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Research Note

Motion of charged particles around a black hole in a magnetic field

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Abstract. The motion of electrically charged test particles in the vicinity of a nonrotating, uncharged black hole in the magnetic field is studied. Regions of orbits stable with respect to small perturbations in the equatorial plane are found within the framework of an exact solution of the Einstein-Maxwell equations. The relativistic concept of the Rayleigh stability criterion is discussed.

Key words: black holes – relativity – gravitation – magnetic field

1. Introduction

Black holes surrounded by magnetic fields are believed to play the crucial role in the theory of active galactic nuclei. Relativistic effects in the vicinity of the event horizon are often studied in the Schwarzschild or Kerr metric, but the influence of magnetic fields on the spacetime curvature may also be important.

Before astrophysically relevant models with a viscous disc can be studied, we need to understand the motion of an individual test particle. Trajectories of electrically neutral particles around a magnetized nonrotating black hole were treated by Karas & Vokrouhlický (1990). This paper deals with the equatorial circular motion of charged particles. A stability criterion derived from the effective potential is compared with the relativistic generalization of the Rayleigh criterion, which was established by Abramowicz & Prasanna (1990). Compared with the Schwarzschild case, the spacetime of the magnetized black hole offers a nontrivial example of the Rayleigh criterion reversals, because the Lorentz force acting on charged particles is present. In addition, due to the magnetic field, two photon orbits exist in the equatorial plane, where the centrifugal force action reverses.

2. Equatorial orbits of charged particles

The spacetime element describing a nonrotating (Schwarzschild) black hole immersed in a magnetic field was given by Ernst (1975). The metric can be rewritten in the form

$$g = \Phi\,(\tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j - \mathrm{d}t^2)\,, \tag{1}$$

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where $\Phi = \Lambda^2 \Delta r^{-2}$, $\Lambda = 1 + \frac{1}{4}(B_0 r \sin\theta)^2$ and $\Delta = r^2 - 2Mr$; $\tilde{g}_{ij}$ describes the optical reference geometry (see Abramowicz et al. 1988)

$$\tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \Delta^{-2} r^4\,\mathrm{d}r^2 + \Delta^{-1} r^4\,(\mathrm{d}\theta^2 + \Lambda^{-4}\sin^2\theta\,\mathrm{d}\varphi^2)\,. \tag{2}$$

We use geometrical units $c = G = 1$; for the black hole mass parameter we have $M\,[\mathrm{cm}] \doteq 1.5\;10^5\,(M/M_\odot)$ and for the magnetic field parameter $B_0\,[\mathrm{cm}^{-1}] \doteq 2.9\;10^{-25}\,B_0\,[\mathrm{gauss}]$. The magnetic field is parallel to the polar axis $\theta = 0$, π and defines the equatorial plane of the hole, $\theta = \pi/2$.

Two constants of motion, the energy $\mathscr{E}$ and the angular momentum $\mathscr{L}$, can be found. In the case of the equatorial circular motion of a particle with the mass μ and electric charge q, we obtain

$$\mathscr{E}^2 = \mu^2 \Lambda^2 \Delta r^{-2}\,[1 + \tfrac{1}{4} r^2 (\pm\sigma - \alpha)^2 \phi^{-2}]\,, \tag{3}$$

$$\mathscr{L}^2 = \tfrac{1}{4}\mu^2 r^4\,[(\pm\sigma - \alpha)\,\phi^{-1}\Lambda^{-1} + \alpha]^2\,, \tag{4}$$

where

$$\sigma = [\alpha^2 + 4M\phi r^{-1}\Delta^{-1}\,(1 + 2\delta\Delta M^{-1} r^{-1})]^{1/2}\,,$$
$$\phi = 4\Lambda^{-1} - (3r - 5M)/(r - 2M)\,,$$
$$\delta = 1 - \Lambda^{-1}, \quad \alpha = q B_0 \mu^{-1} \Lambda^{-1}\,.$$

In (3)–(4), the negative and positive signs correspond to two different classes of motion. A change of sign of the charge is equivalent to the transition to the other branch with $\mathscr{L} \to -\mathscr{L}$. (For definiteness, we assume $q > 0$ in the following.) Maximum and minimum radii of these trajectories, $r_\pm$, are given by $\sigma = 0$. By analogy, radii r_1, r_2 of the photon circular orbits are given by $\phi = 0$. For $r_1 < r < r_2$ the two branches correspond to the Larmor and anti-Larmor trajectories. Outside this region, only the anti-Larmor motion with the Lorentz force counteracting the gravity is possible (Fig. 1). [Further details can be found in Karas & Vokrouhlický (1990). In (3) – (4), an error in expression (4) of this reference is corrected.]

