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Relativistic models for the BepiColombo radioscience experiment

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Abstract. To test General Relativity with the tracking data of the BepiColombo Mercury orbiter we need relativistic models for the orbits of Mercury and of the Earth, for the light-time and for all the spatio-temporal reference frames involved, with accuracy corresponding to the measurements: $\simeq 10$ cm in range, $\simeq 2$ micron/s in range-rate, over 2 years.

For the dynamics we start from the Lagrangian post-Newtonian (PN) formulation, using a relativistic equation for the solar system barycenter to avoid rank deficiency. In the determination of the PN parameters, the difficulty in disentangling the effects of β from the ones of the Sun's oblateness is confirmed. We have found a consistent formulation for the preferred frame effects, although the center of mass is not an integral. For the identification of strong equivalence principle (SEP) violations we use a formulation containing both direct and indirect effects (through the modified position of the Sun in a barycentric frame).

In the light-time equations, the Shapiro effect is modeled to PN order 1 but with an order 2 correction compatible with (Moyer 2003). The 1.5-PN order corrections containing the Sun's velocity are not relevant at the required level of accuracy.

To model the orbit of the probe, we use a mercury-centric reference frame with its own "Mercury Dynamic Time": this is the largest and the only relativistic correction required, taking into account the major uncertainties introduced by non-gravitational perturbations.

A delicate issue is the compatibility of our solution with the ephemerides for the other planets, and for the Moon, which cannot be improved by the BepiColombo data alone. Conversely, we plan to later export the BepiColombo measurements, as normal points, to contribute with their unprecedented accuracy to the global improvement of the planetary ephemerides.

Keywords. Mercury, radioscience, relativity

1. A radioscience experiment with a Mercury orbiter

BepiColombo is an ESA mission to the planet Mercury, including two spacecrafts (one provided from Japan) to be put in orbit around Mercury; launch is scheduled for 2014, instruments are already being built now. The Mercury Orbiter Radioscience Experiment (MORE) is one of the on board experiments whose goals are:

a) to determine the gravity field of Mercury and its rotation state (to constrain the interior structure of the planet);

b) to determine the orbit of Mercury to constrain the possible theories of gravitation, e.g., by determining the post-Newtonian (PN) parameters;

c) to provide the spacecraft position for geodesy experiments;

d) to contribute to planetary ephemerides improvement.

This is possible thanks to a multi-frequency radio link (in X and Ka bands) allowing to eliminate the uncertainty in the refraction index due to plasma content along the

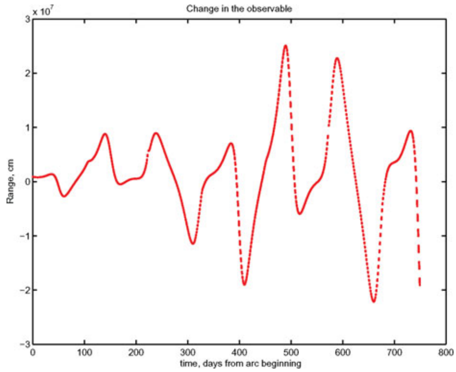

Figure 1. Differences in range using a fully Relativistic and a Newtonian model, over a 2 year Mercury orbiting mission. The total Δr is 4×10^7 cm and $S/N = \Delta r / \sigma(r) \simeq 4 \times 10^6$.

radio waves path. The MORE experiment provides the necessary Ka band transponder and the system to compare the delays in a 5-way link, in combination with instruments installed at the ground stations.

Orders of magnitude for the accuracy which can be achieved in this way are 2 micron/s in range-rate and 10 cm in range: the relative accuracy in range is better than 10^{-12}. This implies the signal to noise ratio (S/N) of all the relativistic effects (both in the dynamics and in the observation equations) is very large, in particular for the range measurements, see Figure 1.

2. The relativistic orbit determination problem

The relativity experiment with MORE needs to solve an orbit determination problem with a full relativistic model (including the terms expressing the violations of general relativity with the PN parameters, such as $\gamma, \beta, \zeta, \eta, \alpha_1, \alpha_2$), not for a generic space-time, but for the one where we are now. Thus we must fit the initial conditions for Mercury and for the Earth-Moon barycenter.

