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# Secular motion in a hierarchic triple stellar system

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## ABSTRACT

A system of equations is derived for the secular motion of a hierarchic triple stellar system. Mean anomalies of the inner binary system and of the outer star orbit are eliminated up to order 9 in the ratio of their Delaunay momenta. Thanks to the use of vectorial elements (angular momenta and Laplace vectors) the equations are non-singular, compact, and independent on the choice of the reference frame axes directions. In the limit of circular outer orbit, the secular Hamiltonian agrees with the forgotten solution by Brown. Being more complete, yet simpler, the present model differs from that of Krymolowski & Mazeh, except for the circular outer orbit limit. A simple extension of the secular equations of motion describing tidal friction is also provided.

**Key words:** methods: analytical – celestial mechanics – stars: kinematics and dynamics.

## 1 INTRODUCTION

About 8 per cent of solar-type stars are triples in the only long-term stable hierarchy of a close (inner) binary accompanied by a third component (e.g. Eggleton & Tokovinin 2008; Tokovinin 2008, 2014). Understanding their origin, as well as understanding the origin of higher multiplicity systems, may be an important clue to the problem of stellar birth in general. A solid determination of triple systems' parameters, such as stellar masses and parameters of their orbital configuration, is a necessary prerequisite to that goal. Astronomical observations allow us to fit these data, depending on variety of available observational techniques. Needless to say that a very helpful circumstance occurs when the inner system's stars, or even all components, are eclipsing each other.

Universality of gravitational interaction between all components in a triple system implies that representation of their motion by a pair of two-body problems – motion of the stars in the inner binary and motion of the third component about the centre-of-mass of the inner binary – is necessarily just a zero-order approximation. In reality, all components move on complicated orbits whose details are revealed, for instance, by numerical integration. In perturbation approach, these intricacies are described by variation of osculating orbital elements of the above-mentioned two-body components in the system's architecture. So far, the temporal evolution in the overall architecture of the triple-star system has not been widely used to constrain its parameters, but this may change in the near future. This provides a motivation for both accurate and efficient theories that would allow us to describe the system's evolution in time.

Observations prompted analysis of the mutual stellar interactions in triple systems already in the early 20th century, though the attention has been primarily paid to the orbital changes in the close binary system only. Thus Brown (1936a,b,c, 1937), using his bold expertise from the theory of lunar motion, developed an interesting approach which allowed us to determine both secular and periodic perturbations in the close binary orbit due to the presence of a third component in the system. Focus on the binary orbit persisted till 1960s (see e.g. Kopal 1959 and references therein), when Harrington (1968, 1969) undertook a more general approach, determining perturbations in both the inner and outer orbits. This work was followed by Söderhjelm (1975, 1982, 1984), who rephrased Harrington's results in a more practical form, as far as comparison with observations is concerned. Both Harrington and Söderhjelm included quadrupole and octupole interaction between the two orbits within the first-order perturbation theory. Focusing on secular terms only, Krymolowski & Mazeh (1999) attempted to push theory to higher order, namely to include the non-linear quadrupole effect (effectively, a part of it). While typically negligible for loosely bound systems, such contribution may become important for compact cases, where the ratio of the orbital period of the outer orbit to that of the inner orbit is close to the stability limit of ∼5. If moreover both orbital periods are short in such cases, the whole architecture of the triple system, as given by orientation of the orbital planes and directions to their pericentres, may evolve quite fast, on a time-scale of decades only. Recent discoveries of numerous such systems by Kepler satellite (e.g. Slawson et al. 2011; Rappaport et al. 2013) motivates to pursue the effort in higher order analytic theories.

In this paper, we aim to develop such a high-order secular theory of a triple stellar system. Our principal goal is to provide a tool that would assist interpretation of observations for compact triple systems. That said, the accuracy in description of these systems' secular evolution on decades to centuries long time-scale is the principal criterion of success. This is because, as also has been pointed out above, for the most interesting systems the denomination 'secular'

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here may not necessarily mean centuries-long time-scale, but rather decades-long time-scale (see also Section 6). Identifying a relevant small parameter, we use standard technique of elimination of fast variables (mean longitude in orbit for both inner and outer orbits in the triple system) to construct a secular system. We push our effort an order higher than previously found in the literature. In particular, we compute the secular interaction of the two orbits up to a complete non-linear quadrupole term, both novelty and major difficulty in this paper. Another quality of our approach is due to a systematic use of non-singular vectorial elements describing the two orbits (Section 2). This removes problems with description of small eccentricities and/or inclinations. Moreover, the use of vector formalism liberates expressions from a particular choice of reference frame, whereas earlier works were firmly attached to the invariable plane. To our knowledge, all previous applications of vectorial elements stopped at the linear perturbations, helping the variables to gain high esteem for simplicity and elegance. The present paper indicates some practical problems with achieving the simplicity at higher orders of perturbations (Section 3.2).

Using the Lie–Hori transformation method as a basic tool, we have implemented a couple of less common devices (partitioned Poisson brackets, purely periodic generator, double normalization with a single generator) that are worth recalling for their efficiency (Section 3, Appendix A).

While no formal constraint on the upper values of the orbital eccentricities is imposed, the assumption of long-term stability of the triple system presents an implicit limit. This is because the secular approach readily implies constant values of the orbital semimajor axes for both orbits. This may be violated if short-range interaction during sufficiently close encounters between the outer star and one in the inner binary occurs. This situation would, however, inevitably result in a instability of the system. Our basic model assumes the participating stars are 'mass monopoles', i.e. their mutual gravitational effects are expressed by a mass-point interaction according to Newton's law. In Section 7, we briefly outline a possibility to include also more complicated interactions in our approach, such as higher-mass multiples of the stars in the inner binary or their tidal effects, but a more involved analysis is beyond the scope of this paper.

## 2 PROBLEM SETUP

Denote stellar masses of the inner binary system $m_0$ and $m_1$, and the mass of the third component $m_2$. The total mass of the inner system is $M_1 = m_0 + m_1$ and the total mass of the triple system $M_2 = m_0 + m_1 + m_2$. Introduce also reduced masses of the inner system $m'_1 = m_0 m_1/M_1$ and for the third component motion about the centre-of-mass of the binary system $m'_2 = m_2 M_1/M_2$. Further notation also profits from introducing $X_0 = m_0/M_1$ and $X_1 = m_1/M_1$, the respective mass contributions of both components of the binary system to it total mass. Note that the equal-mass case has $X_0 = X_1 = \frac{1}{2}$.

Introduce canonical Jacobi variables to describe configuration of the triple system: (i) $\boldsymbol{r}_1$ is the relative position vector of the components in the binary system, and (ii) $\boldsymbol{r}_2$ is the relative position vector of the third star with respect to the centre-of-mass of the binary system. Assuming the whole triple-star system is isolated, the total centre-of-mass performs an ignorable linear motion. Dynamics of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is derived from a Hamiltonian, which in the point-mass approximation reads

$$\mathcal{H}\left(\boldsymbol{r}_1, \boldsymbol{p}_1; \boldsymbol{r}_2, \boldsymbol{p}_2\right) = \mathcal{H}_1\left(\boldsymbol{r}_1, \boldsymbol{p}_1\right) + \mathcal{H}_2\left(\boldsymbol{r}_2, \boldsymbol{p}_2\right) + \mathcal{U}\left(\boldsymbol{r}_1, \boldsymbol{r}_2\right), \tag{1}$$

where $\boldsymbol{p}_j = m'_j \dot{\boldsymbol{r}}_j$ is the momentum associated with $\boldsymbol{r}_j$ coordinate (overdot means time derivative). The Hamiltonian (1) is split in the two-body Hamiltonians of the respective motion of the binary system $\mathcal{H}_1$ and that of the third component $\mathcal{H}_2$ ($k$ is Gaussian constant)

$$\mathcal{H}_1\left(\boldsymbol{r}_1, \boldsymbol{p}_1\right) = \frac{\boldsymbol{p}_1^2}{2m'_1} - k^2 \frac{m'_1 M_1}{r_1}, \tag{2}$$

$$\mathcal{H}_2\left(\boldsymbol{r}_2, \boldsymbol{p}_2\right) = \frac{\boldsymbol{p}_1^2}{2m'_2} - k^2 \frac{m'_2 M_2}{r_2}, \tag{3}$$

and the interaction part

$$\mathcal{U}\left(\boldsymbol{r}_1, \boldsymbol{r}_2\right) = -k^2 \frac{m_2 M_1}{r_2} \sum_{n \geq 2} \chi_n \left(\frac{r_1}{r_2}\right)^n P_n\left(\cos S\right) \tag{4}$$

with $\cos S = \boldsymbol{r}_1 \cdot \boldsymbol{r}_2/(r_1 r_2)$, $P_n$ standing for the Legendre polynomial of degree $n$, and

$$\chi_n = X_0 X_1 \left[X_0^{n-1} - (-X_1)^{n-1}\right]. \tag{5}$$

We assume a long-term stable system for which necessarily $r_1/r_2 < 1$ at all time, making thus the multipole series in (4) well behaved and convergent. A number of studies discuss this question in more depth; a reader may consult e.g. Mardling & Aarseth (2001) or Tokovinin (2014).

As a rule of thumb, (i) odd-degree multipoles are negligible for equal-mass (or near-equal-mass) binary systems for which $X_0 \simeq X_1 \simeq \frac{1}{2}$, and (ii) higher-degree multipoles are quickly negligible for less compact systems, where $r_1/r_2$ is very small.

Instead of the Cartesian coordinates $\boldsymbol{r}_j$ and momenta $\boldsymbol{p}_j$, canonical Delaunay variables can be used. In order to define them in the Jacobian framework, we first set

$$\mathcal{H}_j = -\frac{k^4 \left(m'_j\right)^3 M_j^2}{2L_j^2} = -\frac{k^2 m'_j M_j}{2a_j}. \tag{6}$$

Thus, we define the momentum $L_j$ in terms of both $(\boldsymbol{r}_j, \boldsymbol{p}_j)$ through (2,3), and an osculating semi-axis $a_j$. Linking the latter with orbital periods for $\mathcal{U} = 0$, is done by the Kepler's law $n_j^2 a_j^3 = k^2 M_j$, where the mean motion $n_j = 2\pi/P_j$ is related with a period $P_j$ of the Cartesian Jacobi variables in the motion around $m_0$ for $j = 1$, and around the centre of mass of $m_0$ and $m_1$ for $j = 2$. Of course, then

$$L_j = m'_j \sqrt{k^2 M_j a_j} = m'_j n_j a_j^2, \tag{7}$$

so that the evolution of mean anomaly obeys

$$\dot{\ell}_j = \frac{\partial \mathcal{H}}{\partial L_j} = n_j + \frac{\partial \mathcal{U}}{\partial L_j}.$$

Two angular momentum vectors

$$\boldsymbol{G}_j = \boldsymbol{r}_j \times \boldsymbol{p}_j, \tag{8}$$

serve to define the remaining Delaunay momenta and their relation to osculating eccentricities $e_j$ and inclinations $I_j$

$$G_j = \parallel \boldsymbol{G}_j \parallel = L_j \sqrt{1 - e_j^2}, \qquad H_j = \boldsymbol{G}_j \cdot \hat{\boldsymbol{z}} = G_j \cos I_j. \tag{9}$$

Formally it is best to use the unit vector $\hat{\boldsymbol{z}} = \hat{\boldsymbol{G}}$ along the constant total angular momentum $\boldsymbol{G} = \boldsymbol{G}_1 + \boldsymbol{G}_2$, and benefit from the elimination of the nodes (e.g. Whittaker 1904). But observational conditions may enforce $\hat{\boldsymbol{z}}$ along the line of sight. Moreover, the conservation of $\boldsymbol{G}$ may be violated if additional forces are included. For those reasons, we leave the direction of $\hat{\boldsymbol{z}}$ unspecified in this paper.

The conjugate angles $h_j$ are the longitudes of the ascending nodes in the chosen inertial frame, whereas the arguments of pericentres $g_j$ need two Laplace-Runge-Lenz vectors

$$\boldsymbol{A}_j = L_j \left( \frac{\boldsymbol{p}_j \times \boldsymbol{G}_j}{k^2 \left(m'_j\right)^2 M_j} - \frac{\boldsymbol{r}_j}{r_j} \right), \tag{10}$$

such that

$$\| \boldsymbol{A}_j \| = L_j e_j, \qquad \boldsymbol{A}_j \cdot \boldsymbol{G}_j = 0, \qquad \boldsymbol{A}_j^2 + \boldsymbol{G}_j^2 = L_j^2. \tag{11}$$

The idea of using equations for the evolution of angular momentum and Laplace vectors instead of the usual equations for $g_j$, $h_j$, $G_j$, $H_j$ dates back to Milankovitch (1939). The canonical set is then replaced by

$$\begin{aligned}
\dot{\boldsymbol{G}}_j &= -\boldsymbol{G}_j \times \frac{\partial \mathcal{H}}{\partial \boldsymbol{G}_j} - \boldsymbol{A}_j \times \frac{\partial \mathcal{H}}{\partial \boldsymbol{A}_j}, \\
\dot{\boldsymbol{A}}_j &= -\boldsymbol{A}_j \times \frac{\partial \mathcal{H}}{\partial \boldsymbol{G}_j} - \boldsymbol{G}_j \times \frac{\partial \mathcal{H}}{\partial \boldsymbol{A}_j} - \frac{\partial \mathcal{H}}{\partial \ell_j} \frac{L_j}{A_j^2} \boldsymbol{A}_j, \\
\dot{L}_j &= -\frac{\partial \mathcal{H}}{\partial \ell_j}, \\
\dot{\ell}_j &= \frac{\partial \mathcal{H}}{\partial L_j},
\end{aligned} \tag{12}$$

where $j = 1, 2$. The increased number of equations is expected to be paid back by non-singularity and hopefully a simple algebraic structure.