The radii R of the marginally stable orbits are given by

$$\frac{\partial^2}{\partial r^2} U_{\mathrm{eff}}^2 = 0\,, \quad U_{\mathrm{eff}}^2 = \mu^2 \Delta \Lambda^2 r^{-2}\,[1 + (\mathscr{L} + \alpha r^2/2)^2 \Lambda^2 r^{-2}]\,. \tag{5}$$

For $B_0 = 0$ we obtain $r_- = r_1 = 3M$; r_+, $r_2 \to \infty$; $R = 6M$, as expected. The more complicated case $B_0 \neq 0$ is shown in Fig. 2. Now, in general, radii of the marginally stable orbits are different

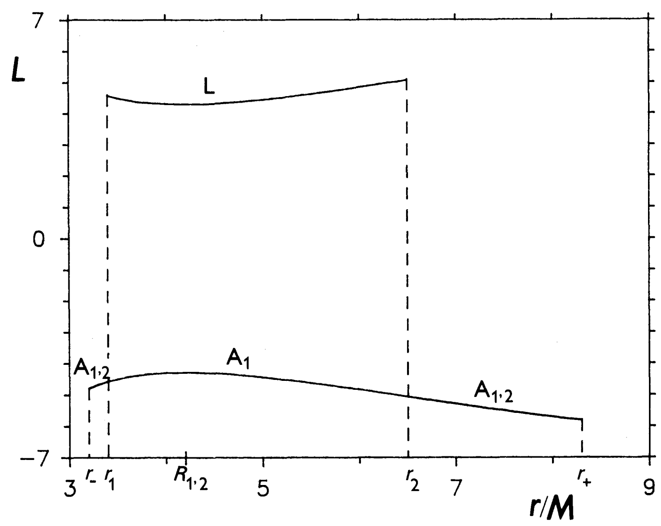

Fig. 1. The constant of motion $L(r) \equiv \mathscr{L}/\mathscr{E}$ on the circular orbits for $B_0 = 0.15\,M^{-1}$, $q/\mu = 1$. The curve A_1 designates the first anti-Larmor branch extending continuously from r_- to r_+. The second branch is discontinuous on the photon orbits and corresponds to the Larmor motion for $r_1 < r < r_2$ (L) and anti-Larmor motion for $r < r_1$ or $r > r_2(A_2$; this curve actually coincides with a part of A_1, though $\mathscr{L}$ and $\mathscr{E}$ are different on the two curves). Notice that for this particular choice of the specific charge $q/\mu = 1$, the marginally stable orbits R_1, R_2 coincide, as described in text

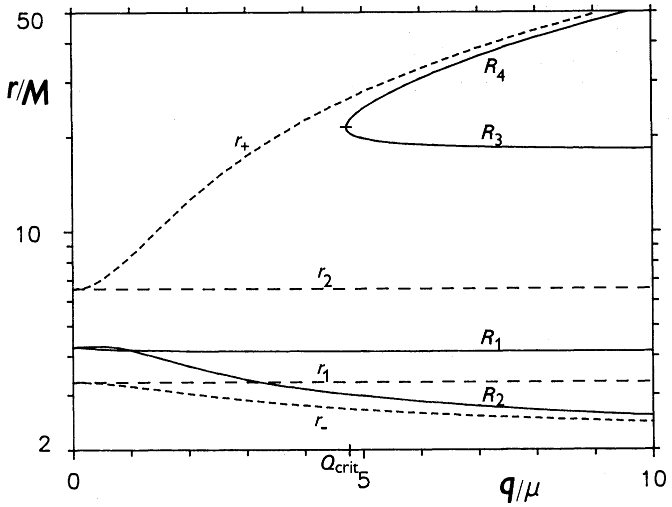

Fig. 2. The regions of stable and unstable orbits for $B_0 = 0.15\,M^{-1}$. The lowest anti-Larmor trajectory, r_-, approaches $2M$ for $q/\mu \to \infty$, while r_+ goes to infinity

for the two branches. The Larmor trajectories [minus sign in (3)–(4)] have only one marginally stable orbit with radius $r = R_1$, above which the motion is stable with respect to small perturbations in the equatorial plane with constant angular momentum (it can be shown that the condition of constant $\mathscr{L}$ is equivalent to constant energy $\mathscr{E}$). The anti-Larmor trajectories have the only marginally stable orbit ($r = R_2$) for specific charge q/μ less then some critical value Q_{crit}. If $q/\mu > Q_{\text{crit}}$ two new marginally stable orbits, R_3 and R_4, occur; in this case stable orbits are bound in the regions $R_2 < r < R_3$ and $R_4 < r < r_+$. For $q/\mu < 1$ we have $R_1 < R_2$, otherwise $R_1 > R_2$. In the case $q/\mu = 1$, which is shown in Fig. 1, the radii R_1 and R_2 coincide.