2.1. *Orbit determination with symmetries*

We shall give the equation of motion by using the parametric post-Newtonian approach: the relativistic equation of motion is linearized with respect to the small parameters v_i^2/c^2 and $G\,m_i/(c^2\, r_{ik})$, where v_i is the barycentric velocity for each of the bodies of mass m_i, c the speed of light and r_{ik} a mutual distance, appearing in the metric of the curved space-time, hence in the equations for geodesic motion. This can be formalized by adding to the Lagrangian $L_{\rm NEW}$ of the N-body problem some corrective terms of PN order 1 in the small parameters. By following the notation of (Moyer 2003)

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i \quad , \quad r_{ij} = |\mathbf{r}_{ij}|\,, \quad \mathbf{v}_{ij} = \dot{\mathbf{r}_j} - \dot{\mathbf{r}_i} = \mathbf{v}_j - \mathbf{v}_i \quad , \quad v_{ij} = |\mathbf{v}_{ij}|\,,$$

for $i, j = 0, \ldots,$ N, where 0 refers to the Sun, the Newtonian Lagrange function is

$$L_{\rm NEW} = \frac{1}{2} \sum_i \mu_i\, v_i^2 + \frac{1}{2} \sum_i \sum_{j \neq i} \frac{\mu_i \mu_j}{r_{ij}}.$$

The usual Lagrangian is multiplied by G, thus only the gravitational masses $\mu_i = G\,m_i$ appear in the Lagrange function. By Noether's theorem, the 3-parameter group of symmetries $\mathbf{r}_j \longrightarrow \mathbf{r}_j + \mathbf{h},\ \mathbf{h} \in \mathbf{R}^3$ results in the vector integral of total linear momentum

$$\mathbf{P} = \sum_i \frac{\partial L_{\rm NEW}}{\partial \mathbf{v}_i} = \sum_i \mu_i\, \mathbf{v}_i .$$

The unobservable linear motion of the center of mass of the (N+1)-body system

$$\mathbf{b}(t) = \frac{1}{M} \sum_i \mu_i\, \mathbf{r}_i(t) = \frac{1}{M}\, \mathbf{P}\, t + \mathbf{b}(0)$$

with $M = \sum_i \mu_i$ implies that an orbit determination with mutual observations has a rank deficiency of 6: the normal matrix of the fit has a kernel of dimension 6. There is only one solution to this problem, *descoping* (Milani & Gronchi 2009, Chapter 6), which can be obtained in two ways: either (1) the center of mass is assumed to be fixed, e.g., $\mathbf{b}(t) = \mathbf{0}$, or (2) it is constrained to remain fixed, by adding a priori observations of the form $\mathbf{b}(0) = \mathbf{0} \pm \sigma_1$ and $\dot{\mathbf{b}}(0) = \mathbf{0} \pm \sigma_2$, with a very small a priori uncertainties σ_i. With solution (1) the equation of motion of the Sun is removed, and the position of the Sun is computed from the center of mass, that is $\mathbf{r}_0$ is replaced by $\mathbf{s}$ with

$$\mathbf{s} = -\frac{1}{\mu_0} \sum_{i=1}^{N} \mu_i \mathbf{r}_i .$$

2.2. *Lagrangian formulation for PN Relativity*

The equations of motion of GR, to 1-PN order, can be deduced from the *relativistic Lagrangian*

$$L = L_{\rm NEW} + L_{\rm GR0} + \beta\, L_\beta + \gamma\, L_\gamma ,$$

where γ, β are the "Eddington parameters", both = 1 in GR, and $L_{\rm GR0}$ is the portion without free parameters (apart from G):

$$\begin{aligned} L_{\rm GR0} \quad = \quad & \frac{1}{8c^2} \sum_i \mu_i\, v_i^4 + \frac{1}{2\,c^2} \sum_i \sum_{j\neq i} \sum_{k\neq i} \frac{\mu_i\, \mu_j\, \mu_k}{r_{ij}\, r_{ik}} \\ & + \frac{1}{2\,c^2} \sum_i \sum_{j\neq i} \frac{\mu_i\, \mu_j}{r_{ij}} \left[\frac{1}{2} \left(v_i^2 + v_j^2\right) - \frac{3}{2} \left(\mathbf{v}_i \cdot \mathbf{v}_j\right) - \frac{1}{2} \left(\mathbf{n}_{ij} \cdot \mathbf{v}_i\right) \left(\mathbf{n}_{ij} \cdot \mathbf{v}_j\right) \right], \end{aligned}$$

where $\mathbf{n}_{ij} = \mathbf{r}_{ij}/r_{ij}$,

$$L_\gamma = \frac{1}{2\,c^2} \sum_i \sum_{j\neq i} \frac{\mu_i \mu_j}{r_{ij}} \left(\mathbf{v}_i - \mathbf{v}_j\right)^2, \quad L_\beta = -\frac{1}{c^2} \sum_i \sum_{j\neq i} \sum_{k\neq i} \frac{\mu_i\, \mu_j\, \mu_k}{r_{ij}\, r_{ik}} .$$