Actually, Milankovich and most of his followers preferred to use dimensionless vectorial elements

$$\boldsymbol{K}_j = \frac{\boldsymbol{G}_j}{L_j}, \qquad \boldsymbol{e}_j = \frac{\boldsymbol{A}_j}{L_j}, \tag{13}$$

with $\| \boldsymbol{K}_j \| = \eta_j = \sqrt{1 - e_j^2}$, and $\| \boldsymbol{e}_j \| = e_j$. They are particularly advantageous in a normalized system with a Hamiltonian function $\mathcal{H}'$ independent on both mean anomalies. Since momenta $L_j$ and semimajor axes $a_j$ are then constant, the equations of motion for the vectors are

$$\dot{\boldsymbol{K}}_j = -\frac{1}{L_j} \left( \boldsymbol{e}_j \times \frac{\partial \mathcal{H}'}{\partial \boldsymbol{e}_j} + \boldsymbol{K}_j \times \frac{\partial \mathcal{H}'}{\partial \boldsymbol{K}_j} \right), \tag{14}$$

$$\dot{\boldsymbol{e}}_j = -\frac{1}{L_j} \left( \boldsymbol{K}_j \times \frac{\partial \mathcal{H}'}{\partial \boldsymbol{e}_j} + \boldsymbol{e}_j \times \frac{\partial \mathcal{H}'}{\partial \boldsymbol{K}_j} \right). \tag{15}$$

A similar set was used, for example, by Farago & Laskar (2010).

## 3 PERTURBATION THEORY WITH VECTORIAL ELEMENTS

Non-integrability of three-body problem enforces the application of perturbation theory. Unlike in the planetary case, the stellar problem cannot be handled using a variables-independent small parameter like the mass ratio. Following Krymolowski & Mazeh (1999), we order terms according to the powers of the ratio

$$\varepsilon = \frac{L_1}{L_2} = \frac{m'_1}{m_2} \sqrt{\frac{M_2\, a_1}{M_1\, a_2}}. \tag{16}$$

Leaving aside the question of masses, for moderate eccentricities this ratio is comparable with $\sqrt{r_1/r_2}$, but one has to be aware, that even for a fixed and sufficiently small $\varepsilon$, the denominator $r_2$ may become small when $e_2$ increases (with an obvious limit given by stability of the system; e.g. Tokovinin 2004). Working with momenta depending $\varepsilon$ has also a considerable practical drawback: operations of differentiation with respect to $L_j$, inevitable in perturbation methods, affect the ordering of terms that should be grouped according to the powers of the small parameter. Our remedy to this problem is the application of a partitioned Poisson bracket in a recursive Hori-type algorithm modified by Breiter & Métris (2002).

### 3.1 Normalization

Let us express the Hamiltonian $\mathcal{H}$ as a sum

$$\mathcal{H} = \sum_{k \geq 0} \mathcal{H}_{0,k}, \tag{17}$$

where the ratio $\mathcal{H}_{0,k}/\mathcal{H}_{0,0} = O(\varepsilon^k)$. Inspecting the orders of magnitude with respect to $\varepsilon$ in $\mathcal{H}$, we find

$$\begin{aligned}
\mathcal{H}_{0,0} &= \mathcal{H}_1(\boldsymbol{r}_1, \boldsymbol{p}_1), \\
\mathcal{H}_{0,2} &= \mathcal{H}_2(\boldsymbol{r}_2, \boldsymbol{p}_2),
\end{aligned} \tag{18}$$

$$\mathcal{H}_{0,2n+2} = -k^2 \frac{m_2 M_1}{r_2} \chi_n \left( \frac{r_1}{r_2} \right)^n P_n(\cos S) = \mathcal{U}_n, \quad n \geq 2, \tag{19}$$

and all the remaining terms $\mathcal{H}_{0,4} = \mathcal{H}_{0,2n+1} = 0$, for $n \geq 0$.

Not interested in the short periodic perturbations, we aim at constructing a canonical transformation such that a transformed Hamiltonian is independent on mean anomalies $\ell_1$ and $\ell_2$. More precisely, we seek a canonical transformation from the original variables $\boldsymbol{x} \in \mathbb{R}^{12}$ to the mean variables $\boldsymbol{x}^* \in \mathbb{R}^{12}$ such that the transformed Hamiltonian

$$\mathcal{K}(\boldsymbol{x}^*) = \mathcal{H}(\boldsymbol{x}(\boldsymbol{x}^*)) = \sum_{k \geq 0} \mathcal{K}_k,$$

is independent on $\ell_1^*$ and $\ell_2^*$, i.e. the Poisson brackets

$$\{\mathcal{K}_0, \mathcal{K}\} = \{\mathcal{K}_2, \mathcal{K}\} = 0, \tag{20}$$

and $\mathcal{K}_0$, $\mathcal{K}_2$ are first integrals in new variables. In our case, it means that $L_1^*$ and $L_2^*$ (hence the mean semi-axes $a_1^*$ and $a_2^*$) will be new constants of motion.

The transformation between the mean and the original variables can be defined by the Hori type Lie generator $\mathcal{S}(\boldsymbol{y})$, where – depending on the context – $\boldsymbol{y}$ may stand for either $\boldsymbol{x}$ or $\boldsymbol{x}^*$. The generator is also a sum of terms with subsequent powers of $\varepsilon$

$$\mathcal{S} = \sum_{k > 0} \mathcal{S}_k, \qquad \mathcal{S}_k = O(\varepsilon^k). \tag{21}$$

The transformation is performed by the exponentiation of the Lie derivative

$$D_{\mathcal{S}} = D_{\mathcal{S}}^1 = \{\ ,\mathcal{S}\}, \qquad D_{\mathcal{S}}^{k+1} = D_{\mathcal{S}} D_{\mathcal{S}}^k, \tag{22}$$

and for any function $F$

$$F(\boldsymbol{x}(\boldsymbol{x}^*)) = \exp(D_{\mathcal{S}})\, F(\boldsymbol{x}^*) = F^*(\boldsymbol{x}^*). \tag{23}$$

A recursive algorithm to create a new Hamiltonian and the generator was designed for the Hori method by Mersman (1970) as a more convenient analogue of the Lie Triangle algorithm of Deprit (1969). The Hori–Mersman (H–M) algorithm creates an upper triangular array starting with a top row containing $\mathcal{H}_{0,k}(\boldsymbol{x}^*)$ up to $k = N$, and then recursively constructs subsequent diagonals. The sum of the diagonal $n$ provides the $n$-th term of a new Hamiltonian. Thus

$$\mathcal{K}_n(\boldsymbol{x}^*) = \sum_{k=0}^{n} \mathcal{H}_{k,n-k}(\boldsymbol{x}^*), \tag{24}$$

where

$$\mathcal{H}_{n,k}(\boldsymbol{x}^*) = \frac{1}{n}\sum_{m=0}^{k}\left\{\mathcal{H}_{n-1,k-m}(\boldsymbol{x}^*), \mathcal{S}_{m+1}(\boldsymbol{x}^*)\right\}, \tag{25}$$

is a term of order $O(\varepsilon^{n+k})$. Choosing at will the new Hamiltonians $\mathcal{K}_n$, we simultaneously create subsequent terms of the generator $\mathcal{S}$.

Two modifications of the standard H–M algorithm are needed in the present case. First, we cannot use directly equation (25), because the resulting $\mathcal{H}_{n,k}$ would be a sum of two parts having different orders of magnitude. The Poisson bracket is a sum of two principal parts

$$\{F, G\} = \{F, G\}_1 + \{F, G\}_2, \tag{26}$$

$$\{F, G\}_j = \frac{\partial F}{\partial \boldsymbol{r}_j}\cdot\frac{\partial G}{\partial \boldsymbol{p}_j} - \frac{\partial F}{\partial \boldsymbol{p}_j}\cdot\frac{\partial G}{\partial \boldsymbol{r}_j}. \tag{27}$$

Recall that by virtue of the invariance with respect to canonical transformations, the partial derivatives in the Poisson bracket can be taken with respect to any canonical set: original or mean, Cartesian or Delaunay, and so on. If $F = O(\varepsilon^{k_1})$ and $G = O(\varepsilon^{k_2})$, the differentiation decreases powers of $L_j$, so instead of obtaining an $O(\varepsilon^{k_1+k_2})$ terms, we have

$$\{F, G\}_1 = L_1^{-1}\, O(\varepsilon^{k_1+k_2}), \quad \text{and}$$

$$\{F, G\}_2 = L_2^{-1}\, O(\varepsilon^{k_1+k_2}) = L_1^{-1}\, O(\varepsilon^{k_1+k_2+1}). \tag{28}$$

It means that in the H–M algorithm, we can push the action of $\{,\}_2$ to the next diagonal with respect to $\{,\}_1$. A similar approach was presented by Breiter & Métris (2002), leading to a modified rule (25)

$$\mathcal{H}_{n,k}(\boldsymbol{x}^*) = \frac{1}{n}\sum_{m=0}^{k}\left\{\mathcal{H}_{n-1,k-m}(\boldsymbol{x}^*), \mathcal{S}_{m+1}(\boldsymbol{x}^*)\right\}_1 + \frac{1}{n}\sum_{m=1}^{k}\left\{\mathcal{H}_{n-1,k-m}(\boldsymbol{x}^*), \mathcal{S}_{m}(\boldsymbol{x}^*)\right\}_2. \tag{29}$$

The second modification helps to remove both mean anomalies in one transformation in spite of the fact, that $\mathcal{H}_1$ and $\mathcal{H}_2$ have different orders of magnitude. Each term of the generator is decomposed into

$$\mathcal{S}_k = \mathcal{S}_k^1 + \mathcal{S}_k^2, \tag{30}$$

where $\mathcal{S}_k^2$ includes all terms of $\mathcal{S}_k$ independent of $\ell_1$, i.e. such that $\{\mathcal{S}_k^2, \mathcal{H}_1\} = 0$. Note that $\mathcal{S}_k^1$ must depend on $\ell_1$, but it may depend on $\ell_2$ as well. Using partial and complete averaging operators

$$\langle F(\boldsymbol{x})\rangle_j = \frac{1}{2\pi}\int_0^{2\pi} F\,\mathrm{d}\ell_j, \qquad \langle F(\boldsymbol{x})\rangle = \langle\langle F(\boldsymbol{x})\rangle_1\rangle_2, \tag{31}$$

we will define $\mathcal{S}$ so that

$$\langle \mathcal{S}_k\rangle_1 = \mathcal{S}_k^2, \quad \text{and} \quad \langle \mathcal{S}_k\rangle = \langle \mathcal{S}_k^1\rangle = \langle \mathcal{S}_k^2\rangle = 0. \tag{32}$$

These conditions, requiring that the generator itself is a purely periodic function of mean anomalies, is stronger than the usual requirement of a bounded $\mathcal{S}$. We impose it for two reasons. First, the zero-average generator inhibits the occurrence of a long periodic or constant offset between the mean variables $\boldsymbol{x}^*$ and the mean values of the osculating variables $\boldsymbol{x}$ (e.g. Metris & Exertier 1995; Ferraz-Mello 1999). Secondly, it allows us to reject the mean values of some Poisson brackets in higher orders of transformation.

