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# Relativistic spin effects in the Earth–Moon system

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**Abstract.** An accurate theory of the Moon's spin-orbital motion has been constructed by numerical integration. Several phenomena capable of producing effects of at least $10^{-4}$ arc seconds in the lunar physical librations have been included, analysed, and for the most part presented in previous papers. The present work deals with the relativistic spin contributions acting on the Earth-Moon system: (i) de Sitter precession of the Earth reference frame, (ii) quasi-Newtonian torques acting on the lunar physical librations. The latter effect does not seem to have been studied yet. The global behaviour of the resulting librations is presented and described. The resulting magnitude of these librations is of the order of a few $10^{-4}$ arc seconds and may reach one milliarcsecond. Unlike the very faint relativistic contribution to the Earth's rotation, the corresponding relativistic terms in the Moon's rotational motion are not negligible with respect to the present observational accuracy of the lunar laser data. Consequently it would be appropriate to consider them in models adjusted to the observations.

**Key words:** celestial mechanics – Moon – gravitation – relativity

## 1. Introduction

The theory of lunar motion has represented one of the most important keystones of celestial mechanics throughout its history. Having this status, it has often served for testing the currently accepted theory of gravitation. Thus it is not surprising, that we already meet important relativistic applications to lunar motion in the first years of Einstein's theory. Namely, de Sitter (1916) discussed the appearance of the combined geodetic and Thomas precessions of the Earth reference frame in the context of lunar motion (nowadays known as 'de Sitter precession'). Secondly, Nordtvedt recognized the possible importance of lunar motion for testing his strong equivalence breaking proposal (Nordtvedt 1968, 1973). Finally, first attempts at lunar laser ranging (LLR) in the late sixties were accompanied by a partial goal consisting in the measurement of the gravitational constant variations [see a review by Mulholland (1980)]. Such variations occur in several tensor-scalar gravitational theories.

On the other hand, it should be understood that currently the importance of lunar motion for testing theories of gravitation has slightly changed. The underlying reasons are as follows:

i) the Earth-Moon(-Sun) system is very 'dirty' from the relativistic point of view because of many Newtonian physical phenomena related to the internal structure of the bodies that are only poorly modelled;
ii) relativistic (PN-)parameters are determined with considerable precision via other solar system experiments (see e.g. Will 1992; the Eddingtonian $\beta$ and $\gamma$ parameters currently coincide with the Einsteinian values with standard errors at the 0.2% level);
iii) the 'true frontiers' of the experimental testing of viable classes of theories of gravitation have moved to systems with a strong field regime, represented mainly by binary pulsars (e.g. Damour & Taylor 1992; Damour & Esposito-Farèse 1992).

Concerning the second point, we can mention that the Einsteinian theory has been largely verified in the weak field regime, at least with the solar system tools currently at our disposal (we are aware of planned Earth vicinity or solar system missions designed to extend the upper limits of the parameters of the post-Newtonian theories; it however appears that currently known constraints on those parameters are sufficiently strong for the possibilities of the LLR experiment). In this situation, one even faces the question: *should the formulation of the lunar motion still serve for testing the Einsteinian theory of gravitation or simply incorporate it as a valid theory?*

It has been demonstrated that there is still a place for good routine work in testing the weak-field limit of the theories of gravitation via lunar data (e.g. Dickey et al. 1989; Müller et al. 1991). Notably, investigations of the Nordtvedt effect still possess interesting theoretical implications (e.g. Nordtvedt 1988; Damour & Esposito-Farèse 1992). Secondly, de Sitter precession of the Earth reference frame shows practical importance well beyond the pure theory of lunar motion.

However, independently of the opinion concerning the answer to the previous items, one has to ask: *which Einsteinian*

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*terms are to be retained in the theory for a given precision?* We recall the important methodological fact, that the Einsteinian theory of gravitation is embodied into the general set of post-Newtonian theories (Damour 1993; Will 1993). Testing relativity, one always distinguishes between null and non-null tests. In the following, we shall have in mind non-null tests of general relativity.

Current state of the art LLR data reach the 3 centimeter precision level, averaged over data sets of the last few years (Dickey et al. 1994). It should be however emphasized that a careful choice of the best performing nights over the last years reaches even better properties – about 1 to 2 centimeters and improvement to a few millimeter level is already envisaged (Ch. Veillet; private communication). In terms of lunar librations this current best data quality means the level of a few milliarcseconds (later referred to as *mas*) and better can be reached in the future.

Generally, one can divide relativistic dynamical contributions into those operating on the (i) translational motion of the bodies, (ii) rotational motion of the bodies with respect to the locally transported frames with the bodies, (iii) orientation of the local (reference) frames transported by the bodies. Up to now, most of the attention (both theoretical and observational) has been focused on the relativistic contributions to the (lunar) translational motion and local frame definitions †. Recent analytical insights can be found e.g. in Soffel (1989) and Nordtvedt (1994). It is interesting to note, that in the case of the Moon, the dominant relativistic range terms (with the amplitude of a few centimeters) are due to the relativistic tidal field of the Sun in the Earth reference frame (e.g. Nordtvedt 1991, 1994; Shahid-Saless 1992). The Earth causes principally an advance of lunar perigee (a so far unmeasurable value of about 0.6 *mas* per year). To our knowledge, none of the previous works touched on relativistic terms related to the lunar rotation. Even more generally: only a small fraction of all tests of relativity theory relates to the spin motion of the bodies (for example see the evidence for the spin-orbital coupling in the binary pulsar systems; Weisberg et al. 1989). From this point of view, and the fact of the existence of high-quality LLR data, it is an interesting task to investigate relativistic contributions to the lunar rotational motion.

In a recent brief communication (Bois & Vokrouhlický 1994), we reported on the possible importance of a relativistic phenomenon connected with the lunar rotational motion not used so far in the LLR data reduction programs. In this paper, we wish to elaborate more on the topic.

## 2. Relativistic spin terms in the Earth-Moon system

The most suitable formulation of the post-Newtonian (PN) theory of motion of a system of N weakly self-gravitating extended bodies for purposes of celestial mechanics has been recently presented in a series of papers by Damour et al. (Damour et al. 1991, 1992, 1993, 1994; hereafter quoted as DSX). We shall essentially follow their approach (the same applies for notation). Success of DSX is based on precise distinguishing between global (barycentric) translational motion of the bodies and their local rotational motion resulting in careful usage of corresponding global and local coordinate systems. Here, we shall deal only with the local rotational motion. As the dynamical state of the local frames is intimately connected with the rotational motion of the corresponding body, we shall also report on the results concerning de Sitter precession of the Earth (dynamically non-rotating) reference frame.

