

Testing for gravitationally preferred directions using the lunar orbit

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As gravity is a long-range force, it is *a priori* conceivable that the Universe's global matter distribution selects a preferred rest frame for local gravitational physics. At the post-Newtonian approximation, the phenomenology of preferred-frame effects is described by two parameters α_1 and α_2 , the second of which is already very tightly constrained. Confirming previous suggestions, we show through a detailed Hill-Brown-type calculation of a perturbed lunar orbit that lunar laser ranging data have the potential of constraining α_1 at the 10^{-4} level. It is found that certain retrograde planar orbits exhibit a resonant sensitivity to external perturbations linked to a fixed direction in space. The lunar orbit being quite far from such a resonance exhibits no significant enhancement due to solar tides. Our Hill-Brown analysis is extended to the perturbation linked to a possible differential acceleration toward the galactic center. It is, however, argued that there are strong *a priori* theoretical constraints on the conceivable magnitude of such an effect. [S0556-2821(96)05912-7]

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I. INTRODUCTION

It has been recognized for many years that lunar motion provides a superb testing ground for relativistic gravity [1–6]. In particular, the Lunar Laser Ranging (LLR) experiment has allowed one to get a very high precision test of the equivalence principle, as well as a 1% test of the Einsteinian spin-orbit coupling [7,8]. However, it has been recently pointed out that the lowest-order perturbation analyses that have been commonly used [4,5,9] to derive theoretical estimates of (null or non-null) relativistic effects are insufficiently accurate in view of the importance of solar tidal effects [10]. Motivated by the results of Ref. [10], we presented in Ref. [11] a full-fledged Hill-Brown theory of the lunar orbit perturbation due to a hypothetical violation of the equivalence principle. We found that the interaction with the quadrupolar tide amplified the results of lowest-order perturbation analyses by a very significant factor: a 60% increase of the naive first-order calculation or a 40% increase of the improved first-order calculations allowing for perigee motion. Such results raise the question of whether similar amplification factors affect other (null or non-null) relativistic effects in the lunar motion. To address this question it is convenient to use the parametrized post-Newtonian (PPN) framework (see, e.g., [9]) in which possible deviations from general relativity in the weak-field regime are described by some parameters $\beta-1$, $\gamma-1$, α_1 , α_2 , etc., which vanish in Einstein's theory.

In the case of the effects linked to the Eddington post-Newtonian parameters β and γ Ref. [6] has indeed shown

that tidal effects are numerically important. An observationally oriented discussion of the influence of the tidal deformation on the main effects linked to β and γ is contained in Ref. [10], while, as we said above, the tidal amplification of equivalence-principle-violation effects was discussed in Refs. [10] and, in more detail, in [11].

In the present paper, we study the influence of the tidal deformation of the lunar orbit on the preferred-frame effects linked to the parametrized post-Newtonian parameter α_1 . We shall also discuss the effect of a hypothetical violation of the equivalence principle of galactic origin (noting, however, that there are strong *a priori* theoretical constraints on the magnitude of such a violation).

II. PREFERRED-FRAME EFFECTS

As gravity is a long-range force, one might *a priori* expect the Universe's global matter distribution to select a preferred rest frame for local gravitational physics. As shown in [12,13,9], all preferred-frame effects in the first post-Newtonian limit are phenomenologically describable by only two parameters α_1 and α_2 . These parameters are associated with the following terms in the Lagrangian describing the gravitational dynamics of N -body systems ($A, B = 1, \dots, N$):

$$L_{\alpha_1} = -\frac{\alpha_1}{4} \sum_{A \neq B} \frac{Gm_A m_B}{r_{AB} c^2} (\mathbf{v}_A^0 \cdot \mathbf{v}_B^0), \quad (2.1a)$$

$$L_{\alpha_2} = \frac{\alpha_2}{4} \sum_{A \neq B} \frac{Gm_A m_B}{r_{ABC}^2} [(\mathbf{v}_A^0 \cdot \mathbf{v}_B^0) - (\mathbf{n}_{AB} \cdot \mathbf{v}_A^0)(\mathbf{n}_{AB} \cdot \mathbf{v}_B^0)]. \quad (2.1b)$$

Here, \mathbf{v}_A^0 represents the velocity of a given body with respect to the gravitationally preferred frame and

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$\mathbf{n}_{AB} = (\mathbf{r}_A - \mathbf{r}_B)/r_{AB}$. Many (though not all) of the observable effects linked to α_1 and α_2 depend on the choice of the gravitationally preferred frame. We shall follow the standard assumption [9] that the latter frame, being of cosmological origin, can be (at least approximately) identified with the rest frame of the cosmic microwave background. This means that the center of mass of the solar system has the velocity \mathbf{w} with respect to the preferred frame of rest, with $|\mathbf{w}| \approx 370 \pm 10$ km/s in the direction $(\alpha, \delta) = (168^\circ, -7^\circ)$ [14].

It has been shown in Ref. [15] that the close alignment of the Sun's spin axis with the solar system angular momentum yields an extremely tight bound on α_2 : $|\alpha_2| \leq 3.9 \times 10^{-7}$ (90% C.L.). This limit on α_2 is much stronger than the existing limits on the other post-Newtonian parameters β , γ , and α_1 . We shall, therefore, neglect all α_2 effects in this work. Concerning α_1 , combined orbital data on the planetary system yield [16]

$$\alpha_1 = (2.1 \pm 3.1) \times 10^{-4} (90\% \text{ C.L.}), \quad (2.2)$$

while binary pulsar data yield comparable or better limits [17]. More precisely, PSR 1855 + 09 data yield $|\alpha_1| < 5.0 \times 10^{-4}$ (90% C.L.) [17], while a recent analysis of PSR J2317 + 1439 data [18] yields

$$|\alpha_1| < 1.7 \times 10^{-4} (90\% \text{ C.L.}). \quad (2.3)$$

The fact that the observational limits on the α_1 parameter are only a factor 10 better than the present limits on the (more conservative) Eddington post-Newtonian parameters $\beta - 1$ and $\gamma - 1$ recently stimulated Damour and Esposito-Farèze [19] to propose several experiments for improving them. Concerning their proposal to use artificial satellite motions, it has been recognized that the currently best laser-tracked satellite Laser Geodynamics Satellite (LAGEOS) cannot yield a better constraint on α_1 because of badly modeled nongravitational forces [20]. Another possibility mentioned in Ref. [19] (and first pointed out in [5]) concerns lunar motion and suggests that LLR data might yield an interesting new limit of the α_1 parameter. However, Refs. [5] and [19] used only first-order perturbation theory to estimate the α_1 effects in the lunar motion. In view of these facts (and the experience of the strong coupling with the solar tides mentioned in Sec. I), we decided to reassess the quantitative value of lunar data for constraining α_1 by building an accurate Hill-Brown theory of the preferred-frame perturbations of the lunar orbit.

