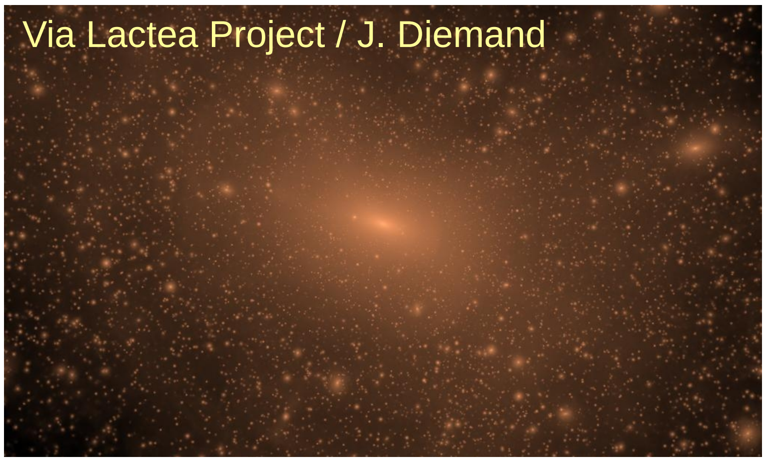
An introduction to dwarf galaxies

