Current topic in Astrophysics – An introduction to dwarf galaxies

Exercise

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January 31, 2018

1 Dynamical friction

This exercise deals with how gravitation affects the movement of a massive body as it travels through a field of lighter bodies. Examples for such systems would be a massive star moving through the many low-mass stars in a massive star cluster, or a satellite galaxy that moves through the putative dark matter halo of a large host galaxy. We start with discussing the gravitional interaction between two particles, draw conclusions on the interaction with many particles from that and thereby arrive at the concept of dynamical friction.

1. Consider a system consisting of a particle with the mass M at the position $\mathbf{r}_M$ and a particle with the mass m at the position $\mathbf{r}_m$ (see Figure 1). The only forces acting on these particles is their mutual gravitational attraction. The force acting on the particle with the mass M is then

$$M\,\ddot{\mathbf{r}}_M = -\frac{GMm\,(\mathbf{r}_m - \mathbf{r}_M)}{\|\mathbf{r}_m - \mathbf{r}_M\|^3} \tag{1}$$

   where the dots represent total derivatives with respect to time and G is the gravitational constant. Use Newton's laws to explain that

$$m\,\ddot{\mathbf{r}}_m + M\,\ddot{\mathbf{r}}_M = 0 \tag{2}$$

   and

$$m\Delta\mathbf{v}_m + M\Delta\mathbf{v}_M = 0 \tag{3}$$

   hold, where $\Delta\mathbf{v}_m$ and $\Delta\mathbf{v}_M$ is the change of the velocities of the two masses through the encounter.

2. Use Newton's laws to show that

$$\ddot{\mathbf{r}} = -\frac{GMm\,\mathbf{r}}{\|\mathbf{r}\|^3} \tag{4}$$

   where $\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M$. Equation describes the motion of a ficticious particle, called the reduced particle, in a fixed gravitational potential created by the mass $M+m$. Why can Equation (4) be rewritten as

$$\ddot{r} = -\frac{GMm}{r^2}, \tag{5}$$

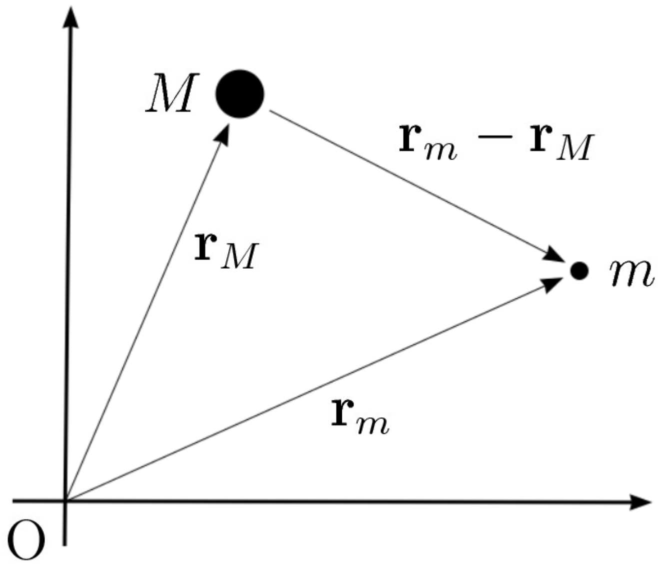

Figure 1: A sketch of a two-body system.

i.e. an equation where the vectors are replaced by their absolute values?

Equation (5) formulates Kepler's problem, and a solution to it is

$$\frac{1}{r} = C\cos(\psi - \psi_0) + \frac{G(M+m)}{L^2}, \tag{6}$$

where ψ is the angle that parameterizes the trajectory of the reduced particle, L is the absolute value of the angular momentum vector, and ψ_0 and C are constants that are determined by the initial conditions.

3. Finding the solution to Equation (5) is not part of this exercise, but those who are interested can find discussions of Kepler's problem in basically any textbook on theoretical mechanics. For the purpose here, only hyperbolic orbits are of interest, i.e. orbits where the relative motion of the masses M and m is so large that they are not bound to each other. Explain why in this case equation (6) becomes

$$\frac{1}{r} = C\cos(\psi - \psi_0) + \frac{G(M+m)}{b^2V_0^2} \tag{7}$$

with the definitions in Figure (2), i.e. b is the impact parameter and V_0 is the absolute value of the initial velocity.