We thus consider the three-body Earth-Moon-Sun system (keeping the notation of Ref. [11]; in particular we use the labels 1 = Moon, 2 = Earth, 3 = Sun). Generally all these three bodies contribute to the sum in Eq. (2.1a); however, we shall restrict ourselves to the ‘‘direct’’ preferred-frame effects with the subscripts A and B spanning only 1 (Moon) and 2 (Earth). It is easy (though not trivial) to verify that ‘‘tidal’’ preferred-frame effects, involving the subscript 3 (Sun) in Eq. (2.1a), are several orders of magnitude smaller than direct effects. Because of their observational irrelevance, we also omit from our discussion several terms in Eq. (2.1a) which are equivalent to a nearly constant redefinition of the locally measured gravitational constant. The dominant preferred-frame effects are then contained in the

following three contributions to the Lagrangian (we factorized the Earth-Moon reduced mass $\mu_{12} \equiv m_1 m_2 / m_0$, $m_0 = m_1 + m_2$, from the Lagrangian):

$$R_{\alpha_1}^{(1)} = -\frac{\alpha_1}{2c^2} \frac{Gm_0}{r} X_{21}(\mathbf{w} \cdot \mathbf{v}), \quad (2.4a)$$

$$R_{\alpha_1}^{(2)} = -\frac{\alpha_1}{2c^2} \frac{Gm_0}{r} X_{21}(\mathbf{v}_0 \cdot \mathbf{v}), \quad (2.4b)$$

$$R_{\alpha_1}^{(3)} = -\frac{\alpha_1}{c^2} \frac{Gm_0}{r} (\mathbf{w} \cdot \mathbf{v}_0). \quad (2.4c)$$

Here, $\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2$ is the geocentric lunar position vector and $\mathbf{v} \equiv d\mathbf{r}/dt$ its velocity, \mathbf{v}_0 is the velocity of the Earth-Moon center-of-mass motion around the Sun, and $X_{21} \equiv X_2 - X_1$, with the mass ratios $X_1 \equiv m_1/m_0$ and $X_2 \equiv m_2/m_0 \equiv 1 - X_1$. In the following section, we treat successively the perturbations of the lunar orbit associated with the three terms (2.4a)–(2.4c).

III. HILL-BROWN TREATMENT OF PREFERRED-FRAME EFFECTS

A. Method in brief

Since the Hill-Brown approach to lunar motion represents a classic tool of celestial mechanics treated with care in the literature (e.g., [21–24]), we outline its concept only very briefly, focusing mainly on the particularities of the method involved in the present study. We also refer the interested reader to Ref. [11] for more details and the notation used.

Following Hill we start by considering the planar Earth-Moon-Sun three-body problem with the Earth-Moon center of mass on a circular orbit around the Sun (the so-called ‘‘main problem’’). The near circular lunar motion is investigated in an Earth-centered coordinate system (X, Y) rotating with the angular velocity n' corresponding to the solar motion around the Earth-Moon center of mass (the Sun thus rests on the axis X). Apart from the Earth direct gravitational action, the quadrupolar piece of the solar (tidal) gravitational potential is also taken into account. The reduced Lagrangian of the lunar motion then reads

$$L_{\text{Hill}} = \frac{1}{2} (\dot{X}^2 + \dot{Y}^2) + n'(X\dot{Y} - Y\dot{X}) + R_{\text{Hill}}, \quad (3.1)$$

with

$$R_{\text{Hill}} = \frac{Gm_0}{\sqrt{X^2 + Y^2}} + \frac{3}{2} n'^2 X^2 \quad (3.2)$$

(the overdot means d/dt). Although simplified, the theory entails the most important part of the solar tidal deformation of lunar motion. Instead of the usual Keplerian ellipse, Hill chooses for the intermediary lunar orbit a periodic solution (with particular symmetries) of (3.1), the so-called ‘‘variational curve.’’ In the following, we shall investigate the forced perturbations of the variational orbit due to the addi-

tional Lagrangian terms¹ (2.4).

A very convenient parametrization of the Hill problem consists of replacing (X, Y) by the complex-conjugated quantities (w, \bar{w}) defined by (i is the complex unit)

$$X + iY = \tilde{a}\zeta(1 + w), \quad (3.3a)$$

$$X - iY = \tilde{a}\zeta^{-1}(1 + \bar{w}), \quad (3.3b)$$

$$\zeta = e^{i\tau}, \quad (3.3c)$$

$$\tau = (n - n')t + \tau_0, \quad (3.3d)$$

where n denotes the mean lunar motion around the Earth and where the τ variable represents the mean geocentric angular separation of the Moon and the Sun. Following Ref. [6], the fiducial lunar semimajor axis \tilde{a} is defined by

$$\frac{Gm_0}{(n - n')^2 \tilde{a}^3} = \kappa(m), \quad (3.4)$$

where

$$\kappa(m) = 1 + 2m + \frac{3}{2}m^2 \quad (3.5)$$

and $m \equiv n'/(n - n')$ is Hill's expansion parameter. (In the actual case of the Moon, $m_{\text{Moon}} = 0.080\,848\,937\,5\dots$) The Lagrange dynamical equations of the variational motion then read

$$L(w, \bar{w}) = W_{\text{Hill}}(w, \bar{w}), \quad (3.6)$$

where we denoted

$$L(w, \bar{w}) = D^2w + 2(1 + m)Dw + \frac{3}{2}\kappa(m)(w + \bar{w}), \quad (3.7)$$

with $D \equiv d/(id\tau) = \zeta d/d\zeta$ and the Hill source terms

$$W_{\text{Hill}}(w, \bar{w}) = -\frac{3}{2}m^2\zeta^{-2}(1 + \bar{w}) + \kappa(m)Q(w, \bar{w}). \quad (3.8)$$

The nonlinear source function $Q(w, \bar{w})$ and its development in terms of (w, \bar{w}) can be found for instance in Ref. [11].