Damour et al. (1993) have succeeded in defining the local spin vector $S_a^{\rm A}$ characterizing rotation of body A (embodied in the isolated system of N extended bodies). Employing first principles they showed that it satisfies Eulerian dynamical equations

$$\frac{dS_a^{\rm A}}{dT_{\rm A}} = \epsilon_{abc}\left[\sum_{k\geq 0}\frac{1}{k!}M_{bK}^{\rm A}G_{cK}^{\rm A} + \frac{1}{2c^2}S_b^{\rm A}H_c^{\rm A}\right] , \tag{1}$$

where $(M_L^{\rm A}, S_L^{\rm A})$ are mass and spin (Blanchet-Damour; BD) multipoles characterizing the PN gravitational field of the extended bodies while $(G_L^{\rm A}, H_L^{\rm A})$ are tidal gravitoelectric and gravitomagnetic PN fields, $c$ is the light velocity. All these quantities are carefully defined and discussed in Damour et al. (1991, 1992, 1993). Let us just note that we adopt definition (4.24) in Damour et al. (1993) for spin multipole moments possessing the simpler form (1) for the dynamical equations. Superindex A denotes a corresponding body and $T_{\rm A}$ stands for the local time measured along the worldline $\mathscr{L}_{\rm A}$ of the origin of the local chart associated with body A (to be identified later with its mass center). Local coordinate time $T_{\rm A}$ is related to the global coordinate time $t$ by the formula

$$\frac{dT_{\rm A}}{dt} = 1 - \frac{1}{c^2}\left(\frac{1}{2}\mathbf{v}_{\rm A}^2 + \bar{w}^{\rm A}\right) + \mathscr{O}(4) , \tag{2}$$

where $\mathbf{v}_{\rm A}$ is the velocity of the local frame origin expressed in the global system and $\bar{w}^{\rm A}$ is the external gravitational potential influencing the motion of body A [see Eq. (4)].

One learns from the DSX series that the first principles lead to dynamical laws for the lowest multipoles only (zero and first-order mass multipoles and first-order spin multipoles). If one wanted to develop a complete theory of the motion one would need to close the laws by some physical assumptions. As the relativity-attributed terms in the dynamical equations contain small parameters (squares of the orbital velocities divided by the light speed, Schwarzschild radii divided by the orbital distances), we shall restrict ourselves to the pole-dipole-quadrupole (PDQ) truncated model (clarified in Damour et al. 1991, 1992). This means that we shall simply neglect all higher multipole moments in the corresponding relativistic terms. Moreover, because we do not dispose of dynamical equations for the quadrupole moments $M_{ab}^{\rm A}$ we shall adopt the 'rigid' model of the extended bodies. Although the notion of rigidity faces conceptual problems in the theory of relativity, we shall formally keep the rigid scheme as known from the Newtonian approach.

† We leave here aside relativistic definition of the time scales in the solar system; Brumberg & Kopejkin 1990, Seidelmann & Fukushima 1992.

Practically this is acceptable since the relativistic quadrupole contributions are very small (see the Appendix for more details).

### *2.1. De Sitter precession of the local dynamically non-rotating reference frames*

We are free to fix several data concerning local comoving systems of the extended bodies. Namely, we shall follow the DSX recommendation to identify the origin of the local system with the mass center of the corresponding body: $M_a^{\rm E} = 0$ (in the following, we shall use index E when specifying the Earth for the previously arbitrary body A; similarly, index M will be reserved for the Moon). This choice advantageously discards the first term in Eq. (1) for $k = 0$. Secondly, in this study we shall use dynamically non-rotating local systems which are characterized by the condition: $H_a^{\rm E} = 0$ (Coriolis effacing). Let us recall, that local dynamically non-rotating frames show a slow (de Sitter) rotation with respect to the kinematically non-rotating frames. The latter ones are practically realized by the VLBI measurements, so that the de Sitter rotation of the body A local chart is measured with respect to the VLBI reference system (see e.g. Soffel & Brumberg 1991; Jacobs et al. 1993).

Damour et al. (1993) showed that the total first order gravitomagnetic tidal moment reads

$$H_a\,(T_{\rm E}) = -4\epsilon_{ijk} R^i_{{\rm E}a}\left[\partial_j \bar{w}^{\rm E}_k + v^j_{\rm E}\partial_j \bar{w}^{\rm E}\right]_{\mathscr{L}_{\rm E}} + \epsilon_{abc}\left[V^{\rm E}_b A^{\rm E}_c + c^2 \frac{dR^i_{{\rm E}b}}{dT_{\rm E}} R^i_{{\rm E}c}\right]_{\mathscr{L}_{\rm E}} , \qquad (3)$$

where $R^i_{{\rm E}a}$ denotes the orthogonal rotational matrix involved in the transformation between local and global systems. Notice the clear theoretical interpretation of the first two terms in (3) as the (PN) gravitational influence of the external bodies, while the last two terms have coordinate origin in the transformation between global and local systems.

In the numerical implementation of this theory (Sect. 3), we shall adopt only the monopole part of the external metric, so that

$$\bar{w}^{\rm E} = \sum_{\rm B\neq E} \frac{GM_{\rm B}}{r_{\rm EB}} \quad , \qquad \bar{w}^{\rm E}_i = \sum_{\rm B\neq E} \frac{GM_{\rm B}}{r_{\rm EB}} v^i_{\rm B} \; , \qquad (4)$$

neglecting 'higher order terms'. It can be easily verified that the contribution of these higher order terms is about 4 orders of magnitude smaller than the one term arising from the monopole part (4).