4. Show that the total time derivative of equation (7) is

$$\dot{r} = CbV_0\sin(\psi - \psi_0). \tag{8}$$

5. Using that $\psi \to 0$ for $t \to -\infty$ with the definitions in Figure (2), show that ψ_0 is given as

$$\tan\psi_0 = -\frac{bV_0^2}{G(M+m)}. \tag{9}$$

6. The point $\psi = \psi_0$ is a very special one on the trajectory of the reduced particle. Why?

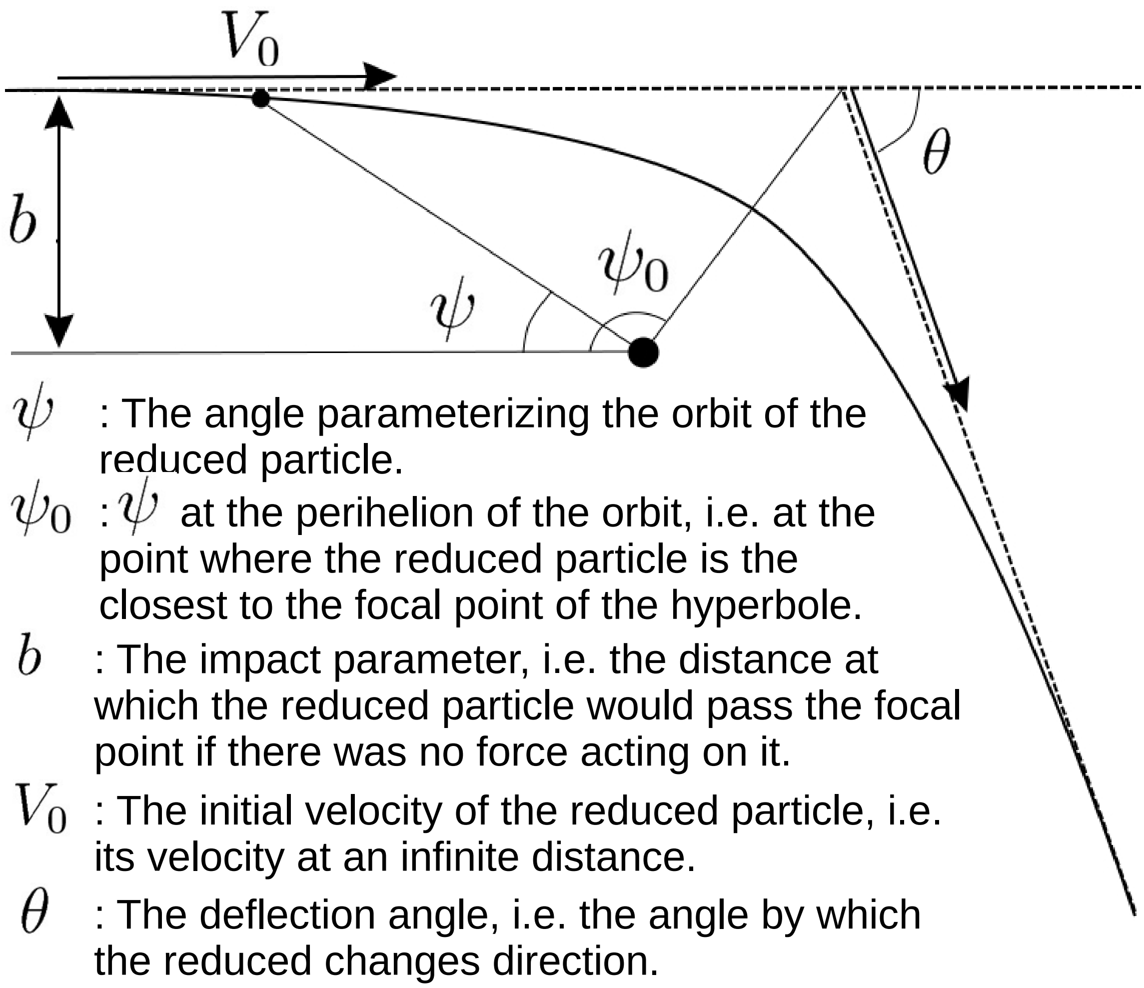

Figure 2: A hyperbolic trajectory of the reduced particle in a gravitational field.