The relativistic Lagrangian L is also invariant by translations, thus by Noether's theorem there is a vector integral

$$\mathbf{P} = \sum_i \frac{\partial L}{\partial \mathbf{v}_i} = \sum_i \mu_i\, \mathbf{v}_i + \sum_i \frac{\partial L_{\rm GR0}}{\partial \mathbf{v}_i},$$

where the contributions from the derivatives of L_β vanish (L_β does not depend on $\mathbf{v}_i$) and the ones from L_γ cancel in the sum over i because they are antisymmetric. Thus

$$\mathbf{P} = \sum_i \mu_i\, \mathbf{v}_i \left[1 + \frac{1}{2} \left(\frac{v_i}{c}\right)^2 - \frac{U_i}{2\,c^2} \right] - \frac{1}{2\,c^2} \sum_i \sum_{k\neq i} \frac{\mu_i\, \mu_k}{r_{ik}} \left(\mathbf{n}_{ik} \cdot \mathbf{v}_k\right) \mathbf{n}_{ik} ,$$

where $U_i = \sum_{k\neq i} \mu_k / r_{ik}$ is the Newtonian potential. To PN order 1, there is a vector integral:

$$\mathbf{B} = \sum_i \mu_i \, \mathbf{r}_i \, \left[1 + \frac{v_i^2}{2c^2} - \frac{U_i}{2c^2}\right], \quad \frac{d\mathbf{B}}{dt} = \mathbf{P}.$$

The relativistic analog of the total mass

$$\mathcal{M} = \sum_i \mu_i \, [1 + (v_i^2 - U_i)/(2c^2)]$$

is an integral to order 1-PN (because the PN order 1 term is the Newtonian energy divided by c^2), thus we can define the relativistic center of mass $\mathbf{b} = \mathbf{B}/\mathcal{M}$ with $d\mathbf{b}/dt$ also a vector integral. The rank deficiency problem is the same as in the Newtonian case. To solve it, we can either set $\mathbf{b}(t) = \mathbf{0}$ and solve for the position of the Sun from the ones of the planets

$$\mathbf{s} = -\frac{1}{\mu_0 \, [1 + (v_0^2 - U_0)/2c^2)]} \sum_{i=1}^{N} \mu_i \, \left[1 + \frac{v_i^2 - U_i}{2c^2}\right] \, \mathbf{r}_i \, .$$

The difference between the position of the Sun from the above formula and the Newtonian one is $\simeq 200$ m, thus it is very significant for our measurement accuracy. On the contrary, the small (< 1 micron/s) differences in the velocity of the Sun are not important, and even formally the changes they introduce in the equations of motion are of PN order 2.

As an alternative, the equation of motion may include the one for the Sun, and a constraint on the center of mass can be added by means of a priori observations: this is somewhat more complicated because the constraints are nonlinear, but it is possible and the results must be the same.

3. Test for parametric post-Newtonian violations

In this Section we will discuss the possible parametric PN violations to understand if we can investigate them with MORE; note that the term containing J_2 is not, strictly speaking, a violation term, but it is closely related because it is strongly correlated with PN parameter β.

3.1. *Three body effects and oblateness of the Sun*

The contribution of the oblateness of the Sun is $J_2 \, L_{J_2}$ with

$$L_{J_2} = -\frac{1}{2} \sum_{i \neq 0} \frac{\mu_0 \, \mu_i}{r_{0i}} \left(\frac{R_0}{r_{0i}}\right)^2 [3(\mathbf{n}_{0i} \cdot \mathbf{e}_0)^2 - 1],$$

where R_0 is the Sun's radius, $\mathbf{e}_0$ is the unit vector along the Sun's rotation axis.

J_2 affects the precession of the longitude of the node, that generates a displacement in the plane of the solar equator, while the main orbital effect of β is a precession of the argument of perihelion, that is a displacement taking place in the plane of the orbit of Mercury. The angle between these two planes is only $\epsilon = 3.3°$ and $\cos\epsilon = 0.998$, thus it is easy to understand how the correlation between β and J_2 can be 0.997, as found in the numerical simulations of (Milani *et al.* 2002). Short of using another test body, with an orbit plane much more inclined than the one of Mercury, this correlation cannot be avoided. One possible way to mitigate this effect is to use the equation derived by (Nordvedt 1970) that relates the SEP η parameter to β, γ and possibly preferred frame parameters within the metric theories:

$$\eta = 4\,\beta - \gamma - 3 - \alpha_1 - \frac{2}{3}\,\alpha_2 .$$

When the values of γ, η and also of the preferred frame parameters (if included in the solution) are well determined, this equation acts essentially as a strong constraint on the value of β, and, as a result, the variance of both β and J_2 is sharply reduced, see Table II in (Milani *et al.* 2002).