Introducing components of the rotational vector $\omega^{\rm E}_i$ characterizing rotation of the dynamically non-rotating frame with respect to the kinematically non-rotating (barycentric) one in the form

$$\omega^{\rm E}_i = -\frac{1}{2}\epsilon_{ijk}\frac{dR^j_{{\rm E}a}}{dT_{\rm E}} R^k_{{\rm E}a} \; , \qquad (5)$$

($\epsilon_{ijk}$ is the Levi-Civita fully antisymmetric tensor) we arrive at

$$\boldsymbol{\omega}^{\rm E} = -\frac{1}{2c^2}\boldsymbol{v}_{\rm E}\times\boldsymbol{a}_{\rm E} + \frac{2}{c^2}\boldsymbol{v}_{\rm E}\times\nabla\bar{w}^{\rm E} + \frac{2}{c^2}\nabla\times\bar{\boldsymbol{w}}^{\rm E} \; , \qquad (6)$$

which is to be compared with results of Kopejkin (1988), Will (1993). Note that some authors (e.g. Will 1993) prefer to include the 'gravitational part' of the barycentric acceleration $\boldsymbol{a}_{\rm E}$ in the second term, arriving then at a factor 3/2 rather than 2.

It should be also noted that Damour et al. (1993) proved that local frames induced by a dynamically non-rotating local chart are Fermi-Walker transported along the worldline $\mathscr{L}_{\rm A}$ of the body A mass center with respect to the external metric. This property shows that DSX also succeeded in constructing Fermi (or generalized Fermi) frames around massive bodies (e.g. Ashby & Bertotti 1986; Fukushima 1988).

### *2.2. Relativistic terms in the quasi-Newtonian torque*

Damour et al. (1993) succeeded in deriving an explicit 'quasi-Newtonian' form for the first relativistic contribution in the rotational motion arising from the first term in Eq. (1) – the only term remaining in the PDQ model. Laborious evaluation of the tidal gravitoelectric multipole $G^{\rm M}_{ab}$ leads to (Damour et al. 1993, 1994)

$$\begin{aligned} G^{\rm M}_{ab} = R^i_{{\rm M}a} R^j_{{\rm M}b} \sum_{\rm B\neq M} \frac{3GM_{\rm B}}{r^3_{\rm MB}} {\rm STF}_{ij}\Big\{ n^{ij}_{\rm MB} + \frac{1}{c^2}\Big[ n^{ij}_{\rm MB}\big(2\mathbf{v}^2_{\rm MB} \\ - 2\bar{w}^{\rm M} - \bar{w}^{\rm B} - \frac{5}{2}(\mathbf{n}_{\rm MB}.\mathbf{v}_{\rm B})^2 - \frac{1}{2}\mathbf{a}_{\rm B}.\mathbf{r}_{\rm MB}\big) \\ + a^i_{\rm MB} r^j_{\rm MB} + v^i_{\rm MB} v^j_{\rm MB} - 2(\mathbf{n}_{\rm MB}.\mathbf{v}_{\rm MB}) n^i_{\rm MB} v^j_{\rm MB} \\ - (\mathbf{n}_{\rm MB}.\mathbf{v}_{\rm M}) n^i_{\rm MB}\big(v^j_{\rm M} - 2v^j_{\rm B}\big)\Big]\Big\} + \mathscr{O}(4) \; , \end{aligned} \qquad (7)$$

$$n^i_{\rm AB} = (r^i_{\rm M} - r^i_{\rm B})/r_{\rm MB} \; , \quad r^i_{\rm MB} = r^i_{\rm M} - r^i_{\rm B} \; ,$$
$$v^i_{\rm MB} = v^i_{\rm M} - v^i_{\rm B} \; , \quad a^i_{\rm MB} = a^i_{\rm M} - a^i_{\rm B} \; ,$$

where the symbol ${\rm STF}_{ij}$ denotes the symmetric and trace-free part of the corresponding tensor quantity (see e.g. Damour et al. 1991). In the first term one might recognize 'the Newtonian torque'. However, one subtle point should be reminded here. Mass (BD) multipoles $M^{\rm A}_L$ are autonomous quantities defined in the PN level and have nothing to do with analogous Newtonian quantities‡ (e.g. zeroth multipole $M_{\rm A}$ representing the mass of the body is trivially constant in the Newtonian approach while variable on the PN level as soon as mass quadrupoles are included; however, in what follows we shall neglect such variations, as they can be shown to be extremely small for the solar system bodies). It should be understood that, conceptually, relativity does not provide *corrections* to the Newtonian solution. Rather one can *identify* some terms, after developing a consistent PN approach, as *Newtonian-like*. This is exactly the case of the similarity of the first term in Eq. (7) with Newtonian torque. Clear appearance of this point is also one important achievement of the DSX theory (more discussion about this point might be found in Damour et al. 1993).

‡ From this point of view all harmonic coefficients corresponding to the multipoles of the Earth-Moon system should be theoretically redefined when considering higher relativistic terms. In practice, due to the smallness of the studied effects, we have accepted values of the Earth-Moon gravity field coefficients as given by previous studies.

Formally, the remaining terms in the right hand sides of Eq. (7) represent the 'relativistic' contribution. The aim of this paper is to investigate numerically their possible importance.

## 3. Numerical results

We have constructed a model of complete lunar motion including relativistic effects discussed previously (more details about our model will be published elsewhere). Our goal is to include all phenomena up to the precision level resulting from the LLR technology, and if possible even better for reasons of consistency (e.g. at least 1 cm for the distance, 1 *mas* for the librations). In particular, several phenomena capable of producing effects of at least 0.1 *mas* in the lunar physical librations have been included and analysed. The effects resulting from planetary actions (essentially Venus), the Earth-Moon figure-figure interactions, and non-rigidities of the Moon and the Earth have been presented in previous papers (Bois et al. 1992; Bois & Journet 1993). In the subsequent sections, we present the results of the relativistic contributions related to the rotational motion of the Moon, as well as the relativistic precession of the Earth reference frame.

The relativistic part of the model has been introduced as a pole-dipole-quadrupole in the previous discussion. However, such truncation would not be sufficient to meet the prescribed precision as mentioned earlier. One has to include higher order mass multipoles (up to $l = 5$ in the Moon case while $l = 4$ in the Earth case) of the bodies. Corresponding 'Newtonian-like' terms do appear in the systematic construction of the PN theory of bodies endowed with higher order multipoles as can be verified using a general scheme in the Appendix of Damour et al. (1992). However, we neglected any terms of purely relativistic origin associated with multipoles higher than second order. Similarly, we took into account several terms arising from the tidal deformations of the bodies (both elastic and anelastic). Again, one can adopt a very broad model (including such tidal effects on the relativistic base; see Damour et al. 1991), but subsequently retain only necessary 'Newtonian-like' terms corresponding to the required precision.