7. The change of the relative velocitiy of the two masses through the encounter is given through

$$\Delta \mathbf{V} = \Delta \mathbf{v}_m - \Delta \mathbf{v}_M. \tag{10}$$

Using the identities

$$\sin(2\psi_0) = \frac{2\tan(\psi_0)}{1+\tan^2(\psi_0)} \tag{11}$$

and

$$\cos(2\psi_0) = \frac{2}{1+\tan^2(\psi_0)}, \tag{12}$$

show that

$$\|\Delta \mathbf{V}_\perp\| = \frac{2bV_0^3}{G(M+m)}\left[1+\frac{b^2V_0^4}{G^2(M+m)^2}\right]^{-1} \tag{13}$$

for the component of $\Delta \mathbf{V}$ vertical to the relative velocity $\mathbf{V_0}$ at the time $t=-\infty$ (i.e. before the encounter), and

$$\|\Delta \mathbf{V}_\|\| = 2V_0\left[1+\frac{b^2V_0^4}{G^2(M+m)^2}\right]^{-1} \tag{14}$$

for the component of $\Delta\mathbf{V}$ parallel to the relative velocity $\mathbf{V_0}$ at the time $t = -\infty$.

8. Show that the components of the change of the velocity of the particle with the mass M are

$$\|\Delta\mathbf{v}_{M\perp}\| = \frac{2mbV_0^3}{G(M+m)^2}\left[1 + \frac{b^2V_0^4}{G^2(M+m)^2}\right]^{-1} \tag{15}$$

for the component of $\Delta\mathbf{v}_M$ vertical to $\mathbf{v}_M$ at the time $t = -\infty$ (i.e. before the encounter), and

$$\|\Delta\mathbf{v}_{M\|}\| = \frac{2mV_0}{M+m}\left[1 + \frac{b^2V_0^4}{G^2(M+m)^2}\right]^{-1}. \tag{16}$$

for the component of $\Delta\mathbf{v}_M$ parallel to $\mathbf{v}_M$ at the time $t = -\infty$ (i.e. before the encounter)

Now imagine that the mass M does not encounter only a single particle with the mass m, but travels through a large homogeneous field of such particles. In this situation, the individual $\Delta\mathbf{v}_{M\perp}$ from the many bodies with mass m cancel out due to symmetry, but the $\Delta\mathbf{v}_{M\|}$ are all in the same direction and result into a non-zero change of the relative velocity $\mathbf{V_0} = \mathbf{v}_m - \mathbf{v}_M$, but in the opposite direction of $\mathbf{V_0}$.

The rate at which the mass M encounters particles with the mass m, velocities in the velocity-space element $d^3\mathbf{v}$ and impact parameters between b and $b + db$ is given as

$$2\pi b\, db \times V_0 \times f(\mathbf{v}_m)\, d^3\mathbf{v}_m, \tag{17}$$

where $f(\mathbf{v}_m)$ is the phase-space density.

9. Using Equations (16) and (17), show that integrating over all relevant inpact parameters $0 < b < b_{\max}$ yields

$$\left.\frac{d\mathbf{v}_M}{dt}\right|_{\mathbf{v}_m} = 2\pi\, \ln(1+\Lambda^2)G^2m(M+m)f(\mathbf{v}_m)\frac{(\mathbf{v}_m - \mathbf{v}_M)}{\|\mathbf{v}_m - \mathbf{v}_M\|^3}\, d^3\mathbf{v}_m \tag{18}$$

where

$$\Lambda \equiv \frac{b_{\max}V_0^2}{G(M+m)}. \tag{19}$$

Equation (18) is the net rate of the velocity change of the mass M due to encounters with the ensemble of particles quantified with Equation (17).

The parameter $b_{\max}$ is taken to be distance from the mass M where the density of particles of particles with mass m has dropped significantly compared to the density at the current location of the mass M. Thus, $b_{\max}$ is not very well defined, but this is a minor concern in practise, as we shall see.