3.2. *Gravitational constant and mass of the Sun*

An interesting goal, especially for cosmologists, would be to measure the time variation of the gravitational constant G. The Lagrange function terms are $(\dot{G}/G)\, L_{\dot{G}/G}$, where

$$L_{\dot{G}/G} = \frac{t - t_0}{2} \sum_{i \neq j} \frac{\mu_i\, \mu_j}{r_{ij}},$$

in practice the only terms with measurable effects contain the mass of the Sun. Hence the parameter which can be determined and the corresponding Lagrange term are

$$\zeta = \frac{d\mu_0}{dt}/\mu_0 \quad , \quad L_\zeta = (t - t_0) \sum_{i \neq 0} \frac{\mu_0 \mu_i}{r_{0i}};$$

we cannot discriminate the change with time of G from change with time of m_0.

Thus this is not a null experiment: what should be measured is $\dot{m}_0/m_0 \simeq -7 \times 10^{-14}$ y^{-1}, due to mass shed as radiation. A smaller contribution is the mass of charged particles emitted by the Sun, but the amount of the latter is not that well constrained (Noerdlinger 2008). If the result of our experiment for ζ was close to 10^{-13} y^{-1}, then it would be hard to discriminate the new physics of a change in G from the standard, but inaccurately known, physical effects.

3.3. *Preferred frame effects*

The preferred frame effects are described by the contribution

$$L_\alpha = \frac{\alpha_2 - \alpha_1}{4\,c^2} \sum_j \sum_{i \neq j} \frac{\mu_i\, \mu_j}{r_{ij}}\,(\mathbf{z}_i \cdot \mathbf{z}_j) - \frac{\alpha_2}{4\,c^2} \sum_j \sum_{i \neq j} \frac{\mu_i\, \mu_j}{r_{ij}}\,[(\mathbf{n}_{ij} \cdot \mathbf{z}_i)\,(\mathbf{n}_{ij} \cdot \mathbf{z}_j)],$$

with two additional post Newtonian parameters α_1, α_2 and with the vector $\mathbf{z}_i = \mathbf{w} + \mathbf{v}_i$, where $\mathbf{w}$ is the velocity of the solar system barycenter with respect to the preferred frame, usually assumed to be the one of the cosmic microwave background, thus $|\mathbf{w}| = 370 \pm 10$ km/s in the direction $(\alpha, \delta) = (168^\circ, 7^\circ)$.

The problem arises from the presence of additional terms in the total linear momentum integral $\mathbf{P}$. By applying again Noether's theorem, after the split of the Lagrangian L into parts with and without the preferred frame effects, $L = L_0 + L_\alpha$, we obtain:

$$\mathbf{P} = \mathbf{P}_0 + \mathbf{P}_\alpha, \quad \mathbf{P}_0 = \sum_j \frac{\partial L_0}{\partial \mathbf{v}_j}, \quad \mathbf{P}_\alpha = \sum_i \frac{\partial L_\alpha}{\partial \mathbf{v}_i}.$$

However, L_α is not invariant with respect to a time-dependent translation with constant velocity, and there is no center of mass integral (Will 1993). That is

$$\frac{d\mathbf{P}}{dt} = \frac{d\mathbf{P}_0}{dt} + \frac{d\mathbf{P}_\alpha}{dt} = \mathbf{0} \implies \mathcal{M}\,\frac{d^2\mathbf{b}}{dt^2} = \frac{d\mathbf{P}_0}{dt} = -\frac{d\mathbf{P}_\alpha}{dt}.$$

The accelerated barycentric frame results in apparent forces, giving the same acceleration $d\mathbf{P}_\alpha/dt \cdot (1/\mathcal{M})$ on all the bodies: these apparent forces are of PN order 1 and

