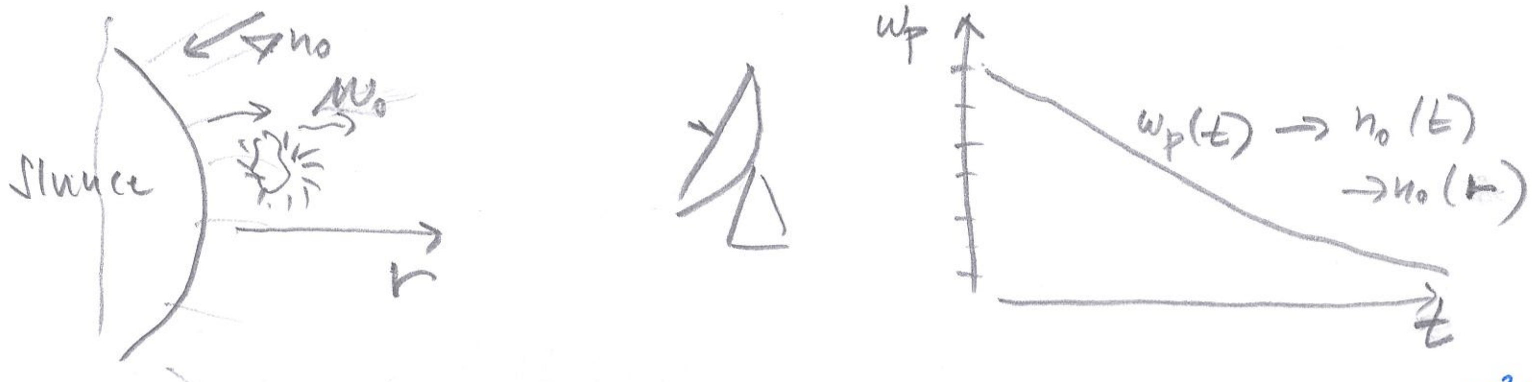
• 

$$\omega_p = \frac{n_0 e^2}{\varepsilon_0 m}$$

primitivní způsob měření hustoty
→ oblak se ozáří spektrem, něco projde,
stanoví se $\omega_p \to n_0$

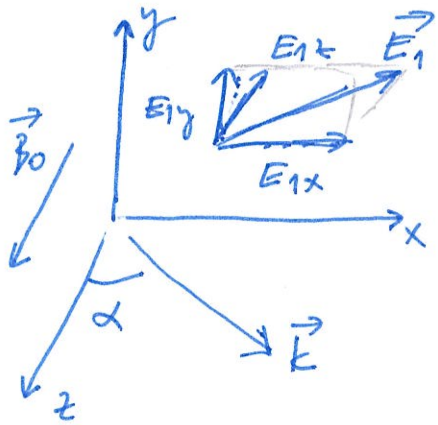


• při překotné plazma stoupá atmosférou hvězdy a je zdrojem elmag. vln



Komplex elmag. vln v plazmatu s B_0

Setup
- jako předtím
- $B_0 \neq 0$



soustadnicový systém

$\vec{B}_0 \parallel \hat{z}$, $\sphericalangle(\vec{k}, \vec{B}_0) = \alpha$

$\vec{b}_0 = \frac{\vec{B}_0}{|B_0|}$

$\vec{b}_0 = (0, 0, 1)$; $\vec{k} = (k\sin\alpha, 0, k\cos\alpha)$

$\vec{E}_1 = (E_{1x}, E_{1y}, E_{1z})$

linearizované rce

$$-i\omega m \vec{u}_{e1} = -e(\vec{E}_1 + \vec{u}_{e1} \times \vec{B}_0)$$

$$+i\vec{k} \times \vec{E}_1 = +i\omega \vec{B}_1$$

$$\frac{i}{\mu_0} \vec{k} \times \vec{B}_1 = -\varepsilon_0 i\omega \vec{E}_1 - e n_0 \vec{u}_{e1}$$

$$\vec{u}_{e1} = -i\frac{e}{m\omega}\vec{E}_1 - i\underbrace{\frac{eB_0}{m\omega}}_{\omega_c} \vec{u}_{e1} \times \vec{b}_0 \qquad c^2 = \frac{1}{\varepsilon_0 \mu_0}$$

$$\vec{E}_1 = -\frac{c^2}{\omega^2}\vec{k} \times (\vec{k} \times \vec{E}_1) + i\frac{e n_0}{\varepsilon_0 \omega}\vec{u}_{e1}$$

$$\vec{u}_{e1} = -i\frac{\varepsilon_0 \omega}{e n_0}\vec{E}_1 - i\frac{c^2 \varepsilon_0 \omega}{\omega^2 e n_0}\vec{k} \times (\vec{k} \times \vec{E}_1)$$

$$-i\frac{\varepsilon_0 \omega}{e n_0}E_1 - i\frac{c^2\varepsilon_0}{\omega e n_0}\left(\vec{k} \times [\vec{k} \times \vec{E}_1]\right) =$$

$$= -i\frac{e}{m\omega}\vec{E}_1 - i\frac{\omega_c}{\omega}\left[-i\frac{\varepsilon_0\omega}{n_0 e}\vec{E}_1 - i\frac{c^2\varepsilon_0}{\omega e n_0}\vec{k} \times (\vec{k} \times \vec{E}_1)\right] \times \vec{b}_0 \qquad \Big| \cdot \frac{-e n_0 \omega}{i\varepsilon_0}$$