When considering an extra perturbation of lunar motion, such as (2.4) in the case of preferred-frame effects, we have to include an additional source function on the right-hand side of Eq. (3.6) given by

$$S(w, \bar{w}; Dw, D\bar{w}) = D \frac{\partial}{\partial Dw} G - \frac{\partial}{\partial D\bar{w}} G, \quad (3.9)$$

in terms of the ‘‘generating function’’ $G(w, \bar{w}; Dw, D\bar{w}) \equiv 2(m/n'\tilde{a})^2 R$.

¹We recall that the variational orbit is not a general solution of the system (3.1). Apart from the ‘‘forced’’ perturbations related to a new physical cause it admits also ‘‘free’’ perturbations covering the classical notion of the lunar orbit eccentricity and its perigee drift due to the solar action. For simplicity, we omit in this study a natural coupling of the two types of perturbations, neglecting thus the (small) lunar eccentricity (and inclination) corrections to the preferred-frame perturbations.

Our method of solution of the system (3.6)–(3.9) consists of consecutive iterations, where at each stage one constructs a particular right-hand-side source based on the results of the previous iterations. Details can be found in Ref. [6] or Appendix B of Ref. [11]. Let us only point out that, contrary to the simpler case of the synodic lunar perturbations due to a hypothetical violation of the equivalence principle studied in Ref. [11], the generic form of the right-hand-side source term now reads

$$W_{\star}(\alpha) = W_{-\alpha}\zeta^{-\alpha} + W_{\alpha}\zeta^{\alpha}, \quad (3.10)$$

where we allow for (i) complex functions $W_{-\alpha}$ and W_{α} and (ii) any *real* (noninteger) values of the powers α . Inversion of the linear problem $L(w_{\star}(\alpha), \bar{w}_{\star}(\alpha)) = W_{\star}(\alpha)$ ($\alpha \neq 0$) has a simple form $w_{\star}(\alpha) \equiv w_{-\alpha}\zeta^{-\alpha} + w_{\alpha}\zeta^{\alpha}$ with

$$w_{\alpha} = \frac{1}{\Delta_{\alpha}(m)} \{ [\alpha^2 - 2(1 + m)\alpha + \frac{3}{2}\kappa] W_{\alpha} - \frac{3}{2}\kappa \bar{W}_{-\alpha} \}, \quad (3.11a)$$

$$w_{-\alpha} = \frac{1}{\Delta_{\alpha}(m)} \{ [\alpha^2 + 2(1 + m)\alpha + \frac{3}{2}\kappa] W_{-\alpha} - \frac{3}{2}\kappa \bar{W}_{\alpha} \}, \quad (3.11b)$$

and

$$\Delta_{\alpha}(m) \equiv \alpha^2 [\alpha^2 + 3\kappa - 4(1 + m)^2]. \quad (3.12)$$

[The solution corresponding to $\alpha = 0$ is identical with that given in Eq. (2.52a) of Ref. [11].] Some values of the power α in (3.12) may lead to a significant amplification of the effect due to the smallness of the corresponding denominator $\Delta_{\alpha}(m)$. Of particular interest for our present work is the case where $\alpha = 1 + m$ which yields the small denominator

$$\Delta_{1+m}(m) \equiv \frac{3}{2}m^2(1 + m)^2. \quad (3.13)$$

In the next section we shall see that it appears in the sidereal excitation of the lunar orbit.

Because of the background motivation of our work, related to the LLR experiment, we are essentially interested in the perturbation of the radial geocentric distance of the Moon given by

$$r^2 = \tilde{a}^2(1 + w)(1 + \bar{w}). \quad (3.14)$$

Performing a variation of this quantity, keeping only linear terms in the perturbation, we obtain

$$\frac{\delta r}{\tilde{a}} = \text{Re} \left[\left(\frac{1 + \bar{w}}{1 + w} \right)^{1/2} \delta w \right] \quad (3.15)$$

for the searched perturbation in radial coordinate. Remembering that $w = O(m^2)$, to lowest order in the m parameter, the radial oscillation can be expressed by the simple formula $\delta r/\tilde{a} \approx (\delta w + \delta \bar{w})/2$.

In the rest of this section we investigate the forced perturbations of the lunar variational orbit related to the three preferred-frame Lagrangian terms (2.4). Finally, we note that, albeit the iteration scheme mentioned previously is straightforward, it represents a huge algebraic manipulation exercise. We thus employed the powerful dedicated algebraic

computer system MINIMS developed by Moons from the University of Namur (Belgium) [25] to perform this task. The lowest two orders of the results have been, however, checked by hand computations.

B. Potential $R_{\alpha_1}^{(1)}$

First, we focus on the source term (2.4a). The corresponding generating function G reads

$$G(w, Dw) = -i\hat{\epsilon}_1 \left(\frac{\tilde{a}}{r} \right) \left\{ [Dw + (1+m)(1+w)] \zeta^{1+m} e^{-i\phi} + [D\bar{w} - (1+m)(1+\bar{w})] \zeta^{-(1+m)} e^{i\phi} \right\}, \quad (3.16)$$

where

$$\hat{\epsilon}_1 = \frac{\alpha_1}{2} X_{21} C \frac{|\mathbf{w}| v_0}{c^2} \left(\frac{\tilde{a}}{a'} \right) \frac{\hat{\kappa}(m)}{m} \quad (3.17)$$

and where one must express r in terms of w through Eq. (3.14). Here, a' is the radius of the (circular) solar orbit in the Earth-Moon center-of-mass frame and $v_0 = n' a'$ its (circular) velocity, $C (\approx 0.98)$ is the cosine of the ecliptic latitude of the unit vector \mathbf{w}^0 , and ϕ is a longitude angle of \mathbf{w}^0 measured from the lunar (and solar) position at time t_0 corresponding to an arbitrary new-moon phase. For instance, if we choose the last new-moon phase in this century, occurring at MJD51 521.2, we obtain $\phi = 267.2^\circ$. Inserting this expression into (3.9) we obtain the source function, to be added to the right-hand side of the Hill equation (3.6), in the form

$$S(w, \bar{w}; Dw, D\bar{w}) = -i \frac{\hat{\epsilon}_1}{2} \left(\frac{\tilde{a}}{r} \right) \left\{ Dw \left[\zeta^{1+m} \frac{e^{-i\phi}}{1+w} - \zeta^{-(1+m)} \frac{e^{i\phi}}{1+w} \right] + (1+m) \times \left[\frac{1+w}{1+\bar{w}} \zeta^{1+m} e^{-i\phi} - \zeta^{-(1+m)} e^{i\phi} \right] \right\}. \quad (3.18)$$