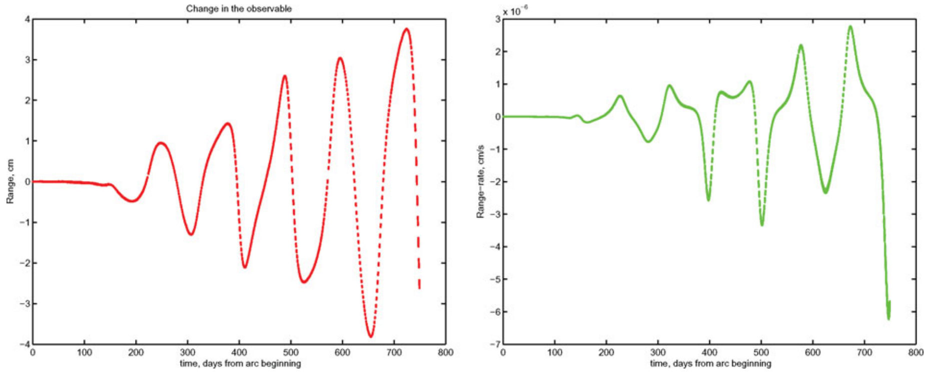

Figure 2. The effect over 2 years of the apparent force associated with preferred frame effects $\alpha_1 = 3 \times 10^{-4}$, $\alpha_2 = 3 \times 10^{-4}$ (range on the left, range rate on the right) has effects small or at the most almost comparable with respect to the measurement accuracy.

are not zero even for $\mathbf{w} = \mathbf{0}$. Even if they results in small effects (Figure 2) they have been included in the model used for simulations. What matters for us is that there is a consistent formulation in "barycentric" coordinates even without a barycenter integral.

3.4. *Violations of the Strong Equivalence Principle*

We can consider that there are for each body i two quantities μ_i and μ_i^I, one in the gravitational potential (including the relativistic part) and the other in the kinetic energy. If there is a violation of the strong equivalence principle involving body i, with a fraction Ω_i of its mass due to gravitational self-energy, then

$$\mu_i = [1 + \eta\Omega_i]\, \mu_i^I \Longleftrightarrow \mu_i^I = [1 - \eta\Omega_i]\, \mu_i + \mathcal{O}(\eta^2),$$

with η a post-Newtonian parameter for this violation. Neglecting $\mathcal{O}(\eta^2)$ terms this is expressed by a Lagrangian term $\eta\, L_\eta$, with an effect on body i:

$$L_\eta = -\frac{1}{2} \sum_i \Omega_i\, \mu_i\, v_i^2 \quad \Longrightarrow \quad \frac{d^2\mathbf{r}_i}{dt^2} = \left.\frac{d^2\mathbf{r}_i}{dt^2}\right|_{\eta=0} [1 + \eta\, \Omega_i].$$

The largest effect of η is a change in the center of mass integral

$$\mathbf{P} = \sum_j \frac{\partial L}{\partial \mathbf{v}_j} = \sum_j [1 - \eta\, \Omega_i]\, \mu_i\, \mathbf{v}_j + \ldots, \quad \frac{d\mathbf{P}}{dt} = \mathbf{0}$$

and if the center of mass is the origin, the position of the Sun has to be corrected:

$$\mathbf{b} = \frac{1}{\mathcal{M}} \sum_j [1 - \eta\Omega_j]\, \mu_j\, \mathbf{r}_j + \ldots = \mathbf{0} \quad \Longrightarrow \quad \mathbf{s} = \frac{-1}{\mu_0\, [1 - \eta\, \Omega_0]} \sum_{j \neq 0} [1 - \eta\Omega_j]\, \mu_j\, \mathbf{r}_j + \ldots.$$

The partial derivative of the acceleration of the body j with respect to η is

$$\frac{\partial}{\partial \eta} \left[\frac{d^2\mathbf{r}_j}{dt^2}\right] = \Omega_j \left[\frac{\mu_0}{r_{j0}^3}\, \mathbf{r}_{j0} + \sum_{i \neq j,0} \frac{\mu_i}{r_{ji}^3}\, \mathbf{r}_{ji}\right] + \frac{\partial}{\partial \mathbf{s}} \left[\frac{\mu_0}{r_{j0}^3}\right] \frac{\partial \mathbf{s}}{\partial \eta},$$

where the first term is the direct, the second the indirect η-perturbation, and where

$$\frac{\partial \mathbf{s}}{\partial \eta} = \sum_{i \neq 0} (\Omega_j - \Omega_0)\, \frac{\mu_i}{\mu_0}\, \mathbf{r}_i.$$