Summarizing previous ideas, we can state that our model is not Newtonian but rather 'Newtonian-like', resulting from truncation of the fully post-Newtonian (DSX) framework. We shall focus on reporting the results related to the relativistic effects mentioned in Sect. 2.

**Table 1.** De Sitter precession – results and comparison with previous works. The first row relates to the principal secular term, the other two to the periodic terms of year and half-year variations.

| term | Fukushima | Bizouard et al. | our results | units |
|---|---|---|---|---|
| secular | 1.919 | 1.914 | 1.9193 | *as/cy* |
| periodic *(1 year)* | 0.153 | 0.1525 | 0.1530 | *mas* |
| periodic ($\frac{1}{2}$ *year*) | $2 \times 10^{-3}$ | $1.9 \times 10^{-3}$ | $1.89 \times 10^{-3}$ | *mas* |

### *3.1. De Sitter precession of the Earth reference frame*

We shall parametrize the vector $\omega_i^{\rm E}$ [see Eq. (5)] corresponding to the de Sitter rotation of the Earth reference frame by three Eulerian angles $(\psi, \theta, \phi)$ referred to the ecliptic barycentric frame ($\psi$ being the precession angle as in usual astronomical practice). As the phenomenon demonstrates precession, we shall focus our attention on the $\psi$ angle. Investigating its behaviour by a Fourier filter, we arrived at the results summarized in table 1 [for comparison we include also results of Fukushima (1991) and Bizouard et al. (1992)].

Comparison of our results with those obtained by Fukushima (1991) and Bizouard et al. (1992) shows a very good agreement, or at least a very close one. It can be seen, that the half-year term amplitude would in reality be affected by higher than monopole contributions in the external potentials $(\bar{w}, \bar{w}_i)$ in Eq. (6). However, its practical importance is small because such a term is below the precision threshold of current astrometric methods (e.g. Kovalevsky 1990; Jacobs et al. 1993).

It should be noticed that the lunar-reference frame undergoes a similar de Sitter precession to the Earth one. An alternative way of representing the two effects is to introduce the de Sitter precession of the common Earth-Moon barycentric reference frame (e.g. Dickey et al. 1989), as it can be easily verified that the principal effects in table 1 originate in the solar action. However, due to the mutual Earth-Moon action, the de Sitter precession of the two reference frames differs slightly. Detailed inspection shows that the lunar reference frame undergoes an additional precession of the order of $30\,mas/cy$, so far not a measurable value. Moreover, bringing out this secular effect is a complicated task due to the other (phenomenologically modeled) secular precessional effects related to the selenophysical phenomena.

Finally, it is to be pointed that we discuss de Sitter (geodetic) precession in 'the narrow sense' following the terminology of Brumberg et al. [1991]. We thus involve in our calculation only first two terms in the fundamental formula (6). Discrepancy between our value and $1.9198\,as/cy$ given by Brumberg et al. [1991] is due to the fact that the latter value includes also contribution from the Lense-Thirring term due to coupling of the solar gravity field with the Earth-Moon angular momentum.

### *3.2. Relativistic terms in the lunar rotational motion*

In what follows, we shall report on the influence of 'the relativistic' terms in (7) on lunar rotational motion. It should be pointed out, that as far as the authors know such terms have not yet been applied in lunar theory. The main reason is that the theoretical description of the relativistic contributions in the rotational motion of bodies was unsatisfactory (see Damour et al. 1991).

Figure 1 shows the behaviour of resulting librations due to the relativistic contributions in Eq. (7). The three Eulerian angles $(\psi, \theta, \phi)$ of the classical 3-1-3 angular sequence represent the physical librations of the Moon's rotational motion with respect to a reference system given by a terrestrial equatorial frame (J2000). As the data drawn in Fig. 1 have been obtained as a dif-