Working out the iterative solution mentioned above one realizes that this perturbation yields a wide spectrum of radial and longitudinal oscillations of the lunar orbit (compare also with the less accurate solution in Ref. [5]). However, a detailed analysis shows that only two of them are sufficiently amplified to give an observably interesting signal: (i) terms with frequency equal to the mean sidereal lunar motion n (having a period of about 27.32 days and (ii) terms with frequency equal to $n - 2n'$ (having a period of about 32.13 days). Both periods are evaluated for the lunar orbit. Hereafter we discuss properties of both of them starting with the sidereal terms.

The perturbation series giving the sidereal-frequency radial oscillations of the lunar orbit reads

$$\frac{\delta_e r}{\tilde{a}} = \frac{2\hat{\epsilon}_1}{3m^2} S_{\alpha_1}^{(1)}(m) \sin[n(t-t_0) - \phi], \quad (3.19)$$

TABLE I. Coefficients s_k of the $S_{\alpha_1}^{(1)}(m)$ series in powers of m . The percentage p_k of the contribution of the listed terms to the series for the lunar orbit, $m = m_{\text{Moon}} = 0.080\,848\,937\,5\dots$, is given in the second column. The last column gives the ratio (s_{k-1}/s_k) .

k	s_k	p_k	(s_{k-1}/s_k)
0	1.0000000000000000	—	—
1	-8.3750000000000000	-67.71	-0.119402985
2	49.3750000000000000	32.27	-0.169620253
3	-268.24739583333333	-14.18	-0.184065161
4	1444.1618923611111	6.17	-0.185746070
5	-7841.277434172453	-2.71	-0.184174314
6	42586.57408613037	1.19	-0.184125575
7	-231375.2814872327	-0.52	-0.184058443
8	1256984.833107015	0.23	-0.184071657
9	-6828885.820821555	-0.10	-0.184068802

with

$$S_{\alpha_1}^{(1)}(m) = 1 - \frac{67}{8}m + \frac{395}{8}m^2 - \frac{103\,007}{384}m^3 + \frac{3\,327\,349}{2304}m^4 + O(m^5). \quad (3.20)$$

Table I gives the coefficients of the series $S_{\alpha_1}^{(1)}(m)$ up to ninth order. The second column of the table indicates the numerical contribution of the corresponding term to the total value of the series for the lunar orbit, i.e., $m = m_{\text{Moon}} = 0.080\,848\,937\,5\dots$ [24]. Two important features are to be noticed: (i) a significant contribution of the higher order corrections to the total value of the series $S_{\alpha_1}^{(1)}(m_{\text{Moon}})$ and (ii) a geometriclike character of this series clearly pronounced after a few terms. The second property suggests the presence of a pole near the value² $m_{cr} \approx -0.184\,07$. Taking a Padé approximant of the series $S_{\alpha_1}^{(1)}(m)$ confirms the presence of a unique root of the denominator located at the value $m_{cr} = -0.184\,07$ with an error of about 10^{-5} .

The physical origin of this pole can be easily understood by using the following argument. When solving the problem by traditional first-order perturbation techniques (see, e.g., [5,19]), one finds that the sidereal orbit oscillation induced by an external force linked to a fixed direction in space contains, in the denominator, the secular rate of the perigee longitude $\dot{\omega}$. This agrees with the intuitive idea that a spatially “frozen” orbit (not moving its pericenter in fixed space) is “resonantly sensitive” to constant forces. As a check of this idea, we have multiplied the $S_{\alpha_1}^{(1)}(m)$ series by $\dot{\omega}(m) = n[(3/4)m^2 + (177/32)m^3 + \dots]$ as given by Andoyer up to order m^9 in Ref. [22]. The resulting series, say, $\bar{S}_{\alpha_1}^{(1)}(m) = 4S_{\alpha_1}^{(1)}(m)\dot{\omega}(m)/(3nm^2) = 1 - m + (313/64)m^2 + \dots$, is much more tame, showing that the main characteristics of $S_{\alpha_1}^{(1)}(m)$ are entailed in the factor $[\dot{\omega}(m)]^{-1}$. The difference

²Such a value corresponds to a retrograde orbit with a sidereal period (for Earth satellites) $T = 2\pi/|n| = 82.4$ days.

between our more precise solution (3.19) and the previous ones [5,19] (when the latter are improved by using the full value of the perigee advance) is essentially contained in the value of the residual series $\bar{S}_{\alpha_1}^{(1)}(m)$. We find that, in the lunar case, $\bar{S}_{\alpha_1}^{(1)}(m_{\text{Moon}}) \approx 0.956$. In contrast with the case of equivalence-principle-violation effects, we see therefore that preferred-frame effects exhibit no significant enhancement genuinely linked to the tidal deformation of the lunar orbit. This result holds for the other effects discussed below and is basically attributable to the fact that the actual lunar orbit is quite different from the ‘‘spatially frozen’’ resonant orbit (while it is rather near the orbit resonant for solar-directed equivalence-principle-violation effects).

The principal quantitative information of the previous analysis is given by the amplitude of the sidereal oscillation of the lunar orbit³ (3.19). Employing the definition (3.4) of the auxiliary lunar semimajor axis \tilde{a} [and a Padé approximant value of $S_{\alpha_1}^{(1)}(m)$: $S_{\alpha_1}^{(1)}(m_{\text{Moon}}) \approx 0.5465$] we have

$$|\delta_\epsilon r| \approx \frac{\alpha_1}{3} X_{21} \mathcal{C} \frac{|w|v_0}{c^2} \left[\frac{\kappa\chi^2}{m^5} \right]^{1/3} S_{\alpha_1}^{(1)}(m) a' \approx 4780 \times \alpha_1 \mathcal{C} [\text{cm}], \quad (3.21)$$

where $\chi \equiv (1 + m_3/m_0)^{-1}$. The current published *accuracy* of the lunar ranging measurements performed by the CERGA team is 14 mm. Recent technical improvements are giving a timing *precision* of about 6 mm [C. Veillet (private communication)]. If the latter precision level can be turned into an accuracy level, the result (3.21) suggests that the LLR data should soon be able to constrain α_1 at the 1×10^{-4} level or better (given the phase information and the presence of several α_1 effects at different frequencies).