A more detailed account of the normalization up to order 10 is given in Appendix A. Here, we provide only the general form of the transformed Hamiltonian up to order 9.

$$\mathcal{K}_0(\boldsymbol{x}^*) = \mathcal{H}_1(L_1^*),$$
$$\mathcal{K}_2(\boldsymbol{x}^*) = \mathcal{H}_2(L_2^*),$$
$$\mathcal{K}_6(\boldsymbol{x}^*) = \langle \mathcal{U}_2(\boldsymbol{x}^*)\rangle,$$
$$\mathcal{K}_8(\boldsymbol{x}^*) = \langle \mathcal{U}_3(\boldsymbol{x}^*)\rangle,$$
$$\mathcal{K}_9(\boldsymbol{x}^*) = \frac{1}{2}\left\langle \left\{\langle \mathcal{U}_2\rangle_1, \mathcal{S}_3^2\right\}_1\right\rangle_2, \tag{33}$$

where

$$\mathcal{S}_3^2(\boldsymbol{x}^*) = \bar{\mathcal{S}}_3^2(\boldsymbol{x}^*) - \langle \bar{\mathcal{S}}_3^2(\boldsymbol{x}^*)\rangle, \tag{34}$$

is a zero-average generator with

$$\bar{\mathcal{S}}_3^2(\boldsymbol{x}^*) = \frac{1}{n_2^*}\int \left(\langle \mathcal{U}_2(\boldsymbol{x}^*)\rangle_1 - \langle \mathcal{U}_2(\boldsymbol{x}^*)\rangle\right)\mathrm{d}\ell_2^*, \tag{35}$$

and all remaining $\mathcal{K}_n = 0$.

Achieving the elimination of both mean anomalies in a single transformation distinguishes the present solution from earlier works on three-body problem involving two steps: first, creating a temporary Hamiltonian independent on $\ell_1$, and then transforming it to remove $\ell_2$. Of course, the final result should be the same, but we find are approach more convenient. Its underlying idea can be traced back to Morrison (1966), who considered a non-linear oscillator with two angles, using the von Zeipel method.

### 3.2 Practical aspects

#### *3.2.1 Averaging the potential*

In order to perform the averaging required in the construction of secular Hamiltonian $\mathcal{K}$, we have to express $\mathcal{U}_k$ in terms of mean, eccentric or true anomalies. In the first step towards this goal, we use the addition theorem to express $P_n(\cos S)$ in (19) as a function of complex spherical harmonics

$$P_n(\cos S) = P_n(\hat{\boldsymbol{r}}_1\cdot\hat{\boldsymbol{r}}_2) = \frac{4\pi}{2n+1}\sum_{k=-n}^{n} Y_{nk}(\hat{\boldsymbol{r}}_1|\,\mathcal{E})\,Y_{nk}^{\star}(\hat{\boldsymbol{r}}_2|\,\mathcal{E}), \tag{36}$$

where $\hat{\boldsymbol{r}}_j$ is a unit vector along $\boldsymbol{r}_j$ given in a reference frame $\mathcal{E}$. The star $\star$ marks a complex conjugate.

Averaging with respect to the mean anomaly of an inner or outer orbit ($\ell_1$ and $\ell_2$, respectively) is most conveniently performed in an associated orbital reference frame $\mathcal{E}_j$ defined by a right-handed, orthogonal triad $(\boldsymbol{e}_j, \boldsymbol{K}_j\times\boldsymbol{e}_j, \boldsymbol{K}_j)$. The two frames can be linked by 3-1-3 Euler angles $(-\omega_1, J, \omega_2)$, where $J$ is the mutual inclination angle, $\omega_1$ and $\omega_2$ are the angles from the common node of two intersecting orbital planes given by $\boldsymbol{K}_1\times\boldsymbol{K}_2$. Note that this is the ascending node for one orbit, and descending for the other, so only one of $\omega_j$ matches the usual argument of pericentre definition, while the second differs from it by $\pi$. The three angles and eccentricities $e_j$ are sufficient to define the scalar products of vectorial elements

$$\boldsymbol{K}_1\cdot\boldsymbol{K}_2 = \eta_1\eta_2\cos J, \tag{37}$$

$$\boldsymbol{e}_1\cdot\boldsymbol{K}_2 = -e_1\eta_2\sin J\sin\omega_1, \tag{38}$$

$$\boldsymbol{e}_2\cdot\boldsymbol{K}_1 = e_2\eta_1\sin J\sin\omega_2, \tag{39}$$

$$\boldsymbol{e}_1\cdot\boldsymbol{e}_2 = e_1e_2\left[\cos^2\frac{J}{2}\cos(\omega_1-\omega_2) + \sin^2\frac{J}{2}\cos(\omega_1+\omega_2)\right], \tag{40}$$

where

$$\eta_j = \sqrt{1 - e_j^2}. \tag{41}$$

Since $\mathcal{E}_1$ is preferred for $\hat{\boldsymbol{r}}_1$ and $\mathcal{E}_2$ for $\hat{\boldsymbol{r}}_2$ as the arguments of spherical harmonics in (36), we set $\mathcal{E} \equiv \mathcal{E}_1$ as a common frame, and then transform the second function according to the rotation rule for spherical functions

$$Y^\star_{nk}\left(\boldsymbol{r}_2 | \, \mathcal{E}_1\right) = \sum_{k'=-n}^{n} \mathrm{i}^{k-k'} d^n_{k,k'}(J) \exp\left[\mathrm{i}\left(k\omega_1 - k'\omega_2\right)\right] Y^\star_{nk'}\left(\boldsymbol{r}_2 | \, \mathcal{E}_2\right), \tag{42}$$

involving the Wigner d-function (or matrix) $d^n_{k,k'}(J)$ (e.g. Biedenharn & Louck 1981). Observing that in the frame $\mathcal{E}_j$ a vector $\hat{\boldsymbol{r}}_j$ points to the equator and its longitude equals to the true anomaly $f_j$, we easily drop the symbols of spherical functions obtaining

$$P_n\left(\cos S\right) = {\sum_{k,k'}}' C_{nkk'} d^n_{k,k'}\left(J\right) \exp\left[\mathrm{i}\left(k\omega_1 - k'\omega_2\right)\right] \times \exp\left[\mathrm{i}\left(kf_1 - k'f_2\right)\right], \tag{43}$$

with numerical coefficients

$$C_{nkk'} = \frac{\sqrt{(n-|k|)!\,(n+|k|)!\,(n-|k'|)!\,(n+|k'|)!}}{2^{2n}\left(\frac{n-|k|}{2}\right)!\left(\frac{n+|k|}{2}\right)!\left(\frac{n-|k'|}{2}\right)!\left(\frac{n+|k'|}{2}\right)!}. \tag{44}$$

The ‘prime’ attached to the summation symbol in (43) denotes that both $n - |k|$ and $n - |k'|$ are to be even and non-negative, which is a consequence of Legendre associated functions with argument set to zero.

After the substitution of (43) into (19), we are ready to perform the main task. Averaging $\mathcal{U}_n$ is neatly executed by means of Hansen coefficients

$$\frac{1}{2\pi}\int\left(\frac{r_1}{a_1}\right)^n \exp\left(\mathrm{i}kf_1\right) \mathrm{d}\ell_1 = X_0^{n,k}\left(e_1\right), \tag{45}$$

and

$$\frac{1}{2\pi}\int\left(\frac{a_2}{r_2}\right)^{n+1} \exp\left(\mathrm{i}k' f_2\right) \mathrm{d}\ell_2 = X_0^{-(n+1),k'}\left(e_2\right). \tag{46}$$

It is worth recalling that $X_0^{-(n+1),n} = 0$. Thus, we obtain

$$\langle\mathcal{U}_n\rangle = -k^2\frac{m_2M_1}{a_2}\chi_n\left(\frac{a_1}{a_2}\right)^n 2{\sum_{k,k'\geq 0}}' \frac{C_{nkk'}\,X_0^{n,k}\left(e_1\right)X_0^{-(n+1),k'}\left(e_2\right)}{\left(1+\delta_{k0}\right)\left(1+\delta_{k'0}\right)} \times\left[d^n_{k,k'}\left(J\right)\cos\left(k\omega_1 - k'\omega_2\right) + d^n_{-k,k'}\left(J\right)\cos\left(k\omega_1 + k'\omega_2\right)\right], \tag{47}$$

where $\delta_{k0}$ and $\delta_{k'0}$ are Kronecker symbols. This formula must be further manipulated to replace trigonometric functions of angles $J$, $\omega_1$ and $\omega_2$ by vectorial elements thanks to identities (37–40). This is a tedious and tricky part, requiring a ‘manual’ final touch even if executed by computer algebra.

### 3.2.2 *Non-linear part*

When it comes to $\mathcal{K}_9$, more effort is needed due to the Poisson bracket present in its definition. First, we have to find $\langle\mathcal{U}_2\rangle_1$. Partial averaging with respect to the orbital motion of the inner subsystem provides

$$\langle\mathcal{U}_2\rangle_1 = \frac{C_2}{6\eta_2^2}\left(\frac{a_2}{r_2}\right)^2\left[F_0 + \sum_{q=1}^{3} q\left(F'_q\cos qf_2 - F_q\sin qf_2\right)\right], \tag{48}$$

with

$$C_2 = \frac{3}{8}k^2\frac{m_2M_1}{a_2}X_0X_1\left(\frac{a_1}{a_2}\right)^2, \tag{49}$$

$$F_0 = 2\left(1 - 6e_1^2 + 3\Psi\right), \tag{50}$$

and

$$\begin{aligned} F_1 &= -6e_2\Theta\,, & F'_1 &= -e_2\left(1 - 6e_1^2 - 3\Psi + 3\Phi\right),\\ F_2 &= -6\,\Theta, & F'_2 &= -\,3\,\left(1 - 6e_1^2 + \;\Psi + \;\Phi\right),\\ F_3 &= -2e_2\Theta, & F'_3 &= -e_2\left(1 - 6e_1^2 + \;\Psi + \;\Phi\right). \end{aligned} \tag{51}$$

These coefficients depend on a fundamental set of functions $\Psi$, $\Phi$ and $\Theta$ given by

$$\Psi = \frac{1}{\eta_2^2}\left[5\left(\boldsymbol{e}_1\cdot\boldsymbol{K}_2\right)^2 - \left(\boldsymbol{K}_1\cdot\boldsymbol{K}_2\right)^2\right], \tag{52}$$

$$\Phi = \frac{2}{e_2^2}\left[5\left(\boldsymbol{e}_1\cdot\boldsymbol{e}_2\right)^2 - \left(\boldsymbol{K}_1\cdot\boldsymbol{e}_2\right)^2\right], \tag{53}$$

$$\Theta = \frac{1}{e_2^2\eta_2}\left[5\left(\boldsymbol{e}_1\cdot\boldsymbol{e}_2\right)\boldsymbol{e}_1 - \left(\boldsymbol{K}_1\cdot\boldsymbol{e}_2\right)\boldsymbol{K}_1\right]\cdot\left(\boldsymbol{e}_2\times\boldsymbol{K}_2\right). \tag{54}$$

Subtracting the complete average

$$\langle\mathcal{U}_2\rangle = \frac{C_2}{6\eta_2^3}F_0, \tag{55}$$

and performing the integration (35), we obtain

$$\bar{\mathcal{S}}_3^2(\boldsymbol{x}) = \frac{C_2}{6n_2\eta_2^3}\left[F_0\left(f_2 - \ell_2\right) + \sum_{q=1}^{3}\left(F_q\cos qf_2 + F'_q\sin qf_2\right)\right], \tag{56}$$

similar to $S_1^*$ of Harrington (1969). The average of this generator is not zero. One easily checks that

$$\left\langle\bar{\mathcal{S}}_3^2\right\rangle = \left\langle\bar{\mathcal{S}}_3^2\right\rangle_2 = \frac{C_2}{6n_2\eta_2^3}\sum_{q=1}^{3}F_q\left\langle\cos qf_2\right\rangle_2 = -\frac{C_2}{6n_2\eta_2^3}\sum_{q=1}^{3}F_qW_q, \tag{57}$$

where, according to Kozai (1962)

$$W_q = -\left(-e_2\right)^q\frac{1+q\eta_2}{\left(1+\eta_2\right)^q}. \tag{58}$$

Specifically, we need

$$W_1 = e_2, \quad W_2 = -e_2^2\frac{1+2\eta_2}{\left(1+\eta_2\right)^2} = \frac{2\eta_2^2}{1+\eta_2} - 1,$$

$$W_3 = e_2^3\frac{1+3\eta_2}{\left(1+\eta_2\right)^3} = e_2\left[1 - \frac{4\eta_2^2}{\left(1+\eta_2\right)^2}\right]. \tag{59}$$

The required Poisson bracket average

$$\left\langle\left\{\langle\mathcal{U}_2\rangle_1, \mathcal{S}_3^2\right\}_1\right\rangle_2 = \left\langle\left\{\langle\mathcal{U}_2\rangle_1, \bar{\mathcal{S}}_3^2\right\}_1\right\rangle_2 - \left\langle\left\{\langle\mathcal{U}_2\rangle_1, \left\langle\bar{\mathcal{S}}_3^2\right\rangle\right\}_1\right\rangle_2, \tag{60}$$

is evaluated by substituting (48) for $\langle\mathcal{U}_2\rangle_1$, (56) for $\bar{\mathcal{S}}_3^2$, and (57) for $\left\langle\bar{\mathcal{S}}_3^2\right\rangle$. Benefiting from linearity and skew symmetry of Poisson brackets and rejecting average values of odd functions, we obtain

$$\left\langle \left\{ \langle \mathcal{U}_2 \rangle_1 , \bar{\mathcal{S}}_3^2 \right\}_1 \right\rangle_2 = \frac{C_2^2}{36\, n_2\, \eta_2^5} \sum_{q=1}^{3} \left[ q \left\langle \left( \frac{a_2}{r_2} \right)^2 \right\rangle_2 \left\{ F_q', F_q \right\}_1 + q \left\langle \left( \frac{a_2}{r_2} \right)^2 (f_2 - \ell_2) \sin q f_2 \right\rangle_2 \left\{ F_0, F_q \right\}_1 \right]. \tag{61}$$