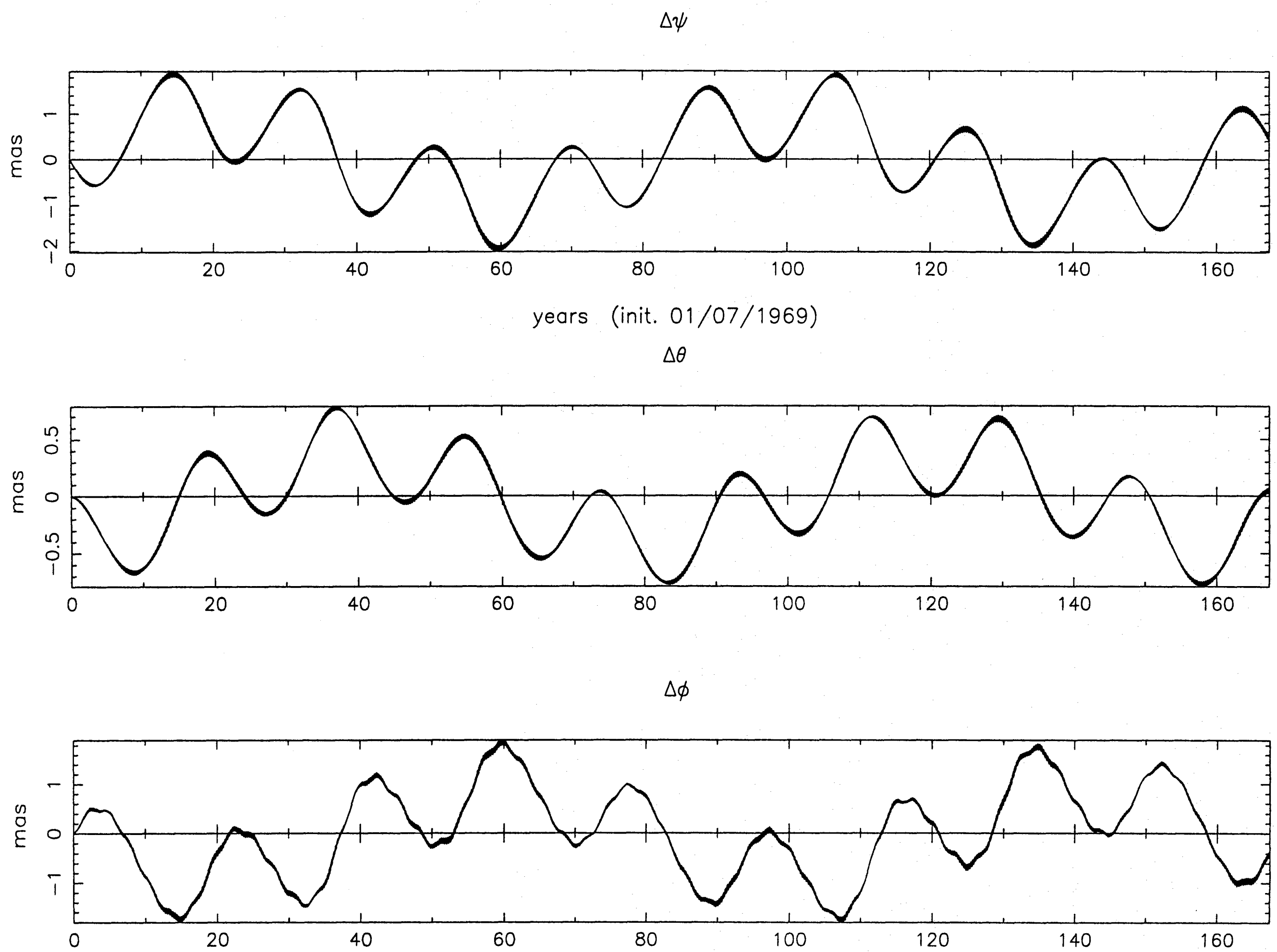


**Fig. 1a–c.** Relativistic lunar librations as they formally appear in the series of the Euler angles characterizing lunar rotation – differences $(\Delta\psi, \Delta\theta, \Delta\phi)$ with respect to the solution free of the relativistic torque (see also Fig. 2). Milliarseconds are on the vertical axis and years on the horizontal axis. Dominant terms of 18.6 and 80.1 year periodicities are in all Euler angles. The angle of lunar proper rotation contains also a component related to the ordinary resonant frequency of 2.9 years for physical librations in longitude

ference of the two numerical integrations of the lunar rotation, first disregarding the relativistic terms, secondly accounting for them, we call them $(\Delta\psi, \Delta\theta, \Delta\phi)$. The global frame time $t$ on the $x$ axis is the solar system barycentric time TCB. This choice is clearly governed by the fact that the solar system barycentric time belongs to the set of the two most important coordinate time scales defined in the solar system, contrary to the lunar (local) coordinate time (e.g. Soffel & Brumberg 1991; Seidelmann & Fukushima 1992). We have thus combined Eqs. (1) and (2) together with (7) in order to obtain the final form of the Euler dynamical equations for lunar librations. Let us also note, that the initial date of the integration presented in Fig. 1 coincides with that of the JPL DE303 ephemeris. In order to justify consistence of our theory, we have adjusted it to the JPL ephemeris on the first 1.5 years up to a level of a few centimeter residuals.

Let us recall the major changes in complete libration angles during the considered time span: the lunar nutation angle $\theta$ changes between 22 and 25 degrees with the major periodicity of 18.6 years being driven by the nodal period of the lunar orbit; the proper rotation of the Moon around its axis of figure is effected by the complete angle $\phi$ with the mean period of 27.3 days; the precession angle $\psi$ is also locked in resonance with the lunar node period of 18.6 years. Further forced behaviour due to the indirect action of the planets and higher harmonics of the Moon can be found in e.g. Eckhardt (1982) or Moons (1984).

We remark long-term components with periods of 18.6 years (nodal rate of the lunar orbit), 80.1 years and 2.9 years (in $\phi$) which are the two resonant frequencies of the spin-orbit motion of the Moon (Kopal 1969; Yoder 1981). Appearance of these terms is not surprising in view of the form of the relativistic torque (7). The magnitude of the relativistic librations is of the order of one milliarcsecond in all Euler angles. The ratio of this magnitude with respect to the corresponding one coming from analysis of the Newtonian-like pole-quadrupole torque [i.e. the first term in (7)] is about $5 \times 10^{-8}$. Such a value corresponds to the order of the small (PN-) relativistic parameters $(v_A/c)^2$ and $\bar{w}^B/c^2$ both of them being of the order of $10^{-8}$. The latter factor also clearly determines the amplitude of the ratio of the relativistic and Newtonian torques associated with pole-quadrupole interaction [see Eq. (7)].

The non-linear features of the differential equations, the correlation degree of the studied effect with respect to its neighbours (in Fourier space) and the spin-orbit resonance, in the lunar case, make it hardly possible to speak about 'pure' effects with their proper behaviour (even after fitting of the initial conditions). The differentiation method may give the right qualitative behaviour of an effect and a good quantification of this effect relative to its neighbours (see Table 2). The effects are not absolutely de-correlated but relatively isolated. It is also possible to make some comparisons with analogous problems (here for instance, with the analogous torques acting on the Earth's rotation). When a rotational effect is simply periodic, a fit of the initial conditions for a set of given parameters only refines without changing completely the effect's behaviour. Nevertheless, the particular status of the relativistic effects may lead us to a precise quantification and will be discussed in a forthcoming paper (see also discussion in the next section).

In order to get an idea about the importance of the relativistic contribution, Table 2 presents a comparison of various effects acting on the Moon's spin motion, obtained by Bois et al. (1992), Bois & Journet (1993), and in the present work. The amplitudes correspond to the maxima reached by one of the three Eulerian angles $(\Delta\psi, \Delta\theta, \Delta\phi)$ in each case. Results are given respectively for two periods of librations when they exist, namely 2.9 years and a period running around 18.6 to 24 years. The acronym $ExMy$ signifies orders of multipoles respectively in interaction, namely the order $x$ of the Earth acting on the order $y$ of the Moon. $V$ means Venus; $LTL$ means lunar tidal librations; $RCL$ refers to the relativistic contributions of (7) represented by the librations given in Fig. 1. We note that the amplitude of the relativistic librations is comparable with the quasi-Newtonian interaction of the Earth quadrupole and the Moon sextupole or the influence of the Venus pole.