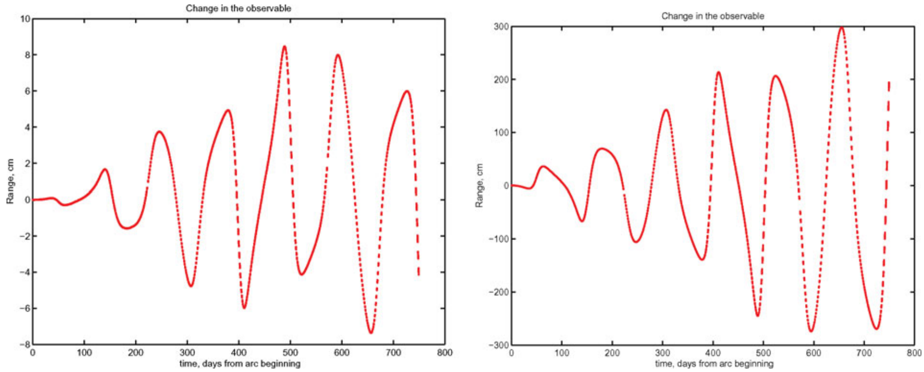

Figure 3. Left: signal in range due to a SEP violation with $\eta = 10^{-5}$, direct part only (acting mostly on Earth); this signal is marginally above measurement accuracy, thus after fitting the initial conditions should not become significant. Right: signal in range due to a SEP violation with $\eta = 10^{-5}$, indirect part only (acting more on Mercury), by assuming the same initial conditions; the fit of initial conditions lowers the signal, but it is anyway significant.

By combining together and omitting smaller terms with $\Omega_i\,\mu_k$ (with $i, k \neq 0$) or η^2

$$\frac{\partial}{\partial\eta}\left[\frac{d^2\mathbf{r}_j}{dt^2}\right] = \Omega_j\,\mu_0\,\frac{\mathbf{r}_{j0}}{r_{j0}^3} - \Omega_0\,\frac{\partial}{\partial\mathbf{r}_0}\left[\frac{1}{r_{j0}^3}\right]\sum_{i\neq 0}\mu_i\,\mathbf{r}_i\,,$$

with a direct (small parameter $\Omega_j\,\mu_0$) and an indirect (small parameter $\Omega_0\,\mu_i$) part. Figure 3 shows the change in the range due to a SEP violation with $\eta = 10^{-5}$: our experiment should be sensible to the indirect part, not to the direct one.

4. The observables

The observables of our experiment are the distance r between the ground antenna and the spacecraft, and its time derivative $\dot{r}$. The range is computed using 5 state vectors:

$$r = |(\mathbf{x}_{\rm sat} + \mathbf{x}_M) - (\mathbf{x}_{EM} + \mathbf{x}_E + \mathbf{x}_{\rm ant})| + S(\gamma),$$

where $\mathbf{x}_{\rm sat}$ is the mercury-centric position of the orbiter, $\mathbf{x}_M$ the solar system barycentric position of Mercury, $\mathbf{x}_{EM}$ the position of the Earth-Moon center of mass in the same reference system, $\mathbf{x}_E$ the vector from the Earth-Moon barycenter to the center of mass of the Earth, $\mathbf{x}_{\rm ant}$ the position of the ground antenna center of phase with respect to the center of mass of the Earth. $S(\gamma)$ is the *Shapiro effect*, the difference between distance in a flat space and the geodesic length in the curved space-time, depending upon the post-Newtonian parameter γ, which has a special role in relativistic orbit determination, since it appears in both the dynamics and the equations of observation.

4.1. *Shapiro effect in range*

The Shapiro effect at the 1-PN level is (e.g. Will 1993, Moyer 2003)

$$S(\gamma) = \frac{(1+\gamma)\,\mu_0}{c^2}\,\ln\left(\frac{r_r + \mathbf{k}\cdot\mathbf{r_r}}{r_t + \mathbf{k}\cdot\mathbf{r_t}}\right) = \frac{(1+\gamma)\,\mu_0}{c^2}\,\ln\left(\frac{r_t + r_r + r}{r_t + r_r - r}\right),$$

where $\mathbf{r_r}$ and $\mathbf{r_t}$ are heliocentric positions of the transmitter and the receiver at the corresponding time instants of photon transmission and reception, $\mathbf{k}$ is the unit vector in the direction of the radio waves. Note that the planetary terms, similar to the solar in $S(\gamma)$ above, can also be included but they prove to be smaller than the accuracy needed for our measurements. However a question arises whether the very high signal to noise in the range requires other terms in the solar gravity influence: (i) motion of the source, or (ii) higher-order PN corrections when the radio waves are passing near the Sun, at just a few solar radii (and thus the denominator in the ln-function of the Shapiro formula is small). The former are of the PN order 1.5, while the latter are of PN order 2.

The PN order 1.5 correction has been widely discussed after the Cassini experiment significantly increased accuracy for the γ determination; see (Bertotti *et al.* 2003) for the Cassini results and (Will 2003, Klioner & Peip 2003, Kopeikin 2008) for further discussion. Figure 4, on the left, shows the incidence of this 1.5-PN term on the range computation: the correction is less than 1 centimeter in range, well below our accuracy.