It can be readily verified that the equatorial trajectories are stable with respect to small perturbations in the θ-direction. To elucidate the meaning of marginally stable orbits in a different way, we look, in Sect. 3, for places where the Rayleigh stability criterion is reversed.

3. Rayleigh criterion reversal

The relativistic centrifugal force acting on a particle can be defined by the equation (Abramowicz & Prasanna 1990; Abramowicz 1990)

$$C_i = -\mu \tilde{v}^2 \tilde{\tau}^j \tilde{V}_j \tilde{\tau}_i\,, \tag{6}$$

where $\tilde{\tau}_i$ denotes the unit three-vector tangent to the particle's trajectory in the optical reference geometry (2) and $\tilde{v}$ the speed along the trajectory. On a circular equatorial orbit, only the radial component C_r is nonzero. The value of the centrifugal force is then

$$C = \frac{\mu \ell^2}{r^3} \frac{\Delta^2 \Lambda^6}{r^4 - \ell^2 \Delta \Lambda^4}\,[1 - 3M/r - B_0^2\,\Delta/\Lambda]\,, \tag{7}$$

where $\ell = (\mathscr{L} + \alpha r^2/2)/\mathscr{E}$. Charged particles moving on the continuous anti-Larmor branch can orbit the hole on both photon orbits. Evaluating the centrifugal force (7), one can verify that the force vanishes here, in accordance with the results of Abramowicz & Prasanna (1990). On the other hand, Larmor motion along the photon orbits is not possible (the speed of a particle with respect to local static observers reaches the speed of light when $r \to r_{1,2}$).

The equilibrium condition on a circular equatorial orbit is given by the balance of forces

$$C(r, \ell^2) + \mathscr{G}(r) + \mathscr{T}(r, \mathscr{L}) = 0\,, \tag{8}$$

where $\mathscr{G}(r)$ denotes the gravitational force

$$\mathscr{G}(r) = -\tfrac{1}{2}\mu \Phi \Lambda^{-2} \tilde{V}_r \Phi = -\mu M \Lambda^2 \Delta r^{-4}\,(1 + 2\delta \Delta M^{-1} r^{-1}) \tag{9}$$

and $\mathscr{T}(r, \mathscr{L})$ is the Lorentz force acting on a particle with electric charge q. In terms of the electromagnetic field tensor $F_{\mu\nu}$ and particle's four-velocity U_μ, we have

$$\mathscr{T}(r, \mathscr{L}) = q \Phi^2 \Lambda^{-2} F_{r\varphi} U^\varphi = q \Lambda^2 B_0 \Delta^2 r^{-5}\,(\mathscr{L} + \alpha r^2/2)\,. \tag{10}$$

We studied the generalized Rayleigh criterion for local stability with respect to small, axisymmetric, quasi-stationary perturbations in the equatorial plane:

$$\left[\frac{\partial C}{\partial \ell^2}\frac{\mathrm{d}\ell^2}{\mathrm{d}r} + \frac{\partial \mathscr{T}}{\partial \mathscr{L}}\frac{\mathrm{d}\mathscr{L}}{\mathrm{d}r}\right] > 0\,. \tag{11}$$

It is well known that in a differentially rotating medium with a distribution of angular momentum increasing outwards, matter is stable with respect to small perturbations preserving the angular momentum. But in our model, the criterion is more complicated due to the centrifugal force reversals and the effect of the Lorentz force depending on the angular momentum. Vanishing of the left-hand side of Eq. (11) indicates a transition from linearly stable orbits to unstable ones. One can verify that these boundaries of stability coincide with the radii $R_1, \ldots, R_4$ from Sect. 2. Evaluating the expression (11), we find that these values can equally be obtained from the simple equation

$$\frac{\mathrm{d}\mathscr{L}}{\mathrm{d}r} = 0\,. \tag{12}$$

In Figs. 1 and 2, to show the influence of the magnetic field on the stability of the orbits clearly, we chose an unrealistically strong value $B_0 = 0.15\,M^{-1}$. However, the stability portrait remains valid for more realistic B_0 as well. Let us note that electromagnetic effects depend on the ratio qB_0/μ, which for electrons is approximately equal to $10^{21}\,B_0$, and the trajectories are significantly affected even by a weak field. On the other hand, magnetic field perturbations of the metric depend on B_0 alone. Therefore, in the weak-field limit, one can adopt the approximation of the asymptotically uniform magnetic field on the Schwarzschild background (Galt'sov 1986). The approximation is valid in the astrophysically most interesting region near the black hole, $r < B_0^{-1} \cong r_2$, but it cannot be used far from the hole, because the influence of the magnetic field on the spacetime curvature becomes significant. Also, the motion of electrically neutral particles is beyond this approximation, because uncharged particles interact with the magnetic field only through its influence on the geometry of the spacetime. The exact expressions given in this paper are then essential.

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