**Table 2.** Comparison of various effects on the Moon's librations. See the text for terminology.

| effect | 2.9 years | 18.6/24 years | units |
|---|---|---|---|
| V0M2 | | $\leq 3$ | *mas* |
| E0M5 | 10 | $\leq 20$ | *mas* |
| E2M2 | | $\leq 80$ | *mas* |
| E2M3 | 0.5 | $\leq 1$ | *mas* |
| LTL | 1 | $\leq 4$ | *mas* |
| RCL | 0.1 | $\leq 2$ | *mas* |

## 4. Discussion and conclusion

The main results of this communication can be summarized as follows:

i) numerical simulation of the de Sitter precession of the Earth reference frame yields results very close to those of Fukushima (1991) or Bizouard et al. (1992);
ii) preliminary tests of importance of the relativistic torque in the case of the Moon's rotational motion suggest that this effect ought to be included in the lunar models.

Let us point out that the first result (i) is to be understood more as an inner test of our model (program), because this phenomenon has already been studied both theoretically (e.g. Fukushima 1991; Bizouard et al. 1992) and even when interpreting the LLR data (e.g. Bertotti et al. 1987; Shapiro et al. 1988; Dickey et al. 1989; Müller et al. 1991). On the contrary, the second result (ii) does not seem to have been studied yet. Bizouard et al. (1992) recently investigated the role of analogous terms in the case of the Earth's rotation. They concluded that the magnitude of this effect was negligible. Interestingly, in the lunar case the resulting contribution does not seem to be totally negligible. Similar results were reported for the influence of the higher order multipole coupling on the 'Newtonian level' (dominated by the $J_2 - J_2$ interaction). Souchay (1994; private communication) has studied such effects for the Earth's rotation, while Bois et al. (1992) considered the lunar case. The resulting behaviour is mainly explained by the ratio of the resulting torque intensity to the angular velocity via the degree of asymmetry of the body. This ratio is larger for the Moon than for the Earth.

It should be emphasized that results presented in Sect. 3.2 (notably, direct differences of the Euler angles) place only *the upper limit* for the magnitude of the discussed relativistic lunar librations with respect to their observability. Three reasons account for this: (i) a subclass of infinitesimal rotations is not suitably represented by the 3-1-3 Eulerian parametrization as discussed in the next paragraph; (ii) initial conditions of the integration have not been adjusted; (iii) interpretation of the relativistic effects requires special care due to possible coordinate dependence of the effects. We comment on these problems in the following paragraphs.

Differences of the Euler angles of the two configurations reported in Sect. 3.2 (Fig. 1) show how the relativistic librations formally appear in the lunar rotation. However, we must face the question of their possible observability through the LLR facility, the problem being qualitatively more difficult. The first step to reach the observability task would require reparametrization of the kinematic description of a rotational motion. Instead of the difference of the two series of Euler angles [accounting and not-accounting for the influence of the relativistic torque (1)], we shall actually construct the rotation of the reference frame defined by the lunar rotation solution, without the relativistic term to the reference frame given by the lunar rotation including the relativistic contribution. It appears that a suitable representation of the rotation we are looking for is obtained by specifying the instantaneous (virtual) rotation axis and corresponding angle $\delta$ of rotation around the axis (see Fig. 2 for this concept). The position of the virtual axis is actually not important for our order of magnitude argument, as we are basically seeking the amplitude of the additional lunar frame rotation imposed by the appearance of the relativistic term in the right hand sides of the Euler Eq. (1). Using elements of the rotation group algebra (for information on the mathematical techniques of rigid body rotation

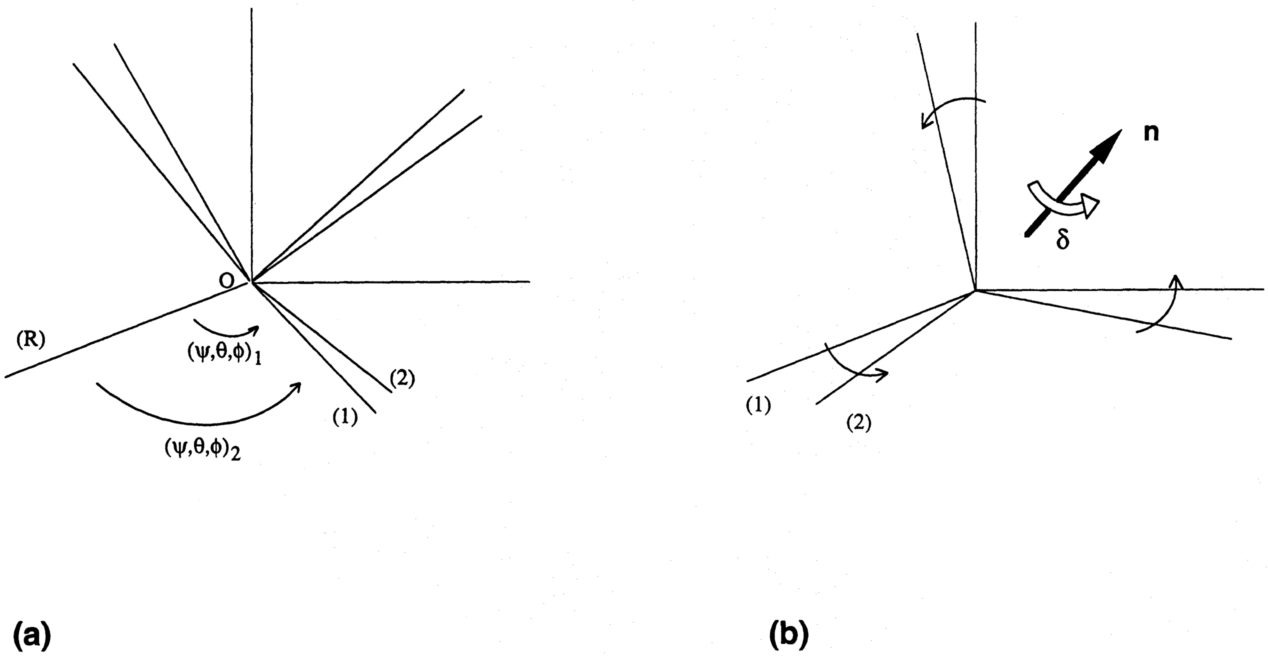


**Fig. 2.** The concepts of representation of the difference of the two rotated frames (1 and 2) with respect to the reference one (R): **a** simple difference of the Euler angles relating the two moving frames with respect to the reference one, **b** $(\delta, \mathbf{n})$ parametrization of the infinitesimal rotation of the frame 2 with respect to the frame 1. In our application, frame 1 is represented by the lunar rotation not accounting for the effect of the relativistic lunar librations, while the rotation of the frame 2 contains contribution due to the relativistic librations. See the text for more details