$$\omega^2 E_1 + c^2(\vec{k} \cdot \vec{E}_1) - c^2 k^2 \vec{E}_1 =$$

$$= \underbrace{\frac{e^2 n_0}{\varepsilon_0 m}}_{\omega_p^2}\vec{E}_1 - i\omega_c \omega \vec{E}_1 \times \vec{b}_0 - i\frac{\omega_c}{\omega}c^2(\vec{k} \cdot \vec{E}_1)(\vec{k} \times \vec{b}_0) +$$

$$+ i\frac{\omega_c}{\omega}c^2 k^2 (E_1 \times b_0)$$

$$\left(\omega^2-\omega_p^2-c^2k^2\right)\vec{E}_1 + i\frac{\omega_c}{\omega}\left(\omega^2-c^2k^2\right)\left(\vec{E}_1\times\vec{b}_0\right) +$$

$$+c^2\left(\vec{k}\cdot\vec{E}_1\right)\vec{k} + i\frac{\omega_c}{\omega}c^2\left(\vec{k}\cdot\vec{E}_1\right)\left(\vec{k}\times\vec{b}_0\right)=0$$

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do komponent podle souřadnicového systému:

$$\left(\omega^2-\omega_p^2-c^2k^2\right)\begin{pmatrix}E_{1x}\\ E_{1y}\\ E_{1z}\end{pmatrix} + i\frac{\omega_c}{\omega}\left(\omega^2-c^2k^2\right)\begin{pmatrix}E_{1y}\\ -E_{1x}\\ 0\end{pmatrix} +$$

$$+c^2\left(kE_{1x}\sin\alpha + kE_{1z}\cos\alpha\right)\begin{pmatrix}k\sin\alpha\\ 0\\ k\cos\alpha\end{pmatrix} +$$

$$+i\frac{\omega_c}{\omega}c^2\left(E_{1x}k\sin\alpha + E_{1z}k\cos\alpha\right)\begin{pmatrix}0\\ -k\sin\alpha\\ 0\end{pmatrix}=0$$

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maticově: $M_{E_1}\cdot E_1 = 0$; prvky M_{E_1} dohledat

$$M_{E_1}=\begin{bmatrix} \omega^2-\omega_p^2-\overbrace{c^2k^2-c^2k^2\sin^2\alpha}^{c^2k^2\cos^2\alpha} & i\frac{\omega_c}{\omega}\left(\omega^2-c^2k^2\right) & c^2k^2\sin\alpha\cos\alpha \\ -i\frac{\omega_c}{\omega}\left(\omega^2-c^2k^2\cos^2\alpha\right) & \omega^2-\omega_p^2-c^2k^2 & -i\frac{\omega_c}{\omega}c^2k^2\cos\alpha\sin\alpha \\ c^2k^2\sin\alpha\cos\alpha & 0 & \omega^2-\omega_p^2-\underbrace{c^2k^2-c^2k^2\cos^2\alpha}_{c^2k^2\sin^2\alpha} \end{bmatrix}$$

netriviální řešení pro $\det M_{E_1}=0$

$$\det M_{E_1} = c^4k^4\sin^2\alpha\cos^2\alpha\left[\frac{\omega_c^2}{\omega^2}\left(\omega^2-c^2k^2\right)-\left(\omega^2-\omega_p^2-c^2k^2\right)\right]+$$

$$+\left(\omega^2-\omega_p^2-c^2k^2\sin^2\alpha\right)\left[\left(\omega^2-\omega_p^2-c^2k^2\cos^2\alpha\right)\left(\omega^2-\omega_p^2-c^2k^2\right)-\right.$$

$$\left.-\frac{\omega_c^2}{\omega^2}\left(\omega^2-c^2k^2\right)\left(\omega^2-c^2k^2\cos^2\alpha\right)\right]=0$$

disperzní relace komplexní elmag. vlny v plazmatu s B_0

speciální případy

① $\vec{k} \perp \vec{B}_0 \Rightarrow \alpha = \pi/2$

$$(\omega^2 - \omega_p^2 - c^2k^2)\left[(\omega^2-\omega_p^2)(\omega^2-\omega_p^2-c^2k^2) - \omega_c^2(\omega^2-c^2k^2)\right] = 0$$

a) $\boxed{\omega^2 = \omega_p^2 + c^2k^2}$ – jako bez mg. pole – řádná (O–) vlna

b) $(\omega^2-\omega_p^2)(\omega^2-\omega_p^2-c^2k^2) = \omega_c^2(\omega^2-c^2k^2)$

$$(\omega^2-\omega_p^2)\cancel{(\omega^2-c^2k^2)}\left(1-\frac{\omega_p^2}{\omega^2-c^2k^2}\right) = \omega_c^2\cancel{(\omega^2-c^2k^2)}$$

$$(\omega^2-\omega_p^2) - \frac{(\omega^2-\omega_p^2)\,\omega_p^2}{\omega^2-c^2k^2} = \omega_c^2$$

$$-\frac{(\omega^2-\omega_p^2)\,\omega_p^2}{\omega^2-c^2k^2} = \omega_c^2 - \omega^2 + \omega_p^2 = -(\omega^2-\omega_h^2)$$

$$\boxed{\omega^2 - c^2k^2 = \frac{\omega_p^2(\omega^2-\omega_p^2)}{\omega^2-\omega_h^2}}$$

→ mimořádná (X–) vlna

↳ částečně podélná, částečně příčná, ale propaguje se ~~podél~~ napříč B_0!