The value $m=0$ of the Hill parameter is apparently another singularity of our ‘‘ranging formula’’ (3.19). However, because we neglected the Earth quadrupole and the other higher multipoles of the Earth gravity field, we cannot extend our solution to near-Earth satellite orbits ($m \approx 0$). This regime has been thoroughly discussed in Ref. [19]. In a first approximation we can, however, match smoothly the solution of Ref. [19], accounting basically for the Earth quadrupole, and the solution presented in the present study, accounting in detail for the third-body (Sun) perturbations, by adding the Earth quadrupole contribution to $\tilde{\omega}(m)$ after having factorized it as denominator of the series $S_{\alpha_1}^{(1)}(m)$. [Actually, as Andoyer’s series is not accurate enough to localize precisely the zero at $m = m_{\text{cr}}$, we found it better to add the quadrupole contribution to the denominator of a Padé approximant of $S_{\alpha_1}^{(1)}(m)$.] Figure 1 shows the synthesis of the two effects. The arrow points to the singularity $m = m_{\text{cr}}$, while the two points M and L stand for the Moon and the artificial (retrograde) satellite LAGEOS, respectively (with-

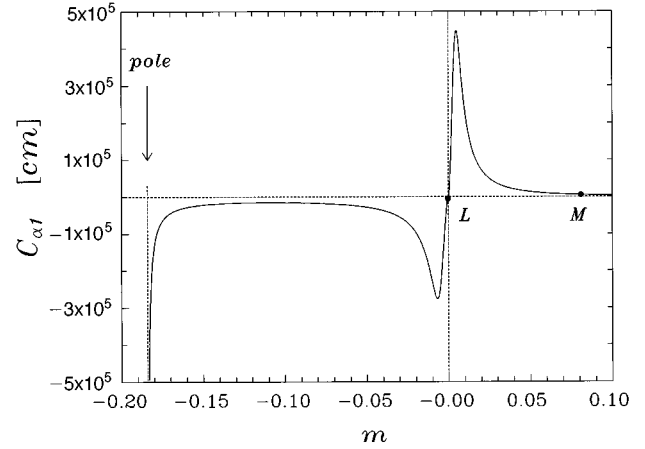


FIG. 1. Amplitude (in centimeters) C_{α_1} of the sidereal oscillation (for $\alpha_1=1$) vs the Hill parameter m (positive for prograde orbits, negative for retrograde orbits). For high orbits $C_{\alpha_1} \approx (2/3)\tilde{a}\hat{\epsilon}_1 S_{\alpha_1}^{(1)}(m)m^{-2}$ as given in Eq. (3.19). In the case of low orbits we introduce the influence of the Earth multipolar structure by adding a quadrupole contribution to the denominator of a Padé approximant of $S_{\alpha_1}^{(1)}(m)$. In addition to the singularly amplified orbit at $m_{\text{cr}} = -0.18407$, two celestial bodies are indicated: (i) the Moon (M) and (ii) the LAGEOS satellite (L).

out taking into account the LAGEOS inclination). We can see that none of the two bodies (an equatorial satellite at the LAGEOS altitude or the Moon) is the best candidate for testing preferred-frame effects, but that (as mentioned in [19]) a high-orbit artificial body with a period of about 30 h optimizes the sensitivity to the α_1 parameter (among prograde orbits). On the other hand, one should be aware of the fact that the motion of artificial bodies is typically influenced by many nongravitational forces, some of which are difficult to be predicted and/or carefully modeled. For instance, this is the reason why the LAGEOS satellite is currently less suitable for constraining the α_1 parameter than the Moon [20], which is a nearly perfect ‘‘drag-free Earth satellite.’’ Therefore the lunar data stand out as a potentially important source for the study of preferred-frame effects.

The intricate interaction of the variational curve perturbations with the underlying tidal deformation leads also to a slowly convergent series for perturbations at the $(n-2n')$ -frequency. The final result for the radial oscillations formula reads

$$\frac{\delta_\epsilon r}{\tilde{a}} = -\frac{5}{4} \frac{\hat{\epsilon}_1}{m} S_{\alpha_1}^{\prime(1)}(m) \sin[(n-2n')(t-t_0) + \phi], \quad (3.22)$$

with

$$S_{\alpha_1}^{\prime(1)}(m) = 1 - \frac{43}{6}m + \frac{28\,867}{720}m^2 - \frac{468\,391}{2160}m^3 + O(m^4). \quad (3.23)$$

The coefficients of $S_{\alpha_1}^{\prime(1)}(m)$ up to the eight order are given in Table II. The intimate coupling of this frequency with the sidereal frequency results in the coincidence of the pole in the m series $S_{\alpha_1}^{\prime(1)}(m)$ and $S_{\alpha_1}^{(1)}(m)$. Numerically $S_{\alpha_1}^{\prime(1)}(m_{\text{Moon}}) \approx 0.6035$, and the lunar orbit sensitivity to the

³Beware, however, that for the final determination of the α_1 constraint through LLR data analysis, the particular phase ϕ is very important. Moreover, we shall see that the preferred-frame perturbations of the lunar orbit act also with several other frequencies and it is their combined influence which determines the full effect to be searched for in the LLR data.

TABLE II. Coefficients s'_k of the $S_{\alpha_1}^{(1)}(m)$ series in powers of m . Other parameters as in Table I.

k	s_k	p_k	(s_{k-1}/s_k)
0	1.0000000000000000	—	—
1	-7.166666666666667	-57.94	-0.139534884
2	40.093055555555556	26.21	-0.178750823
3	-216.8476851851852	-11.46	-0.184890401
4	1173.979870756172	5.02	-0.184711587
5	-6371.734190136318	-2.20	-0.184248093
6	34622.85594255030	0.97	-0.184032600
7	-188086.0224857120	-0.42	-0.184079899
8	1021827.835270500	0.19	-0.184068212

α_1 perturbation on this frequency is given by $|\delta_{\epsilon r}| \approx 800 \times \alpha_1$ [cm], approximately 5.98 times smaller than for the principal sidereal effect. Notice, however, that this term in the spectrum of the preferred-frame lunar perturbations may bound the α_1 parameter as efficiently as the sidereal term if it turns out that there is significantly less noise at this frequency.