Similarly

$$\left\langle \left\{ \langle \mathcal{U}_2 \rangle_1 , \langle \bar{\mathcal{S}}_3^2 \rangle \right\}_1 \right\rangle_2 = -\frac{C_2^2}{36\, n_2\, \eta_2^5} \sum_{q=1}^{3} \left\langle \left( \frac{a_2}{r_2} \right)^2 \right\rangle_2 W_q \left\{ F_0, F_q \right\}_1 . \tag{62}$$

Then, thanks to

$$\left\langle \left( \frac{a_2}{r_2} \right)^2 \right\rangle_2 = \frac{1}{\eta_2}, \qquad \left\langle \left( \frac{a_2}{r_2} \right)^2 (f_2 - \ell_2) \sin q f_2 \right\rangle_2 = \frac{W_q}{q \eta_2}, \tag{63}$$

we find $\mathcal{K}_9$ as a half of the difference of (61) and (62)

$$\mathcal{K}_9 = \frac{C_2^2}{72\, n_2\, \eta_2^6} \sum_{q=1}^{3} \left( q \left\{ F_q', F_q \right\}_1 + 2 W_q \left\{ F_0, F_q \right\}_1 \right). \tag{64}$$

Actually, for a pair of functions independent on $\ell_1$ and expressed in terms of $\boldsymbol{e}_1$ and $\boldsymbol{K}_1$, like $F_0$, $F_q$ and $F_q'$, we can use a reduced Poisson bracket: if $f'$ and $g'$ are two functions with the above properties and each occurrence of $\eta_1$ is replaced by $\sqrt{1-e_1^2}$ for the differentiation, then

$$\left\{ f'(e_1, \boldsymbol{e}_1, \boldsymbol{K}_1),\ g'(e_1, \boldsymbol{e}_1, \boldsymbol{K}_1) \right\}_1 = \frac{1}{L_1} \left( f'; g' \right), \tag{65}$$

where

$$\left( f'; g' \right) = \frac{1}{e_1} (\boldsymbol{e}_1 \times \boldsymbol{K}_1) \cdot \left[ \left( \partial_{e_1} g' \right) \nabla_{\boldsymbol{e}_1} f' - \left( \partial_{e_1} f' \right) \nabla_{\boldsymbol{e}_1} g' \right] + \boldsymbol{K}_1 \cdot \left[ \nabla_{\boldsymbol{e}_1} f' \times \nabla_{\boldsymbol{e}_1} g' + \nabla_{\boldsymbol{K}_1} f' \times \nabla_{\boldsymbol{K}_1} g' \right] + \boldsymbol{e}_1 \cdot \left[ \nabla_{\boldsymbol{K}_1} f' \times \nabla_{\boldsymbol{e}_1} g' + \nabla_{\boldsymbol{e}_1} f' \times \nabla_{\boldsymbol{K}_1} g' \right]. \tag{66}$$

Thus, we obtain

$$\mathcal{K}_9 = \frac{C_2^2}{6 L_1 n_2 \eta_2^6} \left[ \eta_2^2 \frac{5+\eta_2}{1+\eta_2} \left( \Psi - 2 e_1^2 ; \Theta \right) + \left( 5 - 2\eta_2^2 \right) \left( \Phi - 4 e_1^2 ; \Theta \right) \right]. \tag{67}$$

The bracket (66) acts like a bilinear differential operator, so three partial results are sufficient to find the Hamiltonian: $(e_1^2; \Theta)$, $(\Psi; \Theta)$, and $(\Phi; \Theta)$. Thanks to the use of vectorial formalism, the brackets can be cast into a compact form. Let

$$Q_{pq} = \boldsymbol{v}_p \cdot \boldsymbol{w}_q, \tag{68}$$

stand for the scalar products of two vectors belonging to an orthogonal, orbit orienting set:

$$\boldsymbol{v}_1 = \boldsymbol{e}_1, \quad \boldsymbol{v}_2 = \boldsymbol{K}_1 \times \boldsymbol{e}_1, \quad \boldsymbol{v}_3 = \boldsymbol{K}_1, \tag{69}$$

$$\boldsymbol{w}_1 = \boldsymbol{e}_2, \quad \boldsymbol{w}_2 = \boldsymbol{K}_2 \times \boldsymbol{e}_2, \quad \boldsymbol{w}_3 = \boldsymbol{K}_2. \tag{70}$$

Using this notation, we obtain a fairly economic form

$$(e_1^2; \Theta) = \frac{10}{e_2^2 \eta_2} \left[ Q_{11} \left( Q_{13} Q_{31} - Q_{11} Q_{33} \right) - Q_{12} Q_{21} \right], \tag{71}$$

$$(\Psi; \Theta) = \frac{2}{e_2^2 \eta_2^3} \left\{ 20 Q_{13} \left( \eta_2^2 Q_{11} Q_{31} - Q_{12} Q_{32} \right) + Q_{33} \left[ 5 Q_{12}^2 - Q_{32}^2 - \eta_2^2 \left( 5 Q_{11}^2 - Q_{31}^2 \right) \right] \right\}, \tag{72}$$

$$(\Phi; \Theta) = \frac{4}{e_2^2 \eta_2} \left[ 10 Q_{11} Q_{13} Q_{31} - Q_{33} \left( 25 Q_{11}^2 + Q_{31}^2 \right) \right]. \tag{73}$$

### 3.2.3 Removal of apparent singularity

Equations (71)–(73) involve a troubling factor $e_2^{-2}$. It might look like an apparent contradiction to non-singularity of the vectorial elements. One may argue that introducing normalized functions

$$\hat{Q}_{pq} = \frac{Q_{pq}}{\parallel \boldsymbol{v}_p \parallel \parallel \boldsymbol{w}_q \parallel}, \tag{74}$$

solves the problem, because it is possible to extract $e_2^2$ from each product or square present in the equations. But then we are left with e.g. $\hat{Q}_{11} = \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2$ defined in terms of a meaningless unit vector $\hat{\boldsymbol{e}}_2$, because the zero vector cannot be normalized.

There is, however, a number of possible transformation rules stemming from the fact that $\hat{Q}_{pq}$ can be seen as direction cosines of two frames, hence – as the elements of an orthogonal matrix from SO(3, $\mathbb{R}$). Accordingly:

(i) the squared terms of each row or column sum up to 1,
(ii) scalar products of two different rows or columns are null, and
(iii) a cross-product of two rows or columns, taken in correct order, generates a third one.

From technical point of view, the freedom offered by 18 identities is both a blessing and a curse of vectorial elements, because obtaining the simplest (but necessarily non-singular) form of a non-trivial expression is a tricky exercise in the algebra of multivariate polynomials. Nevertheless, working rather by trial and error, we found a non-singular combinations of the brackets (71–73) leading to the Hamiltonian $\mathcal{K}_9$ shown in the next section.

## 4 SECULAR HAMILTONIAN

In this section, we present the final form of the secular Hamiltonian. For brevity we omit the asterisks, but all symbols except physical constants refer to the mean variables. Introducing the non-Keplerian perturbing function $\mathcal{R}$, we have

$$\mathcal{K} = -\frac{k^2 m_1' M_1}{2 a_1} - \frac{k^2 m_2' M_2}{2 a_2} + \mathcal{R}, \tag{75}$$

where, up to $\varepsilon^9$,

$$\mathcal{R} = \mathcal{K}_6 + \mathcal{K}_8 + \mathcal{K}_9. \tag{76}$$

The formal ordering of specific terms can be recovered from equations (33). Their expressions in terms of mean Laplace and angular momentum vectors are the following. The quadrupole potential term is

$$\mathcal{K}_6 = \langle \mathcal{U}_2 \rangle = -\frac{C_2}{\eta_2^5} \left[ \eta_2^2 \left( 2 e_1^2 - \frac{1}{3} \right) + (\boldsymbol{K}_1 \cdot \boldsymbol{K}_2)^2 - 5 (\boldsymbol{e}_1 \cdot \boldsymbol{K}_2)^2 \right] = \frac{C_2}{\eta_2^5} \left[ \frac{1}{3} \eta_2^2 \left( 1 - 6 e_1^2 \right) + 5 Q_{13}^2 - Q_{33}^2 \right], \tag{77}$$

where the $C_2$ coefficient defined by (49) includes the factor $a_1^2/a_2^3$, proportional to $a_1^{-1}(L_1/L_2)^6$. The octupole term is

$$\mathcal{K}_8 = \langle \mathcal{U}_3 \rangle = -\frac{C_3}{\eta_2^7}\Big[\eta_2^2\left(1-8e_1^2\right)(\boldsymbol{e}_1\cdot\boldsymbol{e}_2) - 5(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)(\boldsymbol{K}_1\cdot\boldsymbol{K}_2)^2 + 35(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)(\boldsymbol{e}_1\cdot\boldsymbol{K}_2)^2 - 10(\boldsymbol{e}_1\cdot\boldsymbol{K}_2)(\boldsymbol{e}_2\cdot\boldsymbol{K}_1)(\boldsymbol{K}_1\cdot\boldsymbol{K}_2)\Big]$$
$$= \frac{C_3}{\eta_2^7}\Big[\eta_2^2\left(8e_1^2-1\right)Q_{11} + 5Q_{11}Q_{33}^2 - 35Q_{11}Q_{13}^2 + 10Q_{13}Q_{31}Q_{33}\Big], \tag{78}$$

where

$$C_3 = \frac{15}{64}\frac{k^2 m_2 M_1}{a_2}X_0X_1(X_0-X_1)\left(\frac{a_1}{a_2}\right)^3 = C_2\frac{5}{8}(X_0-X_1)\frac{a_1}{a_2}. \tag{79}$$

Both quadrupole and octupole first-order potentials were already derived in the literature. Notably, $\mathcal{K}_6$ has been given in our form by Farago & Laskar (2010) and $\mathcal{K}_8$ by Liu, Muñoz & Lai (2015). In both cases, the authors use Milankovich variables. Additionally, both $\mathcal{K}_6$ and $\mathcal{K}_8$ were also given in singular variables (e.g. Harrington 1969; Söderhjelm 1975, 1984; Krymolowski & Mazeh 1999; Lee & Peale 2003; Khodykin, Zakharov & Andersen 2004), though the octupole part often in some limiting case (such as small eccentricities $e_1$ and $e_2$ and/or coplanar orbits).

Finally, the first part of non-linear, self-coupled quadrupole contribution, appears at the order $\varepsilon^9$ and reads

$$\mathcal{K}_9 = \frac{C_2'(5+\eta_2)}{(1+\eta_2)^2\eta_2^7}\Bigg\{20\left(1+3\eta_2\frac{3+\eta_2}{5+\eta_2}\right)[\boldsymbol{e}_1\cdot(\boldsymbol{e}_2\times\boldsymbol{K}_2)] \times [\boldsymbol{e}_2\cdot(\boldsymbol{e}_1\times\boldsymbol{K}_1)] - (\boldsymbol{K}_1\cdot\boldsymbol{K}_2)\Bigg[(1+\eta_2)\left(\left(1+24e_1^2\right)\eta_2^2 - (\boldsymbol{K}_1\cdot\boldsymbol{K}_2)^2 - 15(\boldsymbol{e}_1\cdot\boldsymbol{K}_2)^2\right) + 2\left(1+3\eta_2\frac{3+\eta_2}{5+\eta_2}\right) \times \left((\boldsymbol{K}_1\cdot\boldsymbol{e}_2)^2 + 15(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)^2\right)\Bigg]\Bigg\},$$
$$= C_2'B_1\left\{20B_2Q_{12}Q_{21} - Q_{33}\left[\left(1+24e_1^2\right)\eta_2^2 - Q_{33}^2 - 15Q_{13}^2 + 2B_2\left(Q_{31}^2+15Q_{11}^2\right)\right]\right\}, \tag{80}$$

where

$$B_1 = \frac{5+\eta_2}{\eta_2^7(1+\eta_2)}, \qquad B_2 = \frac{5+10\eta_2+3\eta_2^2}{(1+\eta_2)(5+\eta_2)}, \tag{81}$$

and

$$C_2' = \frac{1}{3}\frac{C_2^2}{L_1n_2} = C_2\frac{1}{8}\frac{m_2}{M_1}\frac{n_1}{n_2}\left(\frac{a_1}{a_2}\right)^3 = C_2\frac{1}{8}\frac{m_2}{M_2}\frac{n_2}{n_1}. \tag{82}$$

This term is novel in the literature. Since the aim of Krymolowski & Mazeh (1999) was to derive a potential equivalent to our $\mathcal{K}_9$, we comment on their result in the next section.