→ O– a X– vlny různé polarizace

→ plazma jako dvojlomné prostředí pro elmag. vlny

② podél $B_0 \Rightarrow \alpha = 0$

$$(\omega^2-\omega_p^2)\cdot\left[(\omega^2-\omega_p^2-c^2k^2)^2 - \frac{\omega_c^2}{\omega^2}(\omega^2-c^2k^2)^2\right] = 0$$

a) $\boxed{\omega^2 = \omega_p^2}$ podél pole plazmové oscilace

b) $\omega^2-\omega_p^2-c^2k^2 = \pm\frac{\omega_c}{\omega}(\omega^2-c^2k^2)$

$$\cancel{(\omega^2-c^2k^2)}\left(1-\frac{\omega_p^2}{\omega^2-c^2k^2}\right) = \pm\frac{\omega_c}{\omega}\cancel{(\omega^2-c^2k^2)}$$

$$-\frac{\omega_p^2}{\omega^2-c^2k^2} = \pm\frac{\omega_c}{\omega} - 1 = -\left(1 \mp \frac{\omega_c}{\omega}\right)$$

$$\boxed{\omega^2 - c^2k^2 = \frac{\omega_p^2}{1 \mp \omega_c/\omega}}$$ R-vlna, L-vlna

→ vlny zjevně disperzivní

→ $v_\varphi(R) > v_\varphi(L)$ zjevně

→ polarizace → zpět do $\vec{E}_1$ komponent

$$\begin{pmatrix} \omega^2-\omega_p^2-c^2k^2 & i\frac{\omega_c}{\omega}(\omega^2-c^2k^2) & 0 \\ -i\frac{\omega_c}{\omega}(\omega^2-c^2k^2) & \omega^2-\omega_p^2-c^2k^2 & 0 \\ 0 & 0 & \omega^2-\omega_p^2 \end{pmatrix} \cdot \begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \end{pmatrix} = 0$$

→ $E_{1z} = 0$

→ $(\omega^2-\omega_p^2-c^2k^2)E_{1x} + i\frac{\omega_c}{\omega}(\omega^2-c^2k^2)E_{1y} = 0$

použije disperzní relaci: $c^2k^2 = \omega^2 - \omega_p^2\left(1 \mp \frac{\omega_c}{\omega}\right)^{-1}$

$$\left(\cancel{\omega^2}-\omega_p^2-\cancel{\omega^2}+\frac{\omega_p^2}{1\mp\frac{\omega_c}{\omega}}\right)E_{1x} + i\frac{\omega_c}{\omega}\left(\cancel{\omega^2}-\cancel{\omega^2}+\frac{\omega_p^2}{1\mp\frac{\omega_c}{\omega}}\right)E_{1y} = 0$$

$$-\omega_p^2\left(1-\frac{1}{1\mp\frac{\omega_c}{\omega}}\right)E_{1x} + i\frac{\omega_c}{\omega}\left(\frac{\omega_p^2}{1\mp\frac{\omega_c}{\omega}}\right)E_{1y} = 0$$

$$-\cancel{\omega_p^2}\left(\frac{\cancel{1}\mp\frac{\omega_c}{\omega}-\cancel{1}}{\cancel{1\mp\omega_c/\omega}}\right)E_{1x} + i\frac{\omega_c}{\omega}\frac{\cancel{\omega_p^2}}{\cancel{1\mp\omega_c/\omega}}E_{1y} = 0$$

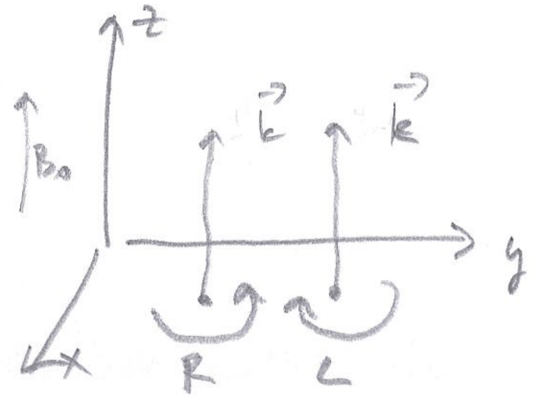
$$\pm\frac{\omega_c}{\cancel{\omega}}E_{1x} + i\frac{\omega_c}{\cancel{\omega}}E_{1y} = 0$$

$$\underline{E_{1x} = \mp i E_{1y}} \Rightarrow E_{1y} = \overset{R}{\underset{L}{\pm}} i E_{1x}$$

⇒ kruhová polarizace

R → E_{1y} napřed před E_{1x} o $\pi/2$ → pravotočivá

L → obráceně → levotočivá



Aplikace : $\boxed{\text{Farradayova rotace}}$

- detekce R a L modů umožňuje např. „změřit" vzdálenost.
- vodné předpokladů – (homogenní prostředí, podélí pole a pod.)

rozpis: $\vec{E}_R = E_0 \exp[i(\vec{k}_R\cdot\vec{r} - \omega t)]\,[\hat{e}_x + i\hat{e}_y]$