The value of Λ is in practise usually very large when typical sets of parameters for V_0, $b_{\max}$ and M are used. Thus we will use the approximation

$$\frac{1}{2}\ln(1+\Lambda^2) \simeq \ln(\Lambda). \tag{20}$$

in the following. Λ being large also implies that $\ln(\Lambda)$ only changes slowly with variations of Λ, and thus quite satisfactory estimates with Equation (18) can also be made without knowing precise values for $b_{\max}$ and V_0 (which in practise is often replaced with a typical velocity $v_{\rm typ}$).

A remarkable property of Equation (18) is that its structure is equivalent to the one that is used to relate the force acting on a particle with mass M due to a surrounding matter density, only that that the density in the case of Equation (18) assumes the form $\rho_v(\mathbf{v}_m) = 4\pi \ln(\Lambda)\, Gm(M+m)\, f(\mathbf{v}_m)$, and that Equation (18) is a function of velocities instead of positions. Thus, one has to integrate over all relevant velocities instead of positions in order to obtain the total force acting on the particle with the mass M.

If the velocity distribution of the particles with mass m is isotropic, then only particles with velocities $v_m < v_M$ slow the moving mass M down (compare to the equivalent problem with a matter distribution!), so that in order to obtain the total acceleration acting to the moving mass M, one needs to integrate over the density $\rho_v(\mathbf{v}_m)$ within a sphere with radius v_M. Thus,

$$\frac{d\mathbf{v}_M}{dt} = -\frac{4\pi\, \ln(\Lambda)\, G^2 m(M+m)}{v_M^3} \left[\int_0^{v_M} f(v_m)\, 4\pi v_m^2\, dv_m \right] \mathbf{v}_M. \tag{21}$$

This is the Chandrasekhar dynamical friction formula.

A well-known isotropic velocity distribution is the Maxwellian velocity distribution, which is given as

$$f(v_m) = \frac{n_0}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{v_m^2}{2\sigma^2}\right) \tag{22}$$

where σ is the velocity dispersion, which measures the width of the velocity distribution. The Chandrasekhar dynamical friction formula then becomes

$$\frac{d\mathbf{v}_M}{dt} = -\frac{4\pi\, \ln(\Lambda)\, G^2 n_0 m(M+m)}{v_M^3} \left[\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}} \exp(-X^2) \right] \mathbf{v}_M, \tag{23}$$

where the expression in the brackets is obtained through partial integration of Equation (22) using the substitution $X \equiv v_M/\sqrt{2}\sigma$, and

$$\mathrm{erf}(X) \equiv \frac{2}{\sqrt{\pi}} \int_0^X \exp(-y^2)\, dy. \tag{24}$$

In the limit $M \gg m$, this equation becomes

$$\frac{d\mathbf{v}_M}{dt} = -\frac{4\pi\, \ln(\Lambda)\, G^2 \rho M}{v_M^3} \left[\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}} \exp(-X^2) \right] \mathbf{v}_M, \tag{25}$$

where also $n_0 m$ has been replaced with the overall background matter density ρ.

10. Far outside the core, the rotation curve of the Milky Way is flat, like it is typical for spiral galaxies. Show that the density distribution that would cause a flat rotation curve is the one of the singular isothermal sphere, which can be written as

$$\rho(r) = \frac{v_{\rm c}^2}{4\pi G r^2}, \tag{26}$$

where v_c is a constant velocity.

11. The velocity dispersion of the singular isothermal sphere is given as

$$2\pi G r^2 \rho(r) = \sigma^2. \tag{27}$$

Thus, what is the relation between $v_{\rm c}$ and σ?

12. Now consider a body with mass M that moves on a circular orbit in this isothermal sphere (a globular cluster or a dwarf galaxy within the dark matter halo of the host galaxy, for instance). Show that the frictional force acting on that body is

$$\mathbf{F}_M = -D\,\ln(\Lambda)\frac{GM^2}{r^2}\frac{\mathbf{v}_{\rm c}}{\|\mathbf{v}_{\rm c}\|} \tag{28}$$

with $D \approx 0.428$. (Hint: There are tabulated values for the error function.)

13. Consider also a body with mass M that moves with a velocity $v_M \gg v_{\rm c}$ through the isothermal sphere (perhaps because it comes from the outside with a large peculiar velocity). What is the limit of Equation (28) for $v \to \infty$? Thus, what is quantified with the factor D?