The important correction is obtained by adding 2-PN terms in the Shapiro formula, due to the bending of the light path. (Moyer 2003) proposed to add a PN 2nd order term $(1+\gamma)\,\mu_0/c^2$ (the radius of a black hole with the mass of the Sun) both in the numerator and denominator of the argument of the natural logarithm:

$$\ln\left(\frac{r_t + r_r + r}{r_r + r_r - r}\right) \to \ln\left(\frac{r_t + r_r + r + \frac{(1+\gamma)\,\mu_0}{c^2}}{r_t + r_r - r + \frac{(1+\gamma)\,\mu_0}{c^2}}\right).$$

Evaluating the argument of the logarithm near the conjunction configuration we obtain an estimate for the 2-PN correction to the Shapiro formula:

$$S_{2PN} \approx \frac{(1+\gamma)\,\mu_0}{c^2}\,\ln\left(1 - \frac{2\,r_r\,r_t}{b^2}\,\frac{R_0^S}{r}\right) \approx -(1+\gamma)^2\left(\frac{\mu_0}{c^2\,b}\right)^2\frac{2\,r_r\,r_t}{r_r + r_t},$$

where b is the impact parameter of the radio wave path passing near the Sun. Interestingly, Moyer's heuristic correction provides the same result as much more theoretically rooted recent derivations of the 2-PN Shapiro terms by (Le Poncin-Lafitte & Teyssandier 2004, Klioner & Zschocke 2007, Teyssandier & Le Poncin-Lafitte 2008, Ashby & Bertotti 2008). Figure 4, on the right, shows that the 2-PN correction is relevant for our experiment, especially when there is a superior conjunction with small b.

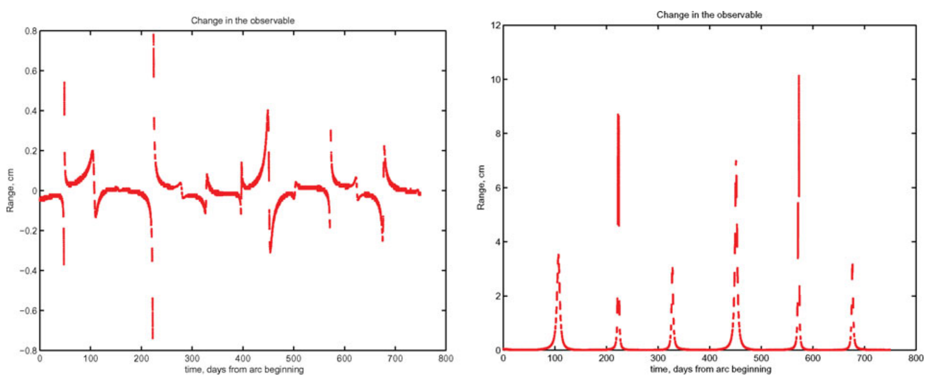

Figure 4. Left: differences in range by using a 1-PN and a 1.5-PN formulation ($\gamma = 1$); the correction has been added with little effort, but does not seem to be important, less than 1 centimeter in range, well below the accuracy level. Right: differences in range by using a 1-PN and a 2-PN formulation ($\gamma = 1$); the correction is relevant for MORE, at least when a superior conjunction results in a small impact parameter b, e.g., in this figure we have plotted data assumed to be available down to $b \simeq 3R_0$. For larger values of b the effect decreases as $1/b^2$.

5. Dynamic Mercury Time

The mercury-centric orbit of the spacecraft is coupled to the orbit of the planet, mostly through the difference between the acceleration from the Sun on the probe and the one on the planet (the Sun tidal term). This coupling is weak because the Sun tide is just 10^{-7} of the monopole acceleration from Mercury. The relativistic perturbations containing the mass of Mercury are small to the point that they are not measurable, being easily absorbed by the much larger non-gravitational perturbations, measured with finite accuracy by the on board accelerometer. Should we conclude that general relativity does not matter in the computation of the mercury-centric orbit? The answer is negative, but the main relativistic effect does not appear in the equation of motion.