$\vec{E}_L = E_0 \exp[i(\vec{k}_L\cdot\vec{r} - \omega t)]\,[\hat{e}_x - i\hat{e}_y]$

stejné amplitudy

R napřed
L pozadu

měření = superpozice

pokud by $k_R = k_L = k$

$$\vec{E} = \frac{1}{2} E_0 e^{-i\omega t}\left[2e^{i\vec{k}\cdot\vec{r}}\hat{e}_x\right] = E_0 \exp[i(\vec{k}\cdot\vec{r} - \omega t)]\hat{e}_x$$

$\Rightarrow$ lineárně polarizovaná vlna ve směru $\hat{x}$

ale opravdu $k_R \neq k_L$

$$k_{R/L} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2/\omega^2}{1 \mp \omega_c/\omega}} \quad \begin{matrix} R \\ L \end{matrix}$$

pro $\omega \gg \omega_c, \omega_p$: $\sqrt{1+x} \sim 1 - \frac{x}{2}$ $\qquad (1+x)^{-1} \sim 1-x$

$$\Rightarrow k_{R/L} \approx \frac{\omega}{c}\left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\frac{1}{1 \mp \omega_c/\omega}\right) \sim \frac{\omega}{c}\left[1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\left(1 \pm \frac{\omega_c}{\omega}\right)\right]$$

$$\rightarrow k_{R/L} \doteq k \begin{matrix} + R \\ - L \end{matrix} \Delta k \quad ; \quad k = \frac{\omega}{c}\left[1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right]$$

$$\Delta k = -\frac{1}{2}\frac{\omega_p^2 \omega_c}{c\,\omega}$$

$$\vec{E} = \frac{1}{2}(\vec{E}_R + \vec{E}_L) = \frac{1}{2}E_0 e^{-i\omega t}\left[e^{ik_R\cdot r}(e_x + ie_y) + e^{ik_L\cdot r}(e_x - ie_y)\right] =$$

$$= \frac{1}{2}E_0 e^{-i\omega t}\left[e^{i\vec{k}\cdot\vec{r}} e^{i\Delta\vec{k}\cdot\vec{r}}(\hat{e}_x + i\hat{e}_y) + e^{i\vec{k}\cdot\vec{r}} e^{-i\Delta\vec{k}\cdot\vec{r}}(\hat{e}_x - i\hat{e}_y)\right] =$$

$$= \underbrace{E_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}}_{\text{lineární polarizace}}\underbrace{\left[\cos(-\Delta k\cdot r)e_x + \sin(-\Delta k\cdot r)e_y\right]}_{\text{rotace polarizace}}$$

$\vec{r} \parallel \vec{k} \Rightarrow$

fázový posun $-\Delta \vec{k} \cdot \vec{r} \Rightarrow -\Delta k z \Rightarrow \varphi$

pak $\underbrace{\frac{d\varphi}{dz}}_{\text{„stáčení"}} = -\Delta k$

$\Rightarrow$ otočení: $\varphi = \varphi_0 + \int_0^d \frac{d\varphi}{dz} dz = \varphi_0 + \int_0^d -\Delta k \, dz =$

$\doteq \varphi_0 + \frac{1}{2cw^2} \int_0^d \cancel{n_0(z) B_0(z) dz}\, \omega_p^2(z)\, \omega_c(z)\, dz$

pokud $n_0 \neq n_0(z) \wedge B_0 \neq B_0(z)$

$\Rightarrow \varphi = \varphi_0 + \frac{e^3}{2m\varepsilon_0 c} \frac{1}{\omega^2} n_0 B_0 d$

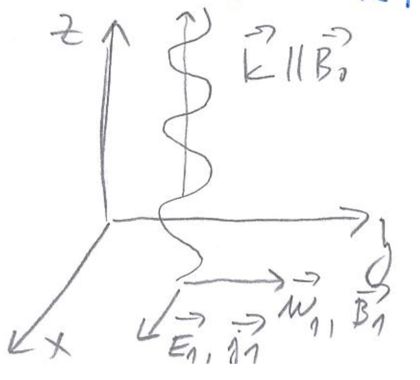
$\varphi = \varphi(\omega) \Rightarrow$ měření stočení polarizační roviny pro různá ω lze fitovat některou z n_0, B_0, d, pokud ostatní 2 známe

Hydromagnetické vlny

- nízkofrekvenční oscilace iontů v přítomnosti magnetického pole

Setup:
- $B_0 \neq 0$, $E_0 = 0$, $B_1 \neq 0$, $E_1 \neq 0$
- pozadí homogenní a stacionární
- ionty i elektrony chladné
- řešíme i ionty i elektrony

Souřadnice: $\hat{e}_x, \hat{e}_y, \hat{e}_z$, aby:



$\vec{B_0} = B_0 \hat{e}_z$

$\vec{E_1} \parallel \vec{j_1} \parallel \hat{e}_x$, $\vec{E_1} \perp \vec{B_0}$

$\vec{B_1} \parallel \vec{u_1} \parallel \hat{e}_y$

$\vec{k} \parallel \vec{B_0}$, $\vec{B_1} \perp \vec{E_1}$

zadneme Maxwellovými rovnicemi

$$\nabla \times E_1 = - \dot{B}_1 \quad \Big| \nabla \times \qquad \text{a spojit přes } \nabla \times \dot{B}_1$$

$$\frac{1}{\mu_0} \nabla \times B_1 = + \varepsilon_0 \dot{E}_1 + j \quad \Big| \partial/\partial t$$