14. Let's return to the body with mass M that initially moves on a circular orbit with the velocity $v_{\rm c}$. What the force induced by the dynamical friction actually does is changing the angular momentum. Show that this is quantified through

$$\dot{L}_M \equiv \frac{d}{dt}\|\mathbf{L}_M\| = -D\,\ln(\Lambda)\frac{GM^2}{r}. \tag{29}$$

This change of L_M will cause the body sink towards the center of the isothermal sphere.

15. As the body moves towards the center, it maintains the velocity v_c. Use the angular momentum the body consequently has to show that

$$r\dot{r} = -D\,\ln(\Lambda)\frac{GM}{v_{\rm c}}. \tag{30}$$

16. Integrate Equation (30) over time with appropriate boundary conditions to arrive at

$$t_{\rm sink} = \frac{1}{2D\,\ln(\Lambda)}\frac{r_{\rm ini}^2 v_{\rm c}}{GM}, \tag{31}$$

where $r_{\rm ini}$ is the initial distance from the center. This equation quantifies the time it takes the body to sink from $r_{\rm ini}$ to the center.

17. Let us now estimate $t_{\rm sink}$ for some components of the Milky Way system, assuming that they are embedded in a halo of cold dark matter whose profile is approximated by a single isothermal sphere. Assume that $v_{\rm c} = v_M = 220$ km/s and $b_{\rm max} = r_{\rm ini}$, where $r_{\rm ini}$ is the present-day distance from the center of the Milky Way (MW), and that the considered bodies indeed move on circular orbits. It is useful to remember that $G = 0.0045\,{\rm pc}^3 {\rm M}_\odot^{-1} {\rm Myr}^{-2}$ and 1 km/s is approximately 1 pc/Myr. Consider:

    - The Large Magellanic Cloud (LMC). The LMC has a dynamical mass $M \approx 10^{10}\ {\rm M}_\odot$ and its current distance from the center of the MW is approximately 50 kpc.

- The Sculptor Dwarf Galaxy, whose current distance from the center of the MW is approximately 80 kpc. According to Strigari et al. (2008, Nature, Volume 454, Issue 7208, pp. 1096-1097), the satellite galaxies of the MW may all be located in haloes of cold dark matter with a common approximate mass of 10^9 $\mathrm{M}_\odot$. Thus, assume this as the mass of Scuptor. (Remember however that it is by no means certain that dwarf galaxies, or even galaxies in general, contain dark matter.)
- The most massive globular cluster of the Milky Way, ωCen, has a dynamical mass $M \approx 2.5 \times 10^6$ $\mathrm{M}_\odot$ and its current distance from the center of the MW is approximately 6.4 kpc.
- ωCen is not only unusually massive, but is also unusually distant from the center of the MW. Thus, also consider the values $M \approx 10^5$ $\mathrm{M}_\odot$ and r_{ini}=1.5 kpc, which are more representative for MW globular clusters.
- The Sun. The distance of the Sun from the galactic center is approxmately 8 kpc.

What do the numbers calculated above imply for the stability of disks of satellite galaxies, systems of globular clusters and stellar orbits over the age of the Universe?

Naturally, the above results are only approximative. However, simulations have shown that the analytic approximation given with Chandrasekhars formula for dynamical friction is remarkably accurate, provided that i) the mass M is less than approximately 20% of the mass of the larger system, and that ii) the orbit is not cofined near the center of the larger system or beyond its outer boundary.

2 A rotating disk of satellites

Consider a major galaxy which is accompanied by a disk of satellites that consists of 15 dwarf elliptical galaxies, as sketched in Figure (3).

1. Assuming that the velocities of satellite galaxies are in principle isotropic with respect to their host, how probable is it then that at least 13 out of the 15 satellites show the same sense of rotation with respect to the host galaxy, i.e. that the scalar product of the normal vector of the best-fitting plane with the angular momentum vectors has the same sign in at least 13 out of 15 cases, and the opposite sign only in the remaining cases?
2. Out of the more than 20 known satellite galaxies of the Andromeda Galaxy, 15 constitute a disk of satellites, which we happen to see almost edge-on from our position (Ibata et al. 2013, Nature, Volume 493, Issue 7430, pp. 62-65). 13 out of these 15 satellites show the same sense of rotation around the Andromeda Galaxy. Is this observational finding consistent with the hypothesis that the motions of satellites around their host are isotropic in principle? What are the implications for the ΛCDM-model, according to which many dwarf galaxies would form in dark-matter haloes that are bound to larger dark-matter haloes?