see e.g. Mignard 1989), one arrives at the following formulas

$$\cos\delta = 2q_0^2 - 1 \ , \tag{8}$$

$$q_0 = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos(\psi_1 - \psi_2 + \phi_1 - \phi_2) + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos(\psi_1 - \psi_2 - \phi_1 + \phi_2) \ ,$$

where $(\psi_1, \theta_1, \phi_1)$ and $(\psi_2, \theta_2, \phi_2)$ are two sets of Eulerian angles related to the two reference frames under consideration. In the case of small differences in precession angles $\Delta\psi = \psi_2 - \psi_1$ and proper rotation angles $\Delta\phi = \phi_2 - \phi_1$ (which is the case of curves in Fig. 1), one can simplify the previous formula to the form

$$\delta^2 \approx \Delta\theta^2 + (\Delta\psi + \Delta\phi)^2 + 2\left(\cos\bar{\theta} - 1\right)\Delta\psi\Delta\phi \ , \tag{9}$$

where $\Delta\theta = \theta_2 - \theta_1$ and $\bar{\theta}$ is the averaged nutation angle of the two discussed configurations. Components of the unit vector $\mathbf{n} = (n_1, n_2, n_3)$ of the rotation axis direction then read

$$n_1 = \left(\Delta\theta\cos\bar{\phi} + \Delta\psi\sin\bar{\theta}\sin\bar{\phi}\right)/\delta \ , \tag{10a}$$

$$n_2 = \left(\Delta\theta\sin\bar{\phi} - \Delta\psi\sin\bar{\theta}\cos\bar{\phi}\right)/\delta \ , \tag{10b}$$

$$n_3 = \left(\Delta\psi\cos\bar{\theta} + \Delta\phi\right)/\delta \ , \tag{10c}$$

where similar quantities have been introduced for the remaining Euler angles: $\Delta\psi = \psi_2 - \psi_1$, $\Delta\phi = \phi_2 - \phi_1$ and $(\bar{\psi}, \bar{\phi})$ for the averaged quantities. One easily recognizes the vector character of composition of the infinitesimal rotations. We have passed data shown in Fig. 1 through this test computing $\delta$-angle and corresponding rotation axis motion.

Figure 3 shows the time dependence of the $\delta$-rotational angle. The resulting amplitude decreased to about 0.8 *mas*, as can be expected from the series of the $\Delta\psi$ and $\Delta\phi$ angles. Notice that there essentially appears only the sum of the two quantities in Eq. (9) corresponding to the fact that the infinitesimal rotation of the reference frame can be expressed by arbitrary precession and proper rotation, provided that these two elementary rotations are nearly opposite (this is a well understood inconvenience of the 3-1-3 Euler sequence, Mignard 1989). One remarks a pure 18.6 year periodicity of this quantity forced by the nodal motion of the Moon. It is to be reminded here that the residual value of about one milliarcsecond is on the level of the observability through LLR if the best quality nights are selected (moreover a nearly half order improvement is planned for the forthcoming years; Ch. Veillet, private communication). The motion of the axis $\mathbf{n}$, showing precession in the lunar equatorial plane, can be decomposed into terms with periods of 2.9, 18.6 and 80.1 years. The third component of $\mathbf{n}$ shows oscillations with a period of 2.9 years.

Regarding the well defined frequencies and the stable character of the signal due to relativistic librations, we hope that our results represent well the true behaviour of the phenomena. However, we plan to investigate in some detail the effect of the initial condition adjustment on the results referred to here. In any case we can conclude, that the relativistic lunar librations introduced in this study are among a class of phenomena influencing lunar rotation, which 'are standing just behind the door of observability in the LLR data' and thus would require greater care in the near future.

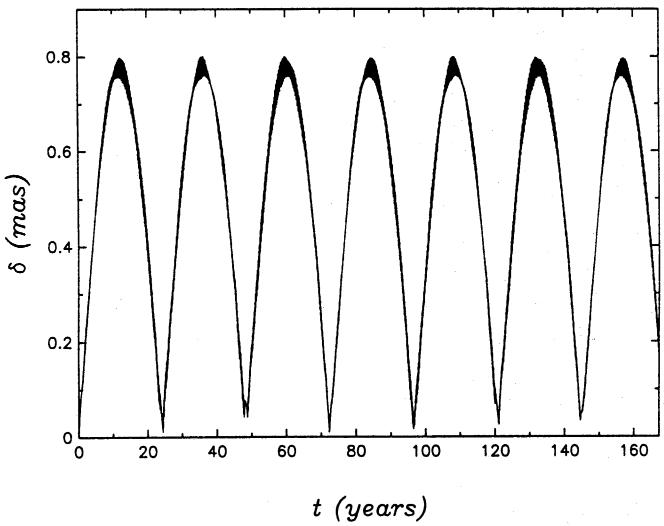


**Fig. 3.** Rotational angle $\delta$ (in *mas*) vs. time in years. The signal essentially decomposes into the 18.6 years long-term component and the short-term 27.3 day component

Concerning the third warning mentioned above [item (iii)], we recall that the final proof of the observability of the relativistic librations is based upon their influence on the two-way laser beam timing, expressed in the proper time of the observer (this being a coordinate independent quantity). Full analysis of such problems lies out of the scope of this paper.

We hope to devote a forthcoming study to the two topics mentioned in the two preceding paragraphs [also items (ii) and (iii) mentioned previously].

Let us finally note that another interesting topic to be tackled in the future concerns the generalization of the DSX formula (7) to the parametrized framework of the PN theory of gravitation (Dickey et al. 1989; Müller et al. 1991)¶.

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## Appendix

In this appendix, we shall comment on our rigid-body Ansatz assumed for the relativistic part of the lunar rotation. As noted in Sect. 2, conceptually we do not dispose of the equations for the quadrupoles $M^{\rm A}_{ab}$ unless a physical concept of the bodies is settled. The most restrictive is rigidity, leaving no free parameter in the description of the body mass multipoles. Necessarily a set of three parameters of the rotation group should be introduced (it is thus rather a geometric concept, being extreme in its physical content). However, in the sense of a formal decomposition, the rigidity notion expresses successfully the behaviour of the mass multipoles of the solar system bodies. It is then conventional to describe more complicated physical body-phenomena (e.g. tides, rotational deformation etc.) as additional terms to those due to the rigid concept (Munk & MacDonald 1975).