$$\nabla \times (\nabla \times E_1) = - \nabla \times \dot{B}_1 = \mu_0 \varepsilon_0 \ddot{E}_1 - \mu_0 \dot{j} = \frac{1}{c^2} \ddot{E}_1 - \frac{1}{\varepsilon_0 c^2} \dot{j}_1$$

do FT

$$- k^2 \underbrace{(k \cdot E_1)}_{=0 \text{ podle předpokladu}} + k^2 E_1 = \frac{i\omega}{\varepsilon_0 c^2} j_1 + \frac{\omega^2}{c^2} E_1$$

$$\circ \quad (\omega^2 - c^2 k^2) \vec{E}_1 = - \frac{i\omega}{\varepsilon_0} \vec{j}_1 \qquad E_1 \parallel j_1 \quad \text{o.k.}$$

$$\square \quad \vec{j}_1 = j_1 \hat{e}_x = n_0 e (u_{N1x} - u_{e1x}) \hat{e}_x$$

z Eulerovy rce:

$$M \frac{\partial \vec{u}_{i1}}{\partial t} = e (\vec{E}_1 + \vec{u}_{i1} \times \vec{B}_0)$$

komponenty ve FT:

$$-i\omega M u_{N1x} = e E_1 + e u_{i1y} B_0$$

$$-i\omega M u_{N1y} = - e u_{N1x} B_0$$

$$-i\omega M u_{N1x} = e E_1 + e B_0 \frac{e B_0}{i\omega M} u_{N1x}$$

$$\Rightarrow u_{N1x} = e E_1 \frac{i}{\omega M} \Big/ \left(1 - \frac{e^2 B_0^2}{M^2 \omega^2}\right) =$$

$$\triangle \quad \Downarrow \quad = \frac{ie}{\omega M} \left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1} E_1$$

$$u_{N1y} = \frac{e}{\omega M} \frac{\Omega_c}{\omega} \left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1} E_1$$

pro elektrony analogicky : $M \to m$, $e \to -e$, $\Omega_c \to -\omega_c$

$$u_{e1x} = - \frac{ie}{\omega m} \left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$u_{e1y} = \frac{ie}{\omega m} \frac{\omega_c}{\omega} \left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

pro $\omega \ll \omega_c$:

$$\diamond \quad u_{e1x} = -\frac{ie}{\omega m}\left(\frac{\cancel{\omega^2}-\omega_c^2}{\omega^2}\right)^{-1} E_1 \sim \frac{ie}{\omega M}\frac{\omega^2}{\omega_c^2}E_1 \to 0 \text{ pro } \omega \ll \omega_c$$

$$u_{e1y} = \frac{e}{\omega m}\frac{\omega_c}{\omega}\left(\frac{\omega^2-\omega_c^2}{\omega^2}\right)^{-1}E_1 \sim \frac{e}{\omega m}\frac{\omega_c}{\omega}\frac{\omega^2}{\omega_c^2}E_1 = -\frac{e}{\omega_c m}E_1 =$$

$$= -\frac{E_1}{B_0}$$

$\diamond + \Delta$ do $\square$ a pak do $\circ$

$$(\omega^2 - c^2k^2)E_1 = -\frac{i\omega}{\varepsilon_0} n_0 e\left[\frac{ie}{\omega M}\left(1-\frac{\Omega_c^2}{\omega^2}\right)^{-1}E_1 - 0\right] =$$

$$= \frac{n_0 e^2}{\varepsilon_0 M}\left(1-\frac{\Omega_c^2}{\omega^2}\right)^{-1}E_1 = \Omega_p^2\left(1-\frac{\Omega_c^2}{\omega^2}\right)^{-1}E_1$$

$$\Rightarrow \boxed{\omega^2 - c^2k^2 = \Omega_p^2\left(1-\frac{\Omega_c^2}{\omega^2}\right)^{-1}}$$

disperzní relace

pro $\omega \ll \Omega_c$:

$$\omega^2 - c^2k^2 = \Omega_p^2\left(\frac{\cancel{\omega^2}-\Omega_c^2}{\omega^2}\right)^{-1} \sim -\omega^2\frac{\Omega_p^2}{\Omega_c^2} = -\omega^2\frac{n_0 M}{\varepsilon_0 B_0^2} =$$

$$= -\omega^2\frac{\rho}{\varepsilon_0 B_0^2}$$

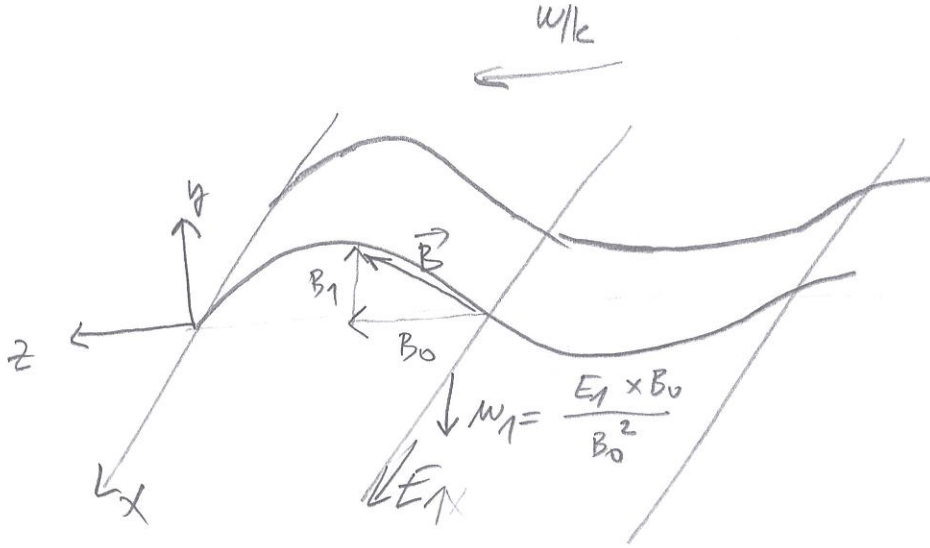
$$\Rightarrow \frac{\omega^2}{k^2} = \frac{c^2}{1+\frac{\rho}{\varepsilon_0 B_0^2}}$$ – význam relativní permitivity