3 The mass of Segue I

Segue I is a faint stellar system in the vicinity of the Milky Way that has been discovered about 10 years ago. It contains of the order of 1000 stars, all of which are resolved at

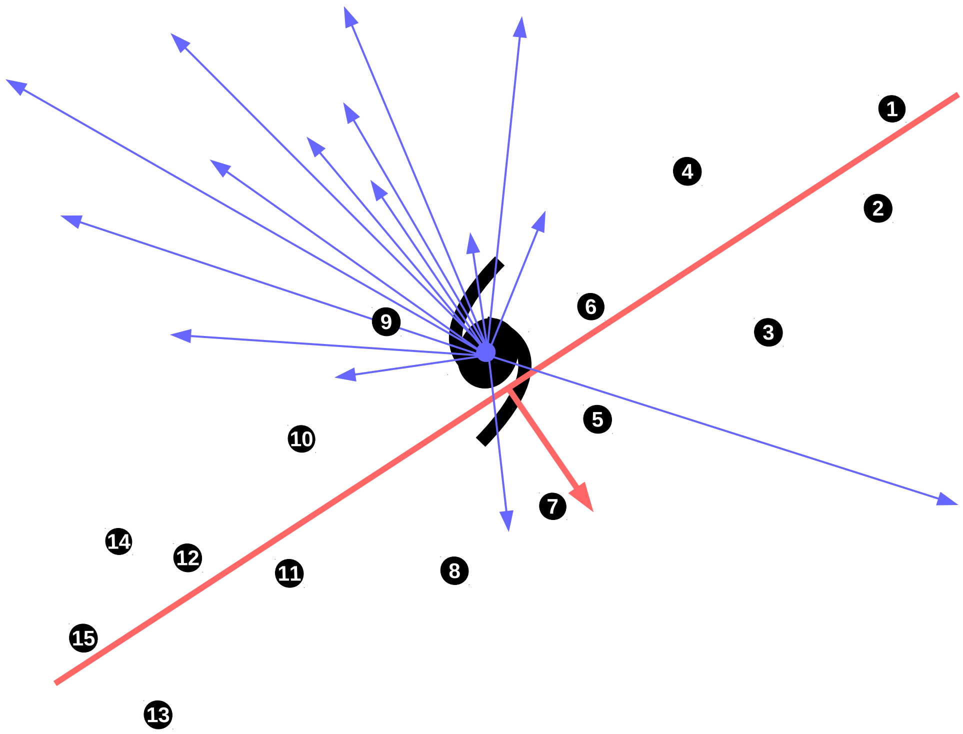

Figure 3: A sketch of an edge-on view of a disk of satellites. The disk consists of 15 dwarf galaxies, which are symbolized by the dots with the numbers, and the best-fitting plane (i.e. the plane around which the satellites exhibit the smallest scatter) is symbolized by the red line. A normal vector to this plane is indicated at a red arrow. The angular momenta of the satellites with respect to the host galaxy, or more accurately their projections along the line of sight, are shown as blue arrows.

the distance of Segue I. Their combined luminosity is about 340 $\mathrm{L}_{\odot}$. The line-of-sight velocity dispersion σ of these stars is estimated to be 3.9 ± 0.8 km/s. The projected half-light radius r_{e} of Segue I is given as 29 (-5/+8) pc.

1. Estimate the dynamical mass of Segue I. Use that $K_{\mathrm{V}} \equiv r_{\mathrm{g}}/r_{\mathrm{e}} \approx 3$ for many self-gravitating stellar systems.

2. Given that the stellar population of Segue I consists of the order of 1000 old stars with a total luminosity of 340 $\mathrm{L}_{\odot}$, make an estimate for the approximate total mass of this stellar population. Is this consistent with the estimate of the mass from the internal dynamics?, If not, can you give possible explanations for this finding?