As the relativistic contribution is small due to the presence of the first-order relativistic parameters (squares of the orbital velocities divided by the light speed, Schwarzschild radii divided by the orbital distances), it is advisable to couple it with the rigid concept quadrupoles [see the first term in Eq. (1)]. In this case, the mass quadrupoles $M^{\rm A}_{ab}$ satisfy equations (see e.g. Dixon 1979)

$$\frac{dM^{\rm A}_{ab}}{dT_{\rm A}} = -{\rm STF}_{ab}\left\{\Omega^{\star}_{{\rm A}ca} M^{\rm A}_{bc}\right\} \;, \qquad (A1)$$

where $\Omega^{\star}_{{\rm A}ab} \equiv \epsilon_{abc}\Omega^{\rm A}_{c}$ is a dual antisymmetric tensor to the vector of the angular velocity of the body A. Though theoretically elegant, this approach is not suitable for astronomical calculations. Luckily the solution of Eq. (A1) is known explicitly and allows us to express quadrupole components directly in terms of the parameters of the proper rotation of the bodies. Moreover, because it is conventional in astronomical and geodynamical research to use spherical harmonics analysis of the gravitational fields with the corresponding notion of harmonic coefficients $(C^{\rm A}_{lm}, S^{\rm A}_{lm})$, we seek expressions of the $M^{\rm A}_{ab}$ in those terms. If the system co-rotating with a particular body is defined by the principal axis of the moment of inertia, then all quadrupole terms vanish with the exception of two: $J^{\rm A}_2 \equiv -C^{\rm A}_{20}$ and $C^{\rm A}_{22}$. Next, assuming that the rotation of body A is parametrized by the set of Euler angles $(\psi_{\rm A}, \theta_{\rm A}, \phi_{\rm A})$, one gets after expansion the following expressions of the mass quadrupoles [to be inserted in Eq. (1)]

$$M^{\rm A}_{ab}(T_{\rm A}) = J^{\rm A}_2 M_{\rm A} a^2_{\rm A} \mathbf{A}_{ab}(\theta_{\rm A}, \psi_{\rm A}) \;, \qquad (A2)$$

where

$$\mathbf{A}_{11} = \frac{1}{\sqrt{2}} \sin^2\theta_{\rm A} \cos 2\psi_{\rm A} + \frac{1}{3}\left(1 - \frac{3}{2}\sin^2\theta_{\rm A}\right) \;,$$

$$\mathbf{A}_{22} = -\frac{1}{\sqrt{2}} \sin^2\theta_{\rm A} \cos 2\psi_{\rm A} + \frac{1}{3}\left(1 - \frac{3}{2}\sin^2\theta_{\rm A}\right) \;,$$

$$\mathbf{A}_{33} = -\frac{2}{3}\left(1 - \frac{3}{2}\sin^2\theta_{\rm A}\right) \;,$$

$$\mathbf{A}_{12} = \frac{1}{\sqrt{2}} \sin^2\theta_{\rm A} \sin 2\psi_{\rm A} \;,$$

$$\mathbf{A}_{13} = -\frac{1}{2} \sin 2\theta_{\rm A} \sin\psi_{\rm A} \;,$$

$$\mathbf{A}_{23} = \frac{1}{2} \sin 2\theta_{\rm A} \sin\psi_{\rm A} \;,$$

for those related to the principal quadrupole terms $J^{\rm A}_2$, and

$$M^{\rm A}_{ab}(T_A) = C^{\rm A}_{22} M_{\rm A} a^2_{\rm A} \mathbf{B}_{ab}(\psi_{\rm A}, \theta_{\rm A}, \phi_{\rm A}) \;, \qquad (A3)$$

¶ Methodological importance of embodying of the Einsteinian general relativity into a wider field of parametrized PN theories for purposes of testing its validity is stressed e.g. by Damour (1992).

where

$$\mathbf{B}_{11} = \left(1+\cos^2\theta_A\right)\cos\psi_A\cos 2\phi_A + 2\cos\theta_A\sin\psi_A\sin 2\phi_A + 6\sin^2\theta_A\cos 2\phi_A\ ,$$

$$\mathbf{B}_{22} = -\left(1+\cos^2\theta_A\right)\cos\psi_A\cos 2\phi_A - 2\cos\theta_A\sin\psi_A\sin 2\phi_A + 6\sin^2\theta_A`\cos 2\phi_A\ ,$$

$$\mathbf{B}_{33} = -12\sin^2\theta_A\cos 2\phi_A\ ,$$

$$\mathbf{B}_{12} = \left(1+\cos^2\theta_A\right)\sin\psi_A\cos 2\phi_A + 2\cos\theta_A\cos\psi_A\sin 2\phi_A\ ,$$

$$\mathbf{B}_{13} = 2\sin\theta_A\left(\sin\psi_A\sin 2\phi_A + \sin\psi_A\cos\theta_A\cos 2\phi_A\right)\ ,$$

$$\mathbf{B}_{23} = 2\sin\theta_A\left(\sin\psi_A\sin 2\phi_A - \sin\psi_A\cos\theta_A\cos 2\phi_A\right)\ ,$$

for the part associated with the $C_{22}^A$ coefficient. Parameter $a_A$ denotes a characteristic dimension of body A related to the definition of the quadrupole harmonic coefficients (it finally drops out from the definition of quadrupole tensor $M_{ab}^A$). Notice that the $J_2$-associated part does not depend on the angle $\phi_A$ of the proper rotation due to axial symmetry.

Finally, let us recall the definition of the symmetric and trace-free part of a second-rank tensor $t_{ij}$ [Eq. (1)]

$$\mathrm{STF}_{ij}\left[t_{ij}\right] = \frac{1}{2}\left(t_{ij}+t_{ji}\right) - \frac{1}{3}\delta_{ij}\mathrm{Tr}\left(\mathbf{t}\right)\ , \qquad (A4)$$

where $\delta_{ij}$ is the Kronecker symbol and $\mathrm{Tr}(\mathbf{t})$ stands for the tensor trace. In particular, we have

$$\mathrm{STF}_{ij}\left[n_{AB}^{ij}\right] = \mathrm{STF}_{ij}\left[n_{AB}^{i}n_{AB}^{j}\right] = n_{AB}^{ij} - \frac{1}{3}\delta_{ij}\ .$$

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