## 5 REFERENCE TO EARLIER WORKS

### 5.1 Krymolowski & Mazeh (1999)

Confronting the results shown in Section 4 with the Hamiltonian of Krymolowski & Mazeh (1999) one should bear in mind that the latter was obtained using the Poincaré-von Zeipel perturbation method, whereas this work is based upon the Lie–Hori technique. But, as shown by Yuasa (1971), the secular Hamiltonians generated by the two methods should be equal; only the generating functions are linked with more sophisticated equivalence relations. Few elementary manipulations may prove that $\mathcal{K}_6$ from equation (77) and $\mathcal{K}_8$ from equation (78), if expressed in terms of orbital elements, are exactly equal to the quadrupole and octupole terms of the Hamiltonian function given by Krymolowski & Mazeh (1999). Thus, an interesting part of the comparison is related to the non-linear or 'indirect' term $\mathcal{K}_9$.

There are two reasons to expect some differences between Krymolowski & Mazeh (1999) and the present work. First, the purely periodic generator $\mathcal{S}_3^2$ is different from the one used in all earlier works and the influence of this choice should manifest in non-linear terms starting from $\mathcal{K}_9$. But, more importantly, our solutions use a different level of approximation. Equation (A7) in the appendix of Krymolowski & Mazeh (1999) is to some extent comparable with our (A19) with all partial derivatives from Poisson brackets written in full length. For unexplained reasons, the authors picked up from their (A7) only two terms out of the five involving their generator $S_2^*$. Since this work uses a complete Poisson bracket, apart from the separation (26), we expected the $\mathcal{K}_9$ to be a more complicated expression than $\mathcal{K}_{\mathrm{KM}}$ – the part of the Hamiltonian factored by $C_{2,\,\mathrm{ind}}$ from Krymolowski & Mazeh (1999). To our surprise, it is quite the contrary, and not just by the use of the compact $Q_{pq}$ notation in (80).

If we reduce $\mathcal{K}_9$ and $\mathcal{K}_{\mathrm{KM}}$ to the same, unique form of a trigonometric polynomial in arguments of pericentres $g_1$ for the inner and $g_2$ for the outer orbit (not to be confused with $\omega_i$ from Section 3.2.1), we find that

$$\mathcal{K}_9 = C_2'\left[b_0 + b_1\cos 2g_1 + b_2\cos 2g_2 + b_3\cos 2(g_1+g_2) + b_4\cos 2(g_1-g_2)\right], \tag{83}$$

whereas

$$\mathcal{K}_{\mathrm{KM}} = C_2'\left[b_0' + b_1'\cos 2g_1 + b_2'\cos 2g_2 + b_3'\cos 2(g_1+g_2) + b_4'\cos 2(g_1-g_2) + b_5'\cos(4g_1+2g_2) + b_6'\cos(4g_1-2g_2)\right], \tag{84}$$

where $b_k$ and $b_k'$ are some functions of eccentricities and of mutual inclination of orbits.

The two Hamiltonians coincide only when the outer orbit is circular ($e_2 = 0$). Otherwise, for any values of the elements, only two amplitudes are equal: $b_0 = b_0'$, and $b_1 = b_1'$. The equivalence at $e_2 = 0$ is actually related with the vanishing of all $b_k$ and $b_k'$ for $k > 1$. We have not undertaken a laborious task of reconstructing the terms rejected by Krymolowski & Mazeh. However, assuming that $\mathcal{K}_9$ is complete and accurate at the order $\varepsilon^9$, we can use the dependence of $\mathcal{K}_9 - \mathcal{K}_{\mathrm{KM}}$ on the factor $e_2^2$ as an explanation why the authors did achieve some improvement when including $\mathcal{K}_{\mathrm{KM}}$ in their numerical tests.

### 5.2 Brown (1936)

It has been overlooked so far that actually the first paper that provides the principal non-linear quadrupole part in $\mathcal{K}_9$ was published by Brown (1936c). Indeed, his formula $\frac{1}{2}\,(R_f\,,\,S)_c$ in page 121 coincides with our $\mathcal{K}_9$ when $e_2 = 0$. Interestingly, according to Brown, this additional term helped him to reconcile a small discrepancy between his lunar theory and observations to-date as far as secular drift of the lunar perigee was concerned. In the next section, we show this is indeed the proper role of the $\mathcal{K}_9$ term in our numerical experiment.

## 6 COMPARISON WITH NUMERICAL INTEGRATION

Here, we briefly confront orbital evolution as provided by the secular three-body Hamiltonian at different levels of approximation with results of detailed numerical integration. In fact, both result from a numerical integration: the secular theory from integration of equations (14) and (15), with an appropriate representation of the Hamiltonian function $\mathcal{H}'$, and the direct model from integration of the Hamilton's equations in Jacobi variables $(\boldsymbol{r}_1, \boldsymbol{r}_2; \boldsymbol{p}_1, \boldsymbol{p}_2)$ (see also Söderhjelm 1982 or Harrington 1991). As usual, the advantage of the secular theory stems from a possibility to choose a much longer timestep at the exchange of impossibility to study orbital variations with frequencies comparable to orbital mean motion values $n_1$ and $n_2$. Typically, the CPU gain of the secular-theory integration is factor of hundred or more. This can allow us to analyse orbital evolution of the system on a much longer time-scale, or study more efficiently its dependence on parameters such as stellar masses or initial orbital data. For reasons we explain below, we choose a synthetic, model case for the comparison between the orbital evolution given by the secular theory and complete numerical integration. However, the parameters are very similar to several real systems of interest, such as V907 Sco, HD 109648 or $\xi$ Tau (e.g. Lacy, Helt & Vaz 1999; Jha et al. 2000; Nemravová et al. 2013).

In order to pursue comparison with the work of Krymolowski & Mazeh (1999), we chose one of their four examples, namely the 'mild inclination' case in their Section 3.2. Specifically, we considered stellar masses $m_0 = 1\ \mathrm{M}_\odot$, $m_1 = 3.7\ \mathrm{M}_\odot$, $m_2 = 2.07\ \mathrm{M}_\odot$, and unperturbed orbital periods $P_1 = 5.33$ d and $P_2 = 149.24$ d. These values satisfy $q_\mathrm{S} = m_0/m_1 = 0.27$, $q_\mathrm{L} = m_2/M_1 = 0.44$ and $P_2/P_1 = 28$ imposed by Krymolowski & Mazeh (1999), and imply orbital semimajor axes $a_1 \simeq 0.1$ au and $a_2 \simeq 1.04$ au. For reference, we mention the value $\varepsilon \simeq 0.14$ of the parameter in equation (16), relevant to ordering of the Hamiltonian contributions. We also assumed initial eccentricity values $e_1 = 0.08$ and $e_2 = 0.27$. The reference system, we would call Laplacian in accord with tradition in planetary dynamics, has been chosen such that the total orbital angular momentum $\boldsymbol{G}$ is directed along the $\hat{z}$ axis. Associated axial symmetry allowed us to choose the ascending node directions of the inner and outer orbits to be $0^\circ$ and $180^\circ$, respectively. In accord with Krymolowski & Mazeh (1999), we took the following values of the argument of pericentre: $g_1 = 0^\circ$ for the inner orbit and $g_2 = 270^\circ$ for the outer orbit. The mutual angle of the orbital planes $J = 20^\circ$ implies, together with the assumed $\varepsilon$ and orbital eccentricities, its partitioning into inclination values $i_1$ and $i_2$ of the inner and outer orbits with respect to the Laplacian plane ($J = i_1 + i_2$). We obtain $i_1 \simeq 17.5^\circ$ and $i_2 \simeq 2.5^\circ$. Finally, the detailed numerical integration requires to assume initial position in orbit: for simplicity we take both orbits in their respective pericentre, hence $\ell_1 = 0^\circ$ and $\ell_2 = 0^\circ$. This choice sets the phase of short-period perturbations and, in principle, affects the needed initial conditions of the numerical integration of the secular system. They can be found either analytically, evaluating short-period terms using the generating function $S \simeq S_3^2$ (equations 21, 56 and 57), or estimated numerically. At this moment, we opt for the easier numerical evaluation. For instance, instead of osculating values of the semimajor axis $a_1$ and $a_2$ from the initial data of the detailed numerical integration of the two orbits, we numerically determine their long-term mean values $\bar{a}_1$ and $\bar{a}_2$. These are considered for integration of the secular system. Similarly, the initial values of other orbital elements are numerically estimated from the initial few orbits of the detailed integration.

Finally, we have a possibility to choose representation of the interaction potential $\mathcal{U}\,(\boldsymbol{r}_1, \boldsymbol{r}_2)$ from equation (1). The point-mass model, with $\mathcal{U}\,(\boldsymbol{r}_1, \boldsymbol{r}_2) = -k^2\, m_0 m_1/r_1 - k^2\, m_0 m_2/r_{02} - k^2\, m_1 m_2/r_{12}$, $r_{02}$ and $r_{12}$ distances between the respective stars, would seem the simplest choice, resulting in equations given by Söderhjelm (1982) or Harrington (1991). Another option is to use the series representation of $\mathcal{U}\,(\boldsymbol{r}_1, \boldsymbol{r}_2)$ from equation (4). We implemented both, but present here results from the second case. This is because this option gives us more possibilities to test our secular theory. For instance, arbitrarily restricting the series representation by the octupole term $n = 3$, we could have checked that our secular implementation with the corresponding term $\mathcal{K}_8$ gives the same orbital evolution (preventing thus any mistypings in the code).

Figs 1 (inner orbit) and 2 (outer orbit) show our results which are to be compared with those in fig. 2 of Krymolowski & Mazeh (1999), who focused on the temporal evolution of the inner eccentricity $e_1$ only. The timespan of 500 yr shown here roughly corresponds to the $4 \times 10^4$ revolutions from Krymolowski & Mazeh (1999) figure, but provides a clearer impression about the time-scale of the effects we describe in terms of orbital evolution of real astronomical systems. Left-hand panels in both figures show a situation, where the secular-theory run contained only potential terms $\mathcal{K}_6$ and $\mathcal{K}_8$, thus first-order averaged interaction expressed to the octupole level. Right-hand panels shows results, where the secular theory was complete to the order 9 in $\varepsilon$, i.e. contained also the non-linear quadrupole term $\mathcal{K}_9$. In order to keep the closest correspondence of the results, we included quadrupole and octupole interaction terms in the detailed numerical integration in both cases. When higher multipoles, or the point-mass potential representation, were included we observed only small change of the results, not relevant to the discussion in this paper.

Fig. 1 indicates that the principal effect of the non-linear term $\mathcal{K}_9$ is in evolution of the dimensionless Laplace vector $\boldsymbol{e}_1$ of the inner orbit; this has been also reported by Krymolowski & Mazeh (1999), though these authors do not discuss other orbital elements and time evolution of the outer orbit. In particular, the main improvement is in correction of the secular drift of the inner orbit longitude of pericentre $\varpi_1 = h_1 + g_1$. Krymolowski & Mazeh (1999) show that long-term periodic effects in $e_1$ come as a combination of terms with $\varpi_1 - \varpi_2$ and $2(\varpi_1 - \varpi_2)$ frequencies. Since $\varpi_2$ is not much affected by higher-order terms, other than $\mathcal{K}_6$, on the given time-scale (Fig. 2), the improvement in $\varpi_1$ is reflected in phase-alignment in $e_1$ oscillations when results of the numerical integration are compared to those of the secular theory. We observed that the phase shift over the 500 yr time improves from $\simeq 1.5$ when only the $\mathcal{K}_6$ and $\mathcal{K}_8$ potentials are taken into account (Fig. 1, left) to $\simeq 0.25$ when the $\mathcal{K}_9$ term is included (Fig. 1, right). This formally represent an improvement of $\simeq 0.25/1.5 \simeq 0.17$, which is indeed of the order $\varepsilon$. This is readily expected when a further term in the development of the potential is accounted for. We expect that further improvements will be achieved by implementation of the