$\frac{\omega}{k} \ll c \Rightarrow 1 + \rho/\varepsilon_0 B_0^2 \gg 1$

$$\Rightarrow \frac{\omega^2}{k^2} \approx \frac{c^2}{\frac{\rho\mu_0 c^2}{B_0^2}} = \frac{B_0^2}{\mu_0\rho}$$

$$\Rightarrow \boxed{\frac{\omega}{k} = \frac{B_0}{\sqrt{\mu_0\rho}} = c_A}$$ Alfvénova rychlost

disperzní relace podélné Alfvénovy vlny



- magnetická komponenta $B_1 \parallel B_y$ → zvlnění mg. pole v y-směru

→ máme drift $v_y = \frac{E_1 \vec{e_x} \times \vec{B_0}}{B_0^2}$ stejný pro $e^\ominus$ i $p^\oplus$

- fázová rychlost poruchy B_1

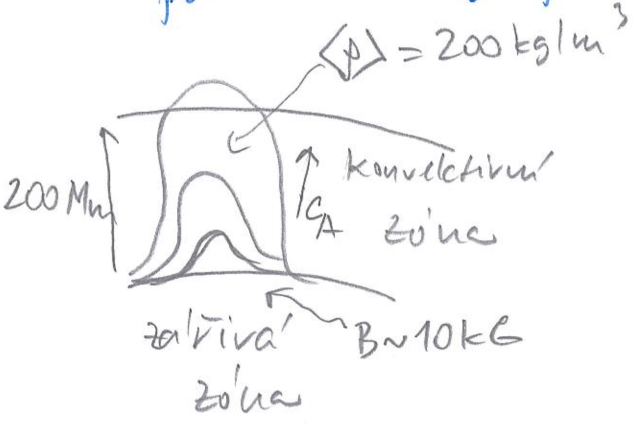
$\omega B_1 = k v_{y,B} B_0$ z indukční rovnice

$k E_1 = -\omega B_1$ z Faradayova zákona

$\Rightarrow v_{y,B} = -\frac{\omega}{k} \frac{k}{\omega} \frac{E_1}{B_0} = -\frac{E_1}{B_0}$, to je stejné

jako driftová rychlost částice
⇒ pole i částice oscilují spolu ⇒
konzistentní se zamrzlým polem

- platí, jen když $E_0 \cdot B_0 = 0$
- Alfvénova rychlost → char. rychlost propagace poruch mg. pole



$$\tau \sim \frac{d}{C_A} \sim \frac{200\,Mm}{\frac{1T}{\sqrt{\mu_0}\sqrt{200\,kg/m^3}}} \sim$$

$$\sim \frac{2\times 10^8}{\frac{1}{\sqrt{4\pi 10^{-7}}\sqrt{200}}}\,s \sim 36\text{ dní}$$

Obecná MHD vlna

1-tekutinová aproximace

systém
$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{w} = 0$$
$$\rho \frac{d\vec{w}}{dt} = -\nabla p + (\nabla \times \vec{B}) \times \vec{B} / \mu_0$$
$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$
$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{w} \times \vec{B})$$
$$\nabla \cdot \vec{B} = 0$$

- pozadí homogenní a stacionární; $\vec{w}_0 = 0$
- Maxwellovy rce → indukční rce bez disipace
  - → $\nabla \cdot B = 0$
  - → $j = \frac{1}{\mu_0} \nabla \times B$
- adiabatická stavová rce

---

linearizace triviální

$$\frac{\partial \rho_1}{\partial t} + \rho_0 (\nabla \cdot w_1) = 0 \qquad (1)$$
$$\rho_0 \frac{\partial w_1}{\partial t} = -\nabla p_1 + (\nabla \times B_1) \times B_0 / \mu_0 \qquad (2)$$
$$\frac{\partial B_1}{\partial t} = \nabla \times (w_1 \times B_0) \qquad (3)$$
$$\nabla \cdot B_1 = 0 \qquad (4)$$

u rovnice stavové složitější

$$0 = \left\{\frac{\partial}{\partial t} + w_1 \cdot \nabla\right\}\left[\frac{p_0 + p_1}{(\rho_0 + \rho_1)^{\gamma}}\right] =$$
$$= \frac{1}{(\rho_0+\rho_1)^{\gamma}}\left[\frac{\partial p_1}{\partial t} + \cancel{(w_1 \cdot \nabla) p_0}\right] + (p_0+p_1)\left\{\frac{-\gamma}{(\rho_0+\rho_1)^{\gamma+1}}\left[\frac{\partial \rho_1}{\partial t} + \cancel{(w_1 \cdot \nabla)\rho_0}\right]\right\} =$$
$$= \frac{1}{(\rho_0+\rho_1)^{\gamma}}\left[\frac{\partial p_1}{\partial t} - \underbrace{\gamma \frac{p_0+p_1}{(\rho_0+\rho_1)}}_{c_s^2} \frac{\partial \rho_1}{\partial t}\right] = \frac{1}{(\rho_0+\rho_1)^{\gamma}}\left[\frac{\partial p_1}{\partial t} - c_s^2 \frac{\partial \rho_1}{\partial t}\right] = 0$$