C. Potential $R_{\alpha_1}^{(2)}$

A special character of this term is due to its independence of the choice of the gravitationally preferred frame. It would be theoretically appealing if this term could significantly contribute to constraining α_1 . Unfortunately, we shall demonstrate that the significance of this perturbation faces two obstacles: (i) Its amplitude is small, and (ii) it acts with a synodic frequency, the same as the other phenomena tested through the LLR experiment (e.g., the classic equivalence-principle-violation effect [9–11]).

In the context of the main lunar problem we consider a circular solar orbit around the Earth. The velocity \mathbf{v}_0 thus becomes $-v_0 \mathbf{e}_Y$ ($v_0 = n' a'$) in the rotating (Hill) coordinate system introduced in Sec. II. The generating function G reads

$$G(w, Dw) = \hat{\epsilon}_2 \left(\frac{\tilde{a}}{r} \right) \{ [Dw + (1+m)(1+w)] \zeta - [D\bar{w} - (1+m)(1+\bar{w})] \zeta^{-1} \}, \quad (3.24)$$

with

$$\hat{\epsilon}_2 = \frac{\alpha_1}{2} X_{21} \left(\frac{v_0}{c} \right)^2 \frac{\kappa(m)}{m} \frac{\tilde{a}}{a'}. \quad (3.25)$$

Then the source function can be easily calculated by using (3.9):

$$S(w, \bar{w}; Dw, D\bar{w}) = \frac{\hat{\epsilon}_2}{2} \left(\frac{\tilde{a}}{r} \right) \left\{ \zeta^{-1}(1-m) + \zeta(1+m) \frac{1+w}{1+\bar{w}} + Dw \left[\frac{\zeta^{-1}}{1+w} + \frac{\zeta}{1+\bar{w}} \right] \right\}. \quad (3.26)$$

The close similarity with the classical equivalence-principle-violation effect studied in [11] consists of the fact that the

source function (3.26) excites the odd powers of ζ , resulting in (radial and longitudinal) oscillations of the lunar orbit with the synodic frequency, aliasing with the equivalence-principle-violation effect [10,11].

We have learned in Refs. [10,11] that any synodic signal is particularly amplified by the presence of a pole singularity occurring for a prograde orbit about 68% larger than the lunar orbit ($m_{\text{cr}} = 0.195\,103\,99\dots$). The corresponding ranging formula reads

$$\frac{\delta_{\epsilon r}}{\tilde{a}} = -\frac{\hat{\epsilon}_2}{2m} S_{\alpha_1}^{(2)}(m) \cos \tau, \quad (3.27)$$

where $S_{\alpha_1}^{(2)}(m)$ is a series in the Hill parameter m , whose numerical value for the lunar orbit is found to be 1.3022. The amplitude of the synodic oscillation (3.27) of the lunar orbit thus reads $|\delta_{\epsilon r}| \approx 57 \times \alpha_1$ [cm], too small to compete with the much greater sensitivity of the lunar orbit to the equivalence-principle-violation term [10,11].

D. Potential $R_{\alpha_1}^{(3)}$

From Eq. (2.4c) we see that this perturbing term is equivalent to a variation of the gravitational constant G [not to be confused with the generating functions $G(w, Dw)$] with a period of one year (see, e.g., [19]). The analysis of this term involves a small denominator $\Delta_{\alpha=m} \propto -m^2$ which, however, cancels out in the radial oscillation [19]. As found in Ref. [5], the remaining signal still exhibits an interesting sensitivity to the α_1 parameter.

The generating function G equivalent to the $R_{\alpha_1}^{(3)}$ reads

$$G(w, Dw) = i \hat{\epsilon}_3 \left(\frac{\tilde{a}}{r} \right) (\zeta^m e^{-i\phi} - \zeta^{-m} e^{i\phi}), \quad (3.28)$$

with

$$\hat{\epsilon}_3 = \alpha_1 \mathcal{C} \frac{|\mathbf{w}| v_0}{c^2} \kappa(m), \quad (3.29)$$

and the resulting source function is

$$S(w, \bar{w}) = \frac{\hat{\epsilon}_3}{2} \left(\frac{\tilde{a}}{r} \right) \frac{i}{1+\bar{w}} (\zeta^m e^{-i\phi} - \zeta^{-m} e^{i\phi}). \quad (3.30)$$

Using the previous scheme of solving the perturbation equations we obtain

$$\frac{\delta_{\epsilon r}}{\tilde{a}} = \hat{\epsilon}_3 S_{\alpha_1}^{(3)}(m) \sin[n'(t-t_0) - \phi] \quad (3.31)$$

for the expected contribution to the radial perturbation of the lunar orbit. The first terms of the series $S_{\alpha_1}^{(3)}(m)$ are listed in Table III. The magnitude of the oscillations (3.31) is about $|\delta_{\epsilon r}| \approx 4550 \times \alpha_1 \mathcal{C}$ [cm], comparable to the sidereal effect coming from $R_{\alpha_1}^{(1)}$. The value $\kappa(m) S_{\alpha_1}^{(3)}(m) \approx 0.9643$ shows that tidal effects are not very important. We learned from J.G. Williams (private communication) that the prospects of decorrelating the effect (3.31) from other annual effects is

TABLE III. The same as in Table I but for the $S_{\alpha_1}^{(3)}(m)$ series (ratio of the consecutive coefficients omitted).

k	s_k	p_k
0	1.0000000000000000	–
1	-2.0000000000000000	-16.17
2	-1.9531250000000000	-1.28
3	-5.8750000000000000	-0.31
4	12.50295003255209	0.05
5	49.86885579427062	0.02
6	73.34703290903986	<
7	-113.6395324400916	<
8	-670.4626529838962	<
9	-1047.599614057912	<

high. If this is confirmed, Eqs. (3.21) and (3.31) would be the two best probes for constraining α_1 .