There are three different time coordinates to be considered. The dynamics of the planets, as described by the Lagrangian, is the solution of differential equations having a time belonging to a space-time reference frame with origin in the SSB as independent variable. There can be different realizations of such a time coordinate: the currently published planetary ephemerides are provided in a time called TDB (Barycentric Dynamic Time). The observations are based on averages of clocks and frequency scales located on the Earth surface: this corresponds to another time coordinate called TT (Terrestrial Time). Thus for each observation the times of transmission and receiving (t_t, t_r) need to be converted from TT to TDB to find the corresponding positions of the planets, e.g., the Earth and the Moon, by combining information from the precomputed ephemerides and the output of the numerical integration for Mercury and the Earth-Moon barycenter. This time conversion step is necessary for the accurate processing of each set of interplanetary tracking data; the main term in the difference TT-TDB is periodic, with period 1 year and amplitude $\simeq 1.6 \times 10^{-3}$ s, while there is essentially no linear trend, as a result of a suitable definition of the TDB.

The equation of motion of a mercury-centric satellite can be approximated, to the required level of accuracy, by a Newtonian equation provided the independent variable is the proper time of Mercury. Thus, for the BepiColombo radioscience experiment, it is necessary to define a new time coordinate TDM (Mercury Dynamic Time) containing terms of 1-PN order depending mostly upon the distance from the Sun r_{10} and velocity v_1 of Mercury. The relationship with the TDB scale, truncated to 1-PN order (we drop the $O(c^{-4})$ terms on the right hand side, that are in principle known, but certainly not needed for our purposes), is given by a differential equation

$$\frac{dt_{\mathrm{TDM}}}{dt_{\mathrm{TDB}}} = 1 - \frac{v_1^2}{2\,c^2} - \sum_{k \neq 1} \frac{G\,m_k}{c^2\,r_{1k}},$$

which can be solved by a quadrature formula once the orbits of Mercury, the Sun and the other planets are known. Figure 5 plots the output of such a computation, showing a drift due to the non-zero average of the post-Newtonian term.

The oscillatory term, having the one of Mercury orbit as main period, has an amplitude $\simeq 0.012$ s. In 0.01 s the spacecraft velocity can change by 3 cm/s, $\simeq 10,000$ times more than the range-rate measurement accuracy, the position by 30 m, $\simeq 300$ times the range measurement accuracy. Thus this effect has to be accurately taken into account for our experiment†.

† The time scale TDB will be replaced in the planetary ephemerides by the new TCB; when this will happen, we will use a suitably defined Mercury Coordinate Time (TCM), such that the differential equation giving the TCB to TCM conversion will be exactly the same as for TDB to TDM.

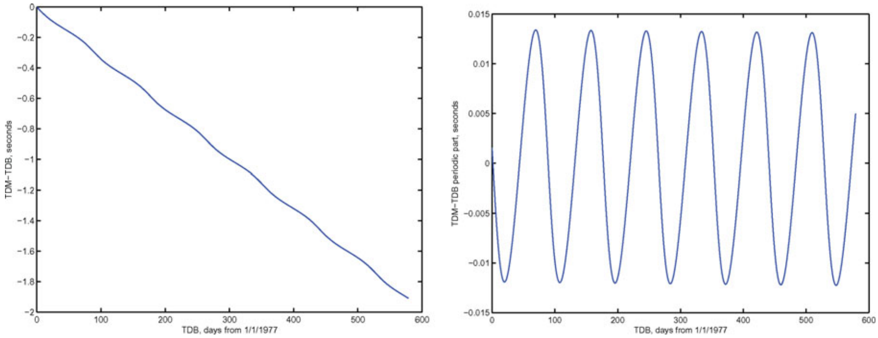

Figure 5. Left: TDM as function of TDB shows a drift due to the non-zero average of the 1-PN term. Right: the oscillatory term is almost an order of magnitude larger than TT-TDB.

6. Restricted ephemerides improvement

The level of accuracy of the measurements of the Mercury orbit radioscience experiment is incompatible with the use of the current planetary ephemerides, which have been solved by using lower accuracy measurements. However the data from BepiColombo by themselves will not allow to improve the ephemerides for all planets. Of the 5 vectors used in the light-time equation, $\mathbf{x}_{ant}$ and $\mathbf{x}_E$ can be assumed known: they cannot be improved by ranging to a Mercury orbiter. For the orbit of the Moon it is more effective to measure the range to the Moon with lunar laser ranging. Navigation satellites and VLBI give more information on the antenna position and on the rotation of the Earth.

The position of the Earth-Moon center of mass and of Mercury can certainly be improved by the range measurements from BepiColombo (the range-rate is less effective: it is more accurate than range only over time scales $\leqslant 50,000$ s). These measurements will also be provided (as normal points) to be used in the planetary ephemerides fit. In this way the BepiColombo data will be available to be used by all the existing and future planetary ephemerides to contribute to more accurate predictions of planetary orbits.

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