$$\Rightarrow \frac{\partial p_1}{\partial t} - c_s^2 \frac{\partial \rho_1}{\partial t} = 0 \qquad (5)$$
je linearizovaná stavová rce

$$\frac{\partial}{\partial t}(2): \quad \frac{\partial^2 w_1}{\partial t^2} = -\frac{1}{\rho_0}\nabla\frac{\partial p_1}{\partial t} + \left(\nabla\times\frac{\partial B_1}{\partial t}\right)\times\frac{B_0}{\rho_0\mu_0}$$

použijeme (5)

$$\frac{\partial^2 w_1}{\partial t^2} = -\frac{1}{\rho_0}\nabla\left(c_s^2\frac{\partial \rho_1}{\partial t}\right) + \left(\nabla\times\frac{\partial B_1}{\partial t}\right)\times\frac{B_0}{\mu_0\rho_0}$$

použijeme (1) použijeme (3)

$$\frac{\partial^2 \vec{w_1}}{\partial t^2} = c_s^2\nabla(\nabla\cdot\vec{w_1}) + \left\{\nabla\times\left[\nabla\times(\vec{w_1}\times\vec{B_0})\right]\right\}\times\frac{\vec{B_0}}{\mu_0\rho_0}$$

$$\frac{\partial}{\partial t}\to -i\omega \qquad \nabla\to ik \qquad \vec{B_0} = B_0\vec{b_0}$$

$$\frac{B_0^2}{\mu_0\rho_0} \equiv C_A^2$$

$$\omega^2\vec{w_1} = \cancel{c_s^2\nabla(\nabla\cdot w_1)+\{\nabla}$$
$$= c_s^2\vec{k}(\vec{k}\cdot\vec{w_1}) + \left\{\vec{k}\times\left[\vec{k}\times(\vec{w_1}\times\vec{B_0})\right]\right\}\times\frac{\vec{B_0}}{\mu_0\rho_0} =$$
$$= c_s^2\vec{k}(\vec{k}\cdot\vec{w_1}) + \left\{\vec{k}\times\left[\vec{k}\times(\vec{w_1}\times\vec{b_0})\right]\right\}\times\vec{b_0}\, C_A^2$$

$$\{k\times[k\times(w_1\times b_0)]\}\times b_0 = \{k\times[w_1(k\cdot b_0) - b_0(k\cdot w_1)]\}\times b_0 =$$
$$= (k\cdot b_0)\{k\times w_1\}\times b_0 - (k\cdot w_1)\{k\times b_0\}\times b_0 =$$
$$= -(k\cdot b_0)\left[k(w_1\cdot b_0) - w_1(k\cdot b_0)\right] + (k\cdot w_1)\left[k\, b_0^2 - b_0(k\cdot b_0)\right] =$$
$$= \vec{w_1}(\vec{k}\cdot\vec{b_0})^2 - (\vec{k}\cdot\vec{w_1})(\vec{k}\cdot\vec{b_0})\vec{b_0} + \vec{k}\left[\vec{k}\cdot\vec{w_1} - (\vec{k}\cdot\vec{b_0})(\vec{w_1}\cdot\vec{b_0})\right]$$

$$\omega^2 w_1 = \overset{(3a)}{c_s^2 k(k\cdot w_1)} + C_A^2\left\{\overset{(1)}{(k\cdot b_0)w_1} - \overset{(2)}{(k\cdot w_1)(k\cdot b_0)b_0} + k\overset{(3b)}{\left[k\cdot w_1 - (k\cdot b_0)(w_1\cdot b_0)\right]}\right\}$$

pokud $k\cdot b_0 = k\cos\alpha \qquad \alpha = \sphericalangle(k, b_0)$

$$\frac{\omega^2}{C_A^2}\vec{w_1} = k^2\cos^2\alpha\,\vec{w_1} - (\vec{k}\cdot\vec{w_1})k\cos\alpha\,\vec{b_0} + \vec{k}\left[\left(1+\frac{c_s^2}{C_A^2}\right)(\vec{k}\cdot\vec{w_1}) - k\cos\alpha\,(\vec{w_1}\cdot\vec{b_0})\right]$$

projekce do $\vec{k}$ a $\vec{b_0}$

$$\vec{k}: \quad \omega^2(k\cdot w_1) = \cancel{k^2C_A^2\cos^2\alpha(w_1\cdot k)} - \cancel{C_A^2k^2\cos^2\alpha(k\cdot w_1)} +$$
$$+ C_A^2k^2(k\cdot w_1) + c_s^2k^2(k\cdot w_1) - C_A^2k^3\cos\alpha\,(w_1\cdot b_0)$$

$$\Rightarrow \left[-\omega^2 + C_A^2k^2 + c_s^2k^2\right](\vec{k}\cdot\vec{w_1}) = C_A^2k^3\cos^2\alpha\,(\vec{w_1}\cdot\vec{b_0})$$

b_0: $\omega^2(b_0 \cdot n_1) = \cancel{c_A^2 k^2 \cos^2\alpha\,(n_1 \cdot b_0)} - \cancel{c_A^2\,(k \cdot n_1)\,k\cos\alpha} +$