IV. POLARIZATION OF THE LUNAR ORBIT BY A GALACTIC DIFFERENTIAL ACCELERATION

In addition to preferred-frame effects, other perturbing forces can be linked to some fixed direction in space. This would be, in particular, the case if the Earth and the Moon would fall with a different acceleration toward the center of the Galaxy. Years ago this possibility has been mentioned in relation with a possible violation of the equivalence principle linked to the gravitational binding energy of planets [26]. More recently, this idea has been revived within the context of possible strong-gravitational-field effects in neutron stars, and has led, through the use of existing binary pulsar data, to new tests of strong-field gravity [27]. Still more recently, the idea surfaced again with a different motivation: the possibility that the coupling between ordinary (visible) matter and galactic dark matter violates the equivalence principle [28]. The latter suggestion led to new galactic-related laboratory tests of the equivalence principle [29,30], as well as to a corresponding reanalysis of lunar laser ranging data [31–33]. Before applying our Hill-Brown algorithm to the latter problem (with the result that we find no unexpected amplification), we wish to emphasize that there are strong *a priori* theoretical constraints (using existing observational data) on the conceivable magnitude of any “dark matter effect.” These constraints diminish, in our opinion, the theoretical significance of the results of Refs. [29,30,32].

We assume a field-theoretic framework (as is always the case in recent discussions concerning possible violations of the equivalence principle, e.g., [28,30]). Within such a framework, any effect on visible matter due to a new (non-Einsteinian) long-range field generated by dark matter is necessarily proportional to the product $\alpha_V \alpha_I$ of two coupling constants: α_V measuring the coupling of the field to visible matter and α_I the coupling to invisible (dark) matter. We normalize these coupling constants with respect to the usual Einsteinian coupling so that the effective gravitational constant between bodies A and B reads $G_{AB} = G_* (1 \pm \alpha_A \alpha_B)$ where G_* is a bare Newtonian constant and where the plus (minus) sign holds for a spin 0 (spin 1) mediating field. To fix the ideas, let us consider the case of a scalar field (our

argument goes through in both cases, but is newest in the scalar case, the vector case having been already discussed in [29], though with a less stringent constraint on α_I). Equivalence principle tests probe the composition dependence of the visible couplings: $\alpha_A^{(V)} - \alpha_B^{(V)} = 0$. Let us denote by f_{AB} the fractional modification of the average coupling to visible matter α_V in a differential composition-dependent experiment: $\alpha_A^{(V)} - \alpha_B^{(V)} = f_{AB} \alpha_V$. The ordinary tests of the equivalence principle (using visible sources) measure the fractional differential acceleration

$$\eta_{AB}^{VV} \equiv \left(\frac{\Delta a}{a} \right)_{AB}^{VV} = (\alpha_A^{(V)} - \alpha_B^{(V)}) \alpha_V = f_{AB} \alpha_V^2. \quad (4.1)$$

These experiments give us a handle (in practice, an upper limit) on the magnitude of α_V : $\alpha_V = (\eta_{AB}^{VV}/f_{AB})^{1/2}$. (For simplicity, we do not put absolute value signs around α , η , and f .) Then the tests of the equivalence principle using the same pair of visible bodies and an invisible source (e.g., the galactic dark matter) measure

$$\eta_{AB}^{VI} \equiv \left(\frac{\Delta a}{a} \right)_{AB}^{VI} = (\alpha_A^{(V)} - \alpha_B^{(V)}) \alpha_I = f_{AB} \alpha_V \alpha_I. \quad (4.2)$$

Inserting the previous value of α_V into (4.2) yields

$$\eta_{AB}^{VI} = (f_{AB})^{1/2} (\eta_{AB}^{VV})^{1/2} \alpha_I. \quad (4.3)$$

The main point is, now, that observable facts give not only very stringent limits on η_{AB}^{VV} , but also mild limits on α_I . Indeed, Damour, Gibbons, and Gundlach [34,35] have shown, in the case of different scalar couplings to visible and invisible matter, that cosmological data were putting the limit $\alpha_I \equiv \sqrt{2} \beta_I < 0.71$. (See also the later work [36] which considered only the case $\alpha_V \equiv 0$, $\alpha_I = 0$.) An even more stringent limit comes from gravitational lenses. Indeed, in some gravitational lenses one measures three different “gravitational masses”: a “virial” mass $G_*(1 + \alpha_I^2)M$ linked to binary interactions, the gravitational mass probed by the x-ray-emitting gas $G_*(1 + \alpha_V \alpha_I)M$, and the lensing mass G_*M (light being uncoupled to the new field, be it scalar or vector). The coincidence, within better than 30%, of these three masses in some systems [37] gives the limit $\alpha_I < (0.30)^{1/2} = 0.55$. (Modulo the sign change $\alpha_I^2 \rightarrow -\alpha_I^2$, this argument applies to the vector case and therefore improves upon the limit $\alpha_I^{(\text{vector})} < 1$ used in [29].) We can apply the above limits to the case of the elemental compositions of the Moon (silica) and the Earth (iron core plus silica mantle). A laboratory approximation of this case (Si/Al versus Cu) has given $\eta_{AB}^{VV} = (5 \pm 7) \times 10^{-12}$ [30] so that, at the 1σ level, $|\eta_{AB}^{VI}|^{1/2} < 3.4 \times 10^{-6}$. Finally, we get for the maximum Moon-Earth (1–2, keeping our previous labels) differential acceleration caused by a possible anomalous coupling to dark matter (taking into account a further factor $m_{\text{core}}/m_{\text{Earth}} = 0.32$),

$$\left(\frac{\Delta a}{a} \right)_{12}^{VI} < 6.0 \times 10^{-7} \sqrt{|f_{AB}|}. \quad (4.4)$$