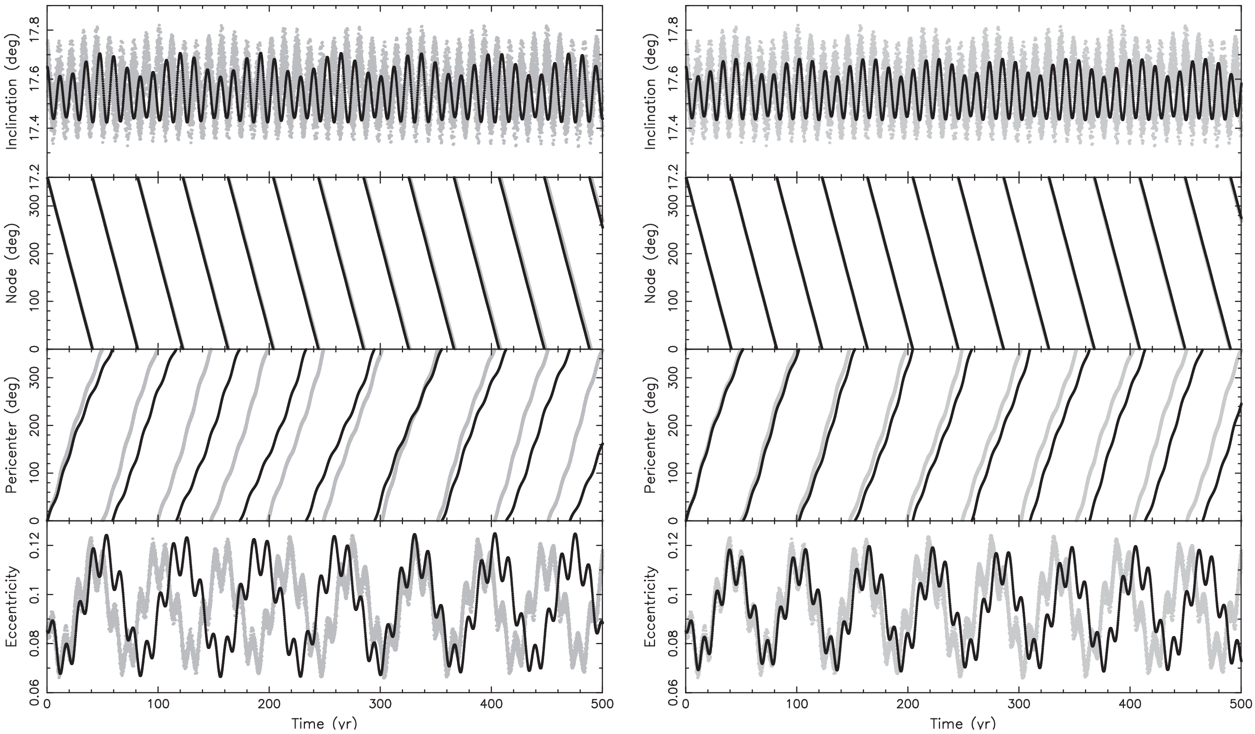


**Figure 1.** Comparison between the numerical integration in Jacobi variables (grey symbols) and in mean variables (black symbols). Orbital elements of the inner binary system are plotted as functions of time: from top to bottom, the panels we show inclination with respect to the invariable plane, longitude of node, longitude of pericentre and eccentricity. Left: the secular theory contains only $\mathcal{K}_6$ and $\mathcal{K}_8$ terms; right: the secular theory contains $\mathcal{K}_9$ in addition. Initial conditions and parameters in the text.

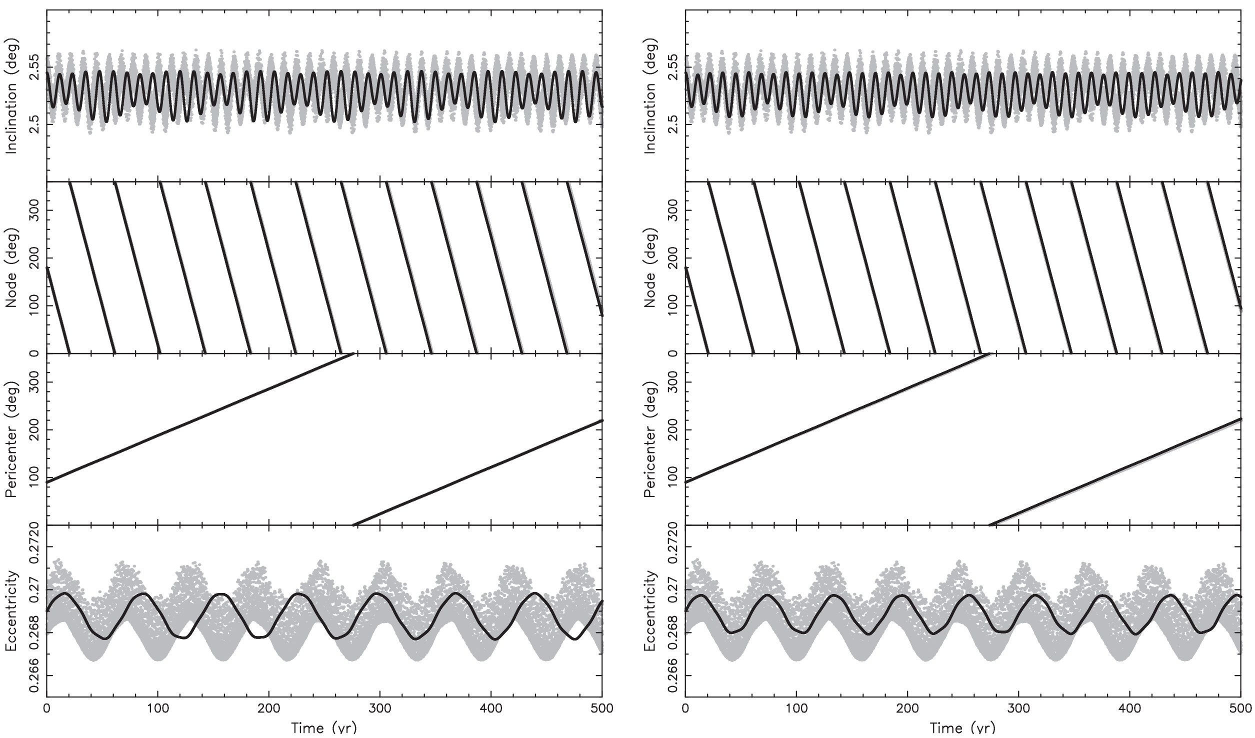


**Figure 2.** The same as in Fig. 1, for orbital elements of the outer orbit.

$\mathcal{K}_{10}$ potential in the secular theory, making then the agreement between the detailed numerical evolution of the triple system with those from the secular theory already very satisfactory on a given time-scale of a few centuries.

We note that the effect of higher order potentials in nodal longitude is minor, as expected from the above mentioned observation of Brown (1936c). Secular drift in the nodes $h_1$, or equivalently $h_2$, are very well expressed by

$$\frac{\dot{h}_1}{n_1} \simeq \frac{3}{4\eta_2^3}\frac{m_2}{M_2}\left(\frac{n_2}{n_1}\right)^2 \cos J\sqrt{1+\gamma^2+2\gamma\cos J}, \tag{85}$$

where $\gamma = \varepsilon/\eta_2$ (e.g. Söderhjelm 1975). This formula is readily obtained from a simplified secular theory containing just the first-order quadrupole term $\mathcal{K}_6$. For low enough mutual inclination $J$ of the inner and outer orbits, the equations admit solution $e_1 = 0$ which is stable (e.g. Farago & Laskar 2010). In this approximation both $e_2$ and $J$ are constant, and orbital nodes drift in Laplacian frame with frequency (85). For parameters of the system considered here, one obtains nodal circulation period of $\simeq 41$ yr, in good accord with Figs 1 and 2. Similarly, the evolution of the outer orbit is well represented by the first-order complete secular theory, including the pericentre longitude $\varpi_2$. Its drift rate is given by

$$\frac{\dot{\varpi}_2}{n_2} \simeq \frac{3}{8\eta_2^4}X_0X_1\left(\frac{a_1}{a_2}\right)^2$$
$$\left(3\cos^2 J - 1 - \frac{\gamma\sin J\sin 2J}{1+\gamma\cos J+\sqrt{1+\gamma^2+2\gamma\cos J}}\right), \tag{86}$$

also provided by the simplified theory with $e_1 = 0$ set (e.g. Söderhjelm 1975). We observe a small improvement in tiny oscillations of the outer orbit eccentricity $e_2$, again related to its dependence on the $\varpi_1 - \varpi_2$ argument (and its multiples). Note, however, that their amplitude is quite small, comparable to that of the short-period terms. This obviously results from the fact that $e_2$ is conserved in $\mathcal{K}_6$-complete secular theory (e.g. Farago & Laskar 2010).

## 7 TIDAL FRICTION FOR CLOSE BINARY SYSTEMS

Long-term stable triple systems have typically (i) the inner binary with an orbital period of 1–10 d, and (ii) the third component revolving about its centre-of-mass with an orbital period of a month or more. The short orbital period of the inner system implies that, for solar-mass components, the separation $a_1$ is only an order of magnitude (or even less) larger than radii of the constituting stars. In this situation, the point-mass model developed above is necessarily only an approximation of their true motion. A vast complexity of dynamical effects related to finite size of the components and possibly a mass-transfer in the binary needs to be taken into account. Moreover, a framework describing only the translational motion of their centre-of-mass has to be complemented by description of their rotational motion in general.

At this moment, our goal is not to provide a complete theory that would cover all such phenomena. We intent to complement our previous formulation by the minimum necessary effects to prevent unphysical solutions. Perhaps the most important feature is an unavoidable excitation of the eccentricity $e_1$ of the inner binary by the third component when $m_0 \neq m_1$ (see, e.g. Section 6). Even if $m_0 = m_1$, the zero $e_1$ solution is not stable whenever the mutual inclination $J$ of the inner and outer orbits exceeds certain critical limit (about 40°). This is the well-known Kozai-Lidov phenomenon. In this case too, $e_1$ is excited to have large excursions until the system, on a long term, either adjusts to a subcritical $J$ value or shrinks enough the semimajor axis $a_1$ of the inner binary so that their tidal interaction helps maintaining $e_1$ small (systems which fail to do so would disrupt, e.g. Fabrycky & Tremaine 2007). In any case, tides effectively work against any state with large $e_1$ in long-term stable systems.

The simplest way to complement our point-mass model with the tidal friction is to directly implement the eccentricity damping for the inner binary. This can be achieved by introducing a fictive acceleration

$$\boldsymbol{f}_{\mathrm{tid}} = -\frac{2}{\tau}\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r^2}\boldsymbol{r} \tag{87}$$

in the motion of the inner binary. Here, $\boldsymbol{r}$ and $\boldsymbol{v}$ are relative position and velocity of the second component with respect to the first component and $\tau$ is a characteristic friction time-scale. With (87) inserted in the Gauss equations of the perturbation theory, one readily obtains

$$\frac{\mathrm{d}\boldsymbol{K}_1}{\mathrm{d}t} = \frac{2}{\tau}\frac{e_1^2}{1+\eta_1}\boldsymbol{K}_1\,, \tag{88}$$

$$\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} = -\frac{2}{\tau}\frac{\eta_1^2}{1+\eta_1}\boldsymbol{e}_1\,, \tag{89}$$

and

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = -\frac{4}{\tau}\frac{a_1e_1^2}{1+\eta_1}\,. \tag{90}$$

For small $e_1$ orbits, $\tau \simeq -e_1/(\mathrm{d}e_1/\mathrm{d}t)$ and thus $\tau$ is directly the circularization time-scale. The value of $\tau$ may be estimated from the first principles within the dynamical model of tidal friction (e.g. Goodman & Dickson 1998; Ivanov, Papaloizou & Chernov 2013, and references therein). Our simple setting provides a characteristic time-scale for semimajor axis decay $-a_1/(\mathrm{d}a_1/\mathrm{d}t) \simeq \tau/(2e_1^2)$, which is significantly longer than the circularization time-scale. Such a disparity between the circularization and orbit-decay time-scales is indeed found for synchronously rotating stars in the binary (e.g. Ivanov et al. 2013; see also Zahn 1977, Hut 1981 and Remus, Mathis & Zahn 2012 for the same result within the equilibrium tide approach). The expected values of $\tau$ range from 0.1 Myr to 10 Gyr for Sun-like stars and orbital periods from 1 to 5 d. More massive components may have significantly smaller $\tau$ values by virtue of mass-dependence in $\tau$ (e.g. Goodman & Dickson 1998; Ivanov et al. 2013, though a different stellar structure may also contribute to an uncertainty in $\tau$ determination).

A more detailed approach is to implement results for equilibrium tide model from Eggleton, Kiseleva & Hut (1998) and Eggleton & Kiseleva-Eggleton (2001); see also Fabrycky & Tremaine (2007). Conveniently, their final form of the equations of motion for the inner binary is given in vectorial elements close to our choice. The exception is their use of angular momentum $\boldsymbol{G}_1$ per unit of reduced mass instead of our $\boldsymbol{K}_1$. We would thus use

$$\frac{\mathrm{d}\boldsymbol{K}_1}{\mathrm{d}t} = \frac{1}{n_1a_1^2}\left(\frac{\mathrm{d}\boldsymbol{G}_1}{\mathrm{d}t} - \frac{1}{2}n_1a_1\frac{\mathrm{d}a_1}{\mathrm{d}t}\boldsymbol{K}_1\right)\,, \tag{91}$$

which relates their rate of change. This choice of fundamental variables also implies that, contrary to the formulation in Eggleton et al. (1998), we need to complement our set of equations describing secular evolution of the system by the rate of change of the semimajor axis $a_1$ due to the tidal friction. This is readily provided by equations (27) in Eggleton & Kiseleva-Eggleton (2001).