$+ \cancel{c_A^2\,(k \cdot n_1)\,k\cos\alpha} + c_S^2\,(k \cdot n_1)\,k\cos\alpha -$

$- \cancel{c_A^2 k^2 \cos^2\alpha\,(n_1 \cdot b_0)}$

$$\Rightarrow \omega^2(\vec{b_0} \cdot \vec{n_1}) = c_S^2\,(\vec{k} \cdot \vec{n_1})\,k\cos\alpha$$

z obou poměr $\dfrac{b \cdot n_1}{n_1 \cdot k}$

$$\frac{b \cdot n_1}{n_1 \cdot k} = \underbrace{\frac{-\omega^2 + c_A^2 k^2 + c_S^2 k^2}{c_A^2 k^2 \cos\alpha} = \frac{c_S^2 k \cos\alpha}{\omega^2}}$$

$$\boxed{\omega^4 - \omega^2 k^2 (c_A^2 + c_S^2) + c_S^2 c_A^2 k^4 \cos^2\alpha = 0}$$

pro $\frac{\omega}{k} > 0$ propagace vln

$$\boxed{\frac{\omega}{k} = \left[\frac{1}{2}(c_S^2 + c_A^2) \pm \frac{1}{2}\sqrt{c_S^4 + c_A^4 - 4 c_S^2 c_A^2 \cos^2\alpha + 2 c_A^2 c_S^2}\right]^{1/2}}$$

disperzní relace pro obecnou magnetoakustickou vlnu

$+$ ... rychlá (fast)

$-$ ... pomalé (slow)

a) podél pole: $\alpha = 0$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(c_S^2 + c_A^2) \pm \frac{1}{2}\sqrt{c_S^4 + c_A^4 - 2 c_S^2 c_A^2} = \frac{1}{2}(c_S^2 + c_A^2) \pm \frac{1}{2}(c_S^2 - c_A^2) =$$

podél pole dvě rychlosti

$$= \begin{cases} c_S^2 \\ c_A^2 \end{cases}$$

b) napříč polem: $\alpha = \pi/2$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(c_S^2 + c_A^2) \pm \frac{1}{2}\sqrt{c_S^4 + c_A^4 + 2 c_S^2 c_A^2} = \frac{1}{2}(c_S^2 + c_A^2) \pm \frac{1}{2}(c_S^2 + c_A^2) =$$

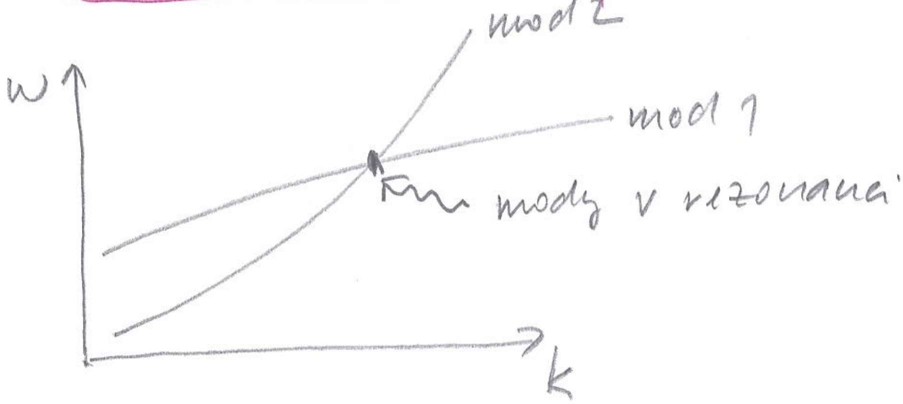
$$= \begin{cases} 0 \\ c_S^2 + c_A^2 \end{cases}$$ tzv. jen rychlá

c) pro $B_0 = 0 \Rightarrow c_A = 0 \Rightarrow$ vždy jen $\left(\frac{\omega}{k}\right)^2 = \begin{cases} 0 \\ c_S^2 \end{cases}$

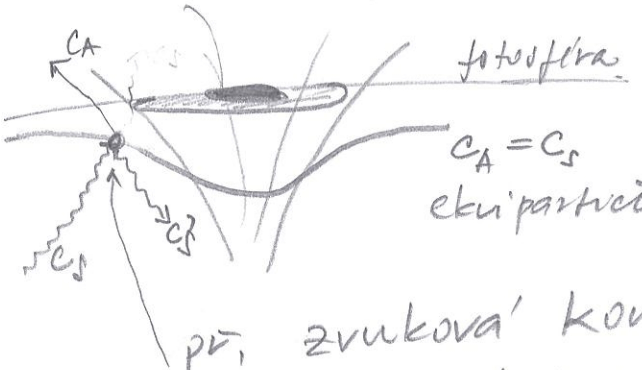
jen zvuková

~~→ konvekce~~

Konverze modů



„potkají-li" se dva mody v k–ω diagramu, mají v daném místě stejnou v_f →
→ může dojít k „přeskoku" → konverze modů



př. zvuková konvertuje částečně na magnetozvukovou nebo Alfvénovu

→ konverzi nelze popsat naším přiblížením
↳ teorie rozptylu