Moreover, the fractional composition dependence f_{AB} (where $A = \text{SiO}_2$, $B = \text{Fe}$) is generically expected to be small compared to one.⁴ For instance, in dilaton models [38] one finds $f_{AB} \approx 1.89 \times 10^{-5} [(E/M)_A - (E/M)_B]$ where $E = Z(Z-1)/(N+Z)^{1/3}$. Here $Z =$ atomic number, $N =$ neutron number, and $M =$ mass in atomic mass units. The only (physically motivated) case, we know of, where α_A would exhibit a significant fractional variation over the periodic table, is the case of appreciable coupling to lepton number L or to $B-L=N$. [In view of the small variation of the baryon to mass ratio $(B/M)_{AB} \sim 10^{-3}$, any coupling involving L with a relative coefficient of order unity leads to essentially the same results.] For instance, for a coupling to $B-L$, one gets $f_{AB} = (2N/M)_{\text{SiO}_2} - (2N/M)_{\text{Fe}} \approx -0.076$. Inserting this figure in Eq. (4.4) and using the full galactic acceleration $a \approx 1.9 \times 10^{-8} \text{ cm/s}^2$, one gets

$$(\Delta a)_{12}^{\text{VI}} < 3.1 \times 10^{-15} \text{ cm/s}^2, \quad (4.5)$$

which is 10 times smaller than the upper limit found in a recent analysis of LLR data [32]. Even if we take $|f_{AB}| \sim 1$, we get $(\Delta a)_{12}^{\text{VI}} \lesssim 1 \times 10^{-14} \text{ cm/s}^2$, which is 3 times smaller than the result of [32]. We conclude that, within what we consider the most natural theoretical framework, LLR data (and *a fortiori* laboratory experiments [30]) do not (yet) probe a theoretically very significant domain of values of possible anomalous couplings to dark matter.

Denoting \mathbf{N}_G the projection of the unit vector directed toward the galactic center on the ecliptic plane and $\mathbf{R} \equiv (X, Y)$, the galactic polarization effect is described by the potential

$$R_G = A_G (\mathbf{N}_G \cdot \mathbf{R}) \quad (4.6)$$

analogous to Eqs. (2.4). The parameter A_G phenomenologically represents a differential acceleration of the Moon and the Earth toward the galactic center. The corresponding generating function G is given by

$$G = \hat{\omega} [(1+w)\zeta^{1+m} e^{-i\phi_G} + (1+\bar{w})\zeta^{-(1+m)} e^{i\phi_G}], \quad (4.7)$$

where

$$\hat{\omega} \equiv m^2 \frac{A_G}{\overline{an}^{\prime 2}}. \quad (4.8)$$

The polar angle $\phi_G = 1.1^\circ$ gives the angular distance of the galactic center from the lunar (and solar) position corresponding to the above chosen new-moon phase at MJD51521.2. Employing Eq. (3.9) we obtain the source term of Hill's problem in the form

$$S = -\hat{\omega} e^{i\phi_G} \zeta^{-(1+m)}. \quad (4.9)$$

Because of the similarity of this function with (3.18), we recover the qualitative conclusions of Sec. III B. The sidereal perturbation of the lunar orbit reads

⁴If f_{AB} were larger than one, one should modify our analysis above, and define more carefully the average value α_V .

TABLE IV. The same as in Table I but for the $S_{\text{gal}}(m)$ series.

k	s_k	p_k	(s_{k-1}/s_k)
0	1.0000000000000000	—	—
1	-9.3750000000000000	-75.80	-0.106666667
2	58.7500000000000000	38.40	-0.159574468
3	-332.38802083333333	-17.57	-0.176751256
4	1810.893663194444	7.74	-0.183549166
5	-9846.043167679392	-3.40	-0.183920955
6	53488.73992091048	1.49	-0.184076932
7	-290602.0005454238	-0.66	-0.184061843
8	1578765.334003927	0.29	-0.184069155
9	-8577020.610670500	-0.13	-0.184069201

$$\frac{\delta_{\omega r}}{\bar{a}} = -2 \frac{\hat{\omega}}{m^2} S_{\text{gal}}(m) \cos[n(t-t_0) - \phi_G], \quad (4.10)$$

with

$$S_{\text{gal}}(m) = 1 - \frac{75}{8}m + \frac{235}{4}m^2 - \frac{127\,637}{384}m^3 + \frac{4\,172\,299}{2304}m^4 + O(m^5). \quad (4.11)$$

A more complete set of the coefficients of this series is given in Table IV. The dominant m dependence of this series is again captured by factorizing $[\hat{\omega}(m)]^{-1}$. The numerical value of the series (4.11) for the lunar orbit is $S_{\text{gal}}(m_{\text{Moon}}) = 0.5050$. Clearly, the dark matter differential coupling contributes also to the $(n-2n')$ frequency of the radial oscillation of the lunar orbit. We do not give here the detailed result, just quoting that its amplitude is about 5.94 times smaller than the amplitude of the principal galactic polarization contribution (4.10). Finally, the series $S_{\text{gal}}(m)$ shows the same pole, near $m_{\text{cr}} \approx -0.184\,07$, as the sidereal series in (3.19).

V. CONCLUSIONS

The main results of this paper may be summarized as follows.

We have confirmed, by more detailed computations, previous suggestions [5,19] that LLR data have the potential of constraining the post-Newtonian parameter α_1 at the 1×10^{-4} level or better. We showed that the preferred-frame perturbations associated with the α_1 parameter contribute a large spectrum of frequencies in the radial oscillation of the lunar orbit. The dominant α_1 effects occur at frequencies n (sidereal effect) and n' (yearly effect) with well-determined phases, and there is a subdominant effect at frequency $n-2n'$. Although the analytical results that we obtained from a high-order Hill-Brown algorithm should be accurate enough for fitting purposes, it may be advisable to resort to a direct numerical integration of the equations of motion (see, e.g., [19] for the α_1 contributions to the equations of motion).

We found that retrograde planar orbits with $n'/(n-n') = -0.184\,07$ (which have fixed perigees in inertial space) exhibit a resonant amplification of preferred-frame effects. Putting an artificial satellite near such an orbit

could be an efficient (though expensive) way of improving the present bounds on α_1 .

We have extended our analysis to another perturbation linked to a fixed direction in space: namely, a possible differential acceleration toward the galactic center. Evidently, this perturbation exhibits also a pole at $m_{\text{cr}} = -0.18407$ corresponding to an orbit “frozen in space.” We argue, however, that there are strong *a priori* theoretical constraints on the conceivable magnitude of such an effect.

A specific suggestion for future work concerns applying our analysis of the singularly perturbed spatially frozen orbits to planetary satellites. It is widely known that solar system satellites are submitted to a complicated cosmogonic tidal evolution. It might be interesting to study if some of these bodies evolved historically through such a frozen orbit configuration, yielding indirect limits on the α_1 parameter. For instance, one easily verifies that several of Jupiter small

satellites do lie close to the frozen configuration. A careful study of these problems is, however, beyond the scope of this paper.

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