## 8 CONCLUSIONS

We have developed and tested non-linearly averaged equations of motion for a hierarchic triple stellar system, complete up to the order 9 of the ratio of Delaunay momenta $L_1/L_2$ in the Jacobi framework. The model can be also supplemented by the terms accounting for tidal friction. The crucial feature of the presented equations is the application of vectorial elements that gained much popularity in recent decade. The results support the common opinion about the advantages of this approach from the point of view of non-singularity and compact form. However, the way to achieve the latter becomes more and more twisted and painful with each order of perturbations due to multitude of possible algebraic transformations. We also pay a comment on the choice of a small parameter $\varepsilon$ from (16) and the related justification of ordering of the consecutive terms in the multipole development of the interaction potential (19), namely $\mathcal{U}_n \propto \varepsilon^{2n+2}$. Denoting $q_{\rm S} = m_1/m_0$, the mass ratio in the close binary system, and $q_{\rm L} = m_2/M_1$, the mass ratio in the outer system, we have

$$\varepsilon = \frac{q_{\rm S}}{(1+q_{\rm S})^2}\,\frac{\sqrt{1+q_{\rm L}}}{q_{\rm L}}\sqrt{\frac{a_1}{a_2}}. \tag{92}$$

Eliminating explicit presence of stellar masses in $\varepsilon$, the latter is now given in terms of mass ratios which are constrained by the observations. Tokovinin (2008) provides a nice overview of $q_{\rm S}$ and $q_{\rm L}$ parameters derived for triple systems. For short period inner systems (periods <10 d, say), $q_{\rm S}$ basically spans the whole interval 0 to 1, with perhaps two groups near ∼0.25 and ∼0.8 values (though their statistical significance is not high). Median value of $q_{\rm S}$ could be taken ∼0.5. The $q_{\rm L}$ have also some scatter with a typical values <0.8 and a median ∼0.4. The typical values of the $q$-dependent factor on the right-hand side of (92) then span the interval 0.4 to 0.75, justifying our ordering rules. Combined with the stability limit of the triple system, namely period ratio of $P_2/P_1 \sim 5$ between outer and inner orbits, $\varepsilon$ is safely smaller then unity with a typical value of ∼0.1. However, care must be paid to possible exceptions. For instance, a somewhat extreme triple system λ Tau (e.g. Fekel & Tomkin 1982) has a low-mass companion star to a very massive inner binary. In this case, one would formally have $\varepsilon \simeq 1.08$, expressing an unusual dominance of the inner system in the orbital angular momentum budget of the system. In such cases, the ordering of the Poisson brackets in equation (28) is not justified; the second term – in this paper pushed to $\mathcal{K}_{10}$ level– would be of the same order as the first term and our treatment of $\mathcal{K}_9$ thus incomplete.

The above example, and persisting small discrepancy between results of the secular theory complete to $\mathcal{K}_9$ level and numerical integration (Section 6) motivates to push analysis given in this paper to the level $\mathcal{K}_{10}$ along the lines shown in the Appendix . We believe this is still tractable, though prospects of going to higher orders are dim. Moreover, it turns out that the really difficult cases like λ Tauri system would anyway be best analysed by direct numerical integration (as also previously advocated by Harrington 1991).

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## APPENDIX A: TRANSFORMATION

The application of algorithm outlined in Section 3.1 involves at each order $n$ two phases. First, the recourse to (24), (29) and (30) establishes the determining equation relating $\mathcal{K}_n$, $\mathcal{S}_n^1$, and $\mathcal{S}_{n-3}^2$. Then an appropriate choice of the three functions is made. After the trivial substitution $\mathcal{K}_0 = \mathcal{H}_1(L_1^*)$, the next orders are as follows.

*Order 1.* With $\mathcal{H}_{0,0} = \mathcal{H}_1$, and $\mathcal{H}_{0,1} = 0$, equation

$$\mathcal{K}_1 = \mathcal{H}_{1,0} + \mathcal{H}_{0,1} = \{\mathcal{H}_1, \mathcal{S}_1^1 + \mathcal{S}_1^2\}_1 + 0, \tag{A1}$$

is easily satisfied by $\mathcal{S}_1^1 = \mathcal{K}_1 = 0$. The part $\mathcal{S}_1^2$ remains undetermined, because $\{\mathcal{H}_1, \mathcal{S}_1^2\}_1 = 0$.

*Order 2.* Rejecting all terms with $\mathcal{S}_1^1 = 0$ (set in order 1) and $\{\mathcal{H}_1, \mathcal{S}_1^2\}_1 = \{\mathcal{H}_1, \mathcal{S}_2^2\}_1 = 0$ (by definition) we are left with

$$\mathcal{K}_2 = \{\mathcal{H}_1, \mathcal{S}_2^1\}_1 + \{\mathcal{H}_1, \mathcal{S}_1^2\}_2 + \mathcal{H}_2. \tag{A2}$$

Once again, $\mathcal{S}_1^2$ is undetermined, because $\{\mathcal{H}_1, F\}_2 = 0$ identically for any function $F$. We choose $\mathcal{K}_2 = \mathcal{H}_2(L_2^*)$ and $\mathcal{S}_2^1 = 0$.

*Order 3*. With $\mathcal{S}_1^1 = \mathcal{S}_2^1 = \mathcal{H}_{0,3} = 0$, we have $\mathcal{K}_3 = \{\mathcal{H}_1, \mathcal{S}_3^1\}_1$, with an obvious choice $\mathcal{K}_3 = \mathcal{S}_3^1 = 0$. $\mathcal{S}_3^2$ remains undetermined, as well as $\mathcal{S}_1^2$ and $\mathcal{S}_2^2$.

*Order 4*. Finally, we can start defining $\mathcal{S}_k^2$ terms as well. From

$$\mathcal{K}_4 = \{\mathcal{H}_1, \mathcal{S}_4^1\}_1 + \{\mathcal{H}_2, \mathcal{S}_1^2\}_2, \tag{A3}$$

we choose $\mathcal{K}_4 = \mathcal{S}_4^1 = \mathcal{S}_1^2 = 0$.

*Order 5*. Similarly to order 4, from

$$\mathcal{K}_5 = \{\mathcal{H}_1, \mathcal{S}_5^1\}_1 + \{\mathcal{H}_2, \mathcal{S}_2^2\}_2, \tag{A4}$$

we choose $\mathcal{K}_5 = \mathcal{S}_5^1 = \mathcal{S}_2^2 = 0$.

*Order 6*. The first non-trivial selection happens in this order of transformation. In the determining equation

$$\mathcal{K}_6 = \{\mathcal{H}_1, \mathcal{S}_6^1\}_1 + \{\mathcal{H}_2, \mathcal{S}_3^2\}_2 + \mathcal{U}_2, \tag{A5}$$

we set

$$\mathcal{K}_6 = \langle \mathcal{U}_2 \rangle, \tag{A6}$$

$$\mathcal{S}_6^1 = \bar{\mathcal{S}}_6^1 - \mathcal{A}_6^1, \tag{A7}$$

$$\mathcal{S}_3^2 = \bar{\mathcal{S}}_3^2 - \mathcal{A}_3^2, \tag{A8}$$

where $\bar{\mathcal{S}}_6^1$ and $\bar{\mathcal{S}}_3^2$ are the usual generators

$$\bar{\mathcal{S}}_6^1 = \frac{1}{n_1^*} \int (\mathcal{U}_2 - \langle \mathcal{U}_2 \rangle_1)\, \mathrm{d}\ell_1^*, \tag{A9}$$

$$\bar{\mathcal{S}}_3^2 = \frac{1}{n_2^*} \int (\langle \mathcal{U}_2 \rangle_1 - \langle \mathcal{U}_2 \rangle)\, \mathrm{d}\ell_2^*, \tag{A10}$$

whereas arbitrary functions $\mathcal{A}_6^1$ and $\mathcal{A}_3^2$ are chosen as

$$\mathcal{A}_6^1 = \left\langle \bar{\mathcal{S}}_6^1 \right\rangle_1, \qquad \mathcal{A}_3^2 = \left\langle \bar{\mathcal{S}}_3^2 \right\rangle_2, \tag{A11}$$

in order to obtain the zero-average generators. With this choice, $\mathcal{S}_6^1$ is a purely periodic function of $\ell_1^*$ (and a periodic function of $\ell_2^*$), while $\mathcal{S}_3^2$ is a purely periodic function of $\ell_2^*$ (independent on $\ell_1^*$), and $\mathcal{K}_6$ is a secular Hamiltonian term, independent on both anomalies.

*Order 7*. In the last trivial order, the equation

$$\mathcal{K}_7 = \{\mathcal{H}_1, \mathcal{S}_7^1\}_1 + \{\mathcal{H}_2, \mathcal{S}_4^2\}_2 \tag{A12}$$

is satisfied by $\mathcal{K}_7 = \mathcal{S}_7^1 = \mathcal{S}_5^2 = 0$.

*Order 8*. The case is quite similar to order 6. Equation

$$\mathcal{K}_8 = \{\mathcal{H}_1, \mathcal{S}_8^1\}_1 + \{\mathcal{H}_2, \mathcal{S}_5^2\}_2 + \mathcal{U}_3, \tag{A13}$$

admits the choice

$$\mathcal{K}_8 = \langle \mathcal{U}_3 \rangle, \tag{A14}$$

$$\mathcal{S}_8^1 = \bar{\mathcal{S}}_8^1 - \left\langle \bar{\mathcal{S}}_8^1 \right\rangle_1, \tag{A15}$$

$$\mathcal{S}_5^2 = \bar{\mathcal{S}}_5^2 - \left\langle \bar{\mathcal{S}}_5^2 \right\rangle_2, \tag{A16}$$

$$\bar{\mathcal{S}}_8^1 = \frac{1}{n_1^*} \int (\mathcal{U}_3 - \langle \mathcal{U}_3 \rangle_1)\, \mathrm{d}\ell_1^*, \tag{A17}$$

$$\bar{\mathcal{S}}_5^2 = \frac{1}{n_2^*} \int (\langle \mathcal{U}_3 \rangle_1 - \langle \mathcal{U}_3 \rangle)\, \mathrm{d}\ell_2^*, \tag{A18}$$

leading to purely periodic generators and a secular Hamiltonian.

*Order 9*. Rejecting some obviously null brackets, we find

$$\mathcal{K}_9 = \frac{1}{2} \left\{ \{\mathcal{H}_1, \mathcal{S}_6^1\}_1 + \{\mathcal{H}_2, \mathcal{S}_3^2\}_2, \mathcal{S}_3^2 \right\}_1 + \{\mathcal{H}_1, \mathcal{S}_9^1\}_1 + \{\mathcal{U}_2, \mathcal{S}_3^2\}_1$$
$$+ \{\mathcal{H}_2, \mathcal{S}_6^1\}_2 + \{\mathcal{H}_2, \mathcal{S}_6^2\}_2. \tag{A19}$$

Recalling (A5), we can substitute $\{\mathcal{H}_1, \mathcal{S}_6^1\}_1 + \{\mathcal{H}_2, \mathcal{S}_3^2\}_2 = \mathcal{K}_6 - \mathcal{U}_2$, so

$$\mathcal{K}_9 = \left[ \frac{1}{2} \left\{ \mathcal{K}_6 + \mathcal{U}_2, \mathcal{S}_3^2 \right\}_1 + \{\mathcal{H}_2, \mathcal{S}_6^1\}_2 \right] + \{\mathcal{H}_1, \mathcal{S}_9^1\}_1 + \{\mathcal{H}_2, \mathcal{S}_6^2\}_2, \tag{A20}$$

where the terms in square brackets involve only functions known from previous orders. Let us choose

$$\mathcal{K}_9 = \left\langle \frac{1}{2} \left\{ \mathcal{K}_6 + \mathcal{U}_2, \mathcal{S}_3^2 \right\}_1 + \{\mathcal{H}_2, \mathcal{S}_6^1\}_2 \right\rangle. \tag{A21}$$

But

$$\left\langle \{\mathcal{H}_2, \mathcal{S}_6^1\}_2 \right\rangle = 0, \qquad \left\langle \left\{ \mathcal{K}_6, \mathcal{S}_3^2 \right\}_1 \right\rangle_1 = \left\{ \mathcal{K}_6, \mathcal{S}_3^2 \right\}_1, \tag{A22}$$

and, benefiting from the zero-average definition (A8),

$$\left\langle \left\{ \mathcal{K}_6, \mathcal{S}_3^2 \right\}_1 \right\rangle_2 = 0, \tag{A23}$$

so the new Hamiltonian is simply

$$\mathcal{K}_9 = \frac{1}{2} \left\langle \left\{ \mathcal{U}_2, \mathcal{S}_3^2 \right\}_1 \right\rangle = \frac{1}{2} \left\langle \left\{ \langle \mathcal{U}_2 \rangle_1, \mathcal{S}_3^2 \right\}_1 \right\rangle_2, \tag{A24}$$

owing its rightmost form to the independence of $\mathcal{S}_3^2$ on $\ell_1^*$.

The generators are defined as

$$\mathcal{S}_9^1 = \bar{\mathcal{S}}_9^1 - \left\langle \bar{\mathcal{S}}_9^1 \right\rangle_1, \tag{A25}$$

$$\mathcal{S}_6^2 = \bar{\mathcal{S}}_6^2 - \left\langle \bar{\mathcal{S}}_6^2 \right\rangle_2, \tag{A26}$$

where

$$\bar{\mathcal{S}}_9^1 = \frac{1}{n_1^*} \int \left( \frac{1}{2} \left[ \left\{ \mathcal{U}_2, \mathcal{S}_3^2 \right\}_1 - \left\langle \left\{ \mathcal{U}_2, \mathcal{S}_3^2 \right\}_1 \right\rangle_1 \right] - n_2^* \frac{\partial \mathcal{S}_6^1}{\partial \ell_2^*} \right) \mathrm{d}\ell_1^*, \tag{A27}$$

$$\bar{\mathcal{S}}_6^2 = \frac{1}{2n_2^*} \int \left( \left\{ \mathcal{K}_6, \mathcal{S}_3^2 \right\}_1 + \left\langle \left\{ \mathcal{U}_2, \mathcal{S}_3^2 \right\}_1 \right\rangle_1 - \left\langle \left\{ \mathcal{U}_2, \mathcal{S}_3^2 \right\}_1 \right\rangle \right) \mathrm{d}\ell_2^*. \tag{A28}$$

*Order 10*. Although we do not proceed thus far in this paper, we give for the record (and possible future use)

$$\mathcal{K}_{10} = \frac{1}{2} \left\{ \{\mathcal{H}_1, \mathcal{S}_6^1\}_1 + \{\mathcal{H}_2, \mathcal{S}_3^2\}_2, \mathcal{S}_3^2 \right\}_2 + \{\mathcal{H}_1, \mathcal{S}_{10}^1\}_1 + \{\mathcal{H}_2, \mathcal{S}_7^2\}_2$$
$$+ \{\mathcal{U}_2, \mathcal{S}_3^2\}_2 + \mathcal{U}_4. \tag{A29}$$

Using (A5) and putting known terms into square brackets we reduce the equation to

$$\mathcal{K}_{10} = \left[ \frac{1}{2} \left\{ \mathcal{K}_6 + \mathcal{U}_2, \mathcal{S}_3^2 \right\}_2 + \mathcal{U}_4 \right] + \{\mathcal{H}_1, \mathcal{S}_{10}^1\}_1 + \{\mathcal{H}_2, \mathcal{S}_7^2\}_2. \tag{A30}$$

The secular Hamiltonian is chosen to be

$$\mathcal{K}_{10} = \left\langle \frac{1}{2} \left\{ \mathcal{U}_2, \mathcal{S}_3^2 \right\}_2 + \mathcal{U}_4 \right\rangle = \frac{1}{2} \left\langle \left\{ \langle \mathcal{U}_2 \rangle_1, \mathcal{S}_3^2 \right\}_2 \right\rangle_2 + \langle \mathcal{U}_4 \rangle, \tag{A31}$$

And then, zero-average generators are

$$\mathcal{S}_{10}^{1} = \bar{\mathcal{S}}_{10}^{1} - \left\langle \bar{\mathcal{S}}_{10}^{1} \right\rangle_1, \qquad \mathcal{S}_7^2 = \bar{\mathcal{S}}_7^2 - \left\langle \bar{\mathcal{S}}_7^2 \right\rangle_2, \tag{A32}$$

where

$$\bar{\mathcal{S}}_{10}^{1} = \frac{1}{n_1^*}\int \left( \frac{1}{2}\left\{\mathcal{U}_2, \mathcal{S}_3^2\right\}_2 + \mathcal{U}_4 - \left\langle \frac{1}{2}\left\{\mathcal{U}_2, \mathcal{S}_3^2\right\}_2 + \mathcal{U}_4 \right\rangle_1 \right) \mathrm{d}\ell_1^*, \tag{A33}$$

$$\bar{\mathcal{S}}_7^2 = \frac{1}{n_2^*}\int \left( \frac{1}{2}\left[ \left\{\mathcal{K}_6, \mathcal{S}_3^2\right\}_2 + \left\langle \left\{\mathcal{U}_2, \mathcal{S}_3^2\right\}_2 \right\rangle_1 - \left\langle \left\{\mathcal{U}_2, \mathcal{S}_3^2\right\}_2 \right\rangle \right] + \langle \mathcal{U}_4 \rangle_1 - \langle \mathcal{U}_4 \rangle \right) \mathrm{d}\ell_2^*. \tag{A34}$$

## APPENDIX B: HAMILTONIAN GRADIENTS

Instead of the final equations of motion (14,15), we provide the gradients of subsequent Hamiltonians which are the main building blocks of the right hand sides. Taking gradients of scalar products $Q_{pq}$ is straightforward, only $Q_{2q}$ or $Q_{p2}$ may require an additional recourse to vector identities. Thus, for example,

$$\frac{\partial Q_{1q}}{\partial \boldsymbol{e}_1} = \boldsymbol{w}_q, \qquad \frac{\partial Q_{2q}}{\partial \boldsymbol{e}_1} = \boldsymbol{w}_q \times \boldsymbol{K}_1, \qquad \frac{\partial Q_{3q}}{\partial \boldsymbol{e}_1} = \boldsymbol{0}, \tag{B1}$$

and so on, with a special case of $Q_{22}$, when,

$$\frac{\partial Q_{22}}{\partial \boldsymbol{e}_1} = (\boldsymbol{K}_2 \times \boldsymbol{e}_2) \times \boldsymbol{K}_1 = Q_{33}\boldsymbol{e}_2 - Q_{31}\boldsymbol{K}_2, \tag{B2}$$

and similarly for remaining vectors. Any explicit occurrence of $e_1$ and $\eta_2$ should be understood as $\sqrt{\boldsymbol{e}_1 \cdot \boldsymbol{e}_1}$, and $\sqrt{\boldsymbol{K}_2 \cdot \boldsymbol{K}_2}$, respectively.

Differentiating equation (77) we find

$$\frac{\partial \mathcal{K}_6}{\partial \boldsymbol{e}_1} = -\frac{2\,C_2}{\eta_2^5}\left[2\eta_2^2\,\boldsymbol{e}_1 - 5Q_{13}\boldsymbol{K}_2\right], \tag{B3}$$

$$\frac{\partial \mathcal{K}_6}{\partial \boldsymbol{K}_1} = -\frac{2\,C_2}{\eta_2^5}Q_{33}\boldsymbol{K}_2, \tag{B4}$$

$$\frac{\partial \mathcal{K}_6}{\partial \boldsymbol{e}_2} = 0, \tag{B5}$$

$$\frac{\partial \mathcal{K}_6}{\partial \boldsymbol{K}_2} = \frac{C_2}{\eta_2^5}\left\{10Q_{13}\boldsymbol{e}_1 - 2Q_{33}\boldsymbol{K}_1 - \left[\left(1-6e_1^2\right) + 5\eta_2^{-2}\left(5Q_{13}^2 - Q_{33}^2\right)\right]\boldsymbol{K}_2\right\}. \tag{B6}$$

Expressions resulting from the octupole (78) are more complicated:

$$\frac{\partial \mathcal{K}_8}{\partial \boldsymbol{e}_1} = \frac{C_3}{\eta_2^7}\left\{16\eta_2^2 Q_{11}\boldsymbol{e}_1 + 10\left[Q_{31}Q_{33} - 7Q_{11}Q_{13}\right]\boldsymbol{K}_2 - \left[\left(1-8e_1^2\right)\eta_2^2 + 35Q_{13}^2 - 5Q_{33}^2\right]\boldsymbol{e}_2\right\}, \tag{B7}$$

$$\frac{\partial \mathcal{K}_8}{\partial \boldsymbol{K}_1} = \frac{10\,C_3}{\eta_2^7}\left[(Q_{13}Q_{31} + Q_{11}Q_{33})\boldsymbol{K}_2 + Q_{13}Q_{33}\boldsymbol{e}_2\right], \tag{B8}$$

$$\frac{\partial \mathcal{K}_8}{\partial \boldsymbol{e}_2} = \frac{C_3}{\eta_2^7}\left\{10Q_{13}Q_{33}\boldsymbol{K}_1 - \left[\left(1-8e_1^2\right)\eta_2^2 + 35Q_{13}^2 - 5Q_{33}^2\right]\boldsymbol{e}_1\right\}, \tag{B9}$$

$$\frac{\partial \mathcal{K}_8}{\partial \boldsymbol{K}_2} = \frac{5\,C_3}{\eta_2^7}\{2\left[Q_{13}Q_{31} + Q_{11}Q_{33}\right]\boldsymbol{K}_1 + 2\left[Q_{31}Q_{33} - 7Q_{11}Q_{13}\right]\boldsymbol{e}_1 + \left[\left(1-8e_1^2\right)Q_{11} + 7\eta_2^{-2}\left(Q_{11}\left(7Q_{13}^2 - Q_{33}^2\right) - 2Q_{13}Q_{31}Q_{33}\right)\right]\boldsymbol{K}_2\}. \tag{B10}$$

The rule of the gradient with respect to $\boldsymbol{K}_2$ being most complicated refers also to expressions resulting from (80):

$$\frac{\partial \mathcal{K}_9}{\partial \boldsymbol{e}_1} = -2C_2'B_1\left\{10B_2\left[Q_{12}(\boldsymbol{K}_1 \times \boldsymbol{e}_2) - Q_{21}(\boldsymbol{K}_2 \times \boldsymbol{e}_2) + 3Q_{11}Q_{33}\boldsymbol{e}_2\right] + 24\eta_2^2 Q_{33}\boldsymbol{e}_1 - 15Q_{13}Q_{33}\boldsymbol{K}_2\right\}, \tag{B11}$$

$$\frac{\partial \mathcal{K}_9}{\partial \boldsymbol{K}_1} = C_2'B_1\left\{4B_2\left[5Q_{12}(\boldsymbol{e}_1 \times \boldsymbol{e}_2) - Q_{31}Q_{33}\boldsymbol{e}_2\right] + \left[15Q_{13}^2 + 3Q_{33}^2 - 2B_2\left(15Q_{11}^2 + Q_{31}^2\right) - \left(1+24e_1^2\right)\eta_2^2\right]\boldsymbol{K}_2\right\}, \tag{B12}$$

$$\frac{\partial \mathcal{K}_9}{\partial \boldsymbol{e}_2} = 4C_2'B_1B_2\left\{5\,Q_{12}(\boldsymbol{K}_1 \times \boldsymbol{e}_1) + 5\,Q_{21}(\boldsymbol{e}_1 \times \boldsymbol{K}_2) - Q_{33}\left(15\,Q_{11}\boldsymbol{e}_1 + Q_{31}\boldsymbol{K}_1\right)\right\}, \tag{B13}$$

$$\frac{\partial \mathcal{K}_9}{\partial \boldsymbol{K}_2} = \frac{\partial \mathcal{K}_9}{\partial \eta_2}\frac{\boldsymbol{K}_2}{\eta_2} + C_2'B_1\left\{30Q_{13}Q_{33}\boldsymbol{e}_1 - 20B_2Q_{21}(\boldsymbol{e}_1 \times \boldsymbol{e}_2) + \left[15Q_{13}^2 + 3Q_{33}^2 - 2B_2\left(15Q_{11}^2 + Q_{31}^2\right) - (1+24e_1^2)\eta_2^2\right]\boldsymbol{K}_1\right\}, \tag{B14}$$

$$\frac{\partial \mathcal{K}_9}{\partial \eta_2} = \frac{C_2'B_1}{\eta_2}\left\{2B_4\left[Q_{33}\left(15Q_{11}^2 + Q_{31}^2\right) - 10Q_{12}Q_{21}\right] + Q_{33}\left[\left(1+24e_1^2\right)\eta_2^2 B_3 - (2+B_3)\left(15Q_{13}^2 + Q_{33}^2\right)\right]\right\}, \tag{B15}$$

where $B_1$, $B_2$ are defined by (81), and

$$B_3 = \frac{25 + 34\eta_2 + 5\eta_2^2}{(1+\eta_2)(5+\eta_2)}, \qquad B_4 = \frac{35 + 105\eta_2 + 95\eta_2^2 + 21\eta_2^3}{(1+\eta_2)^2(5+\eta_2)}. \tag{B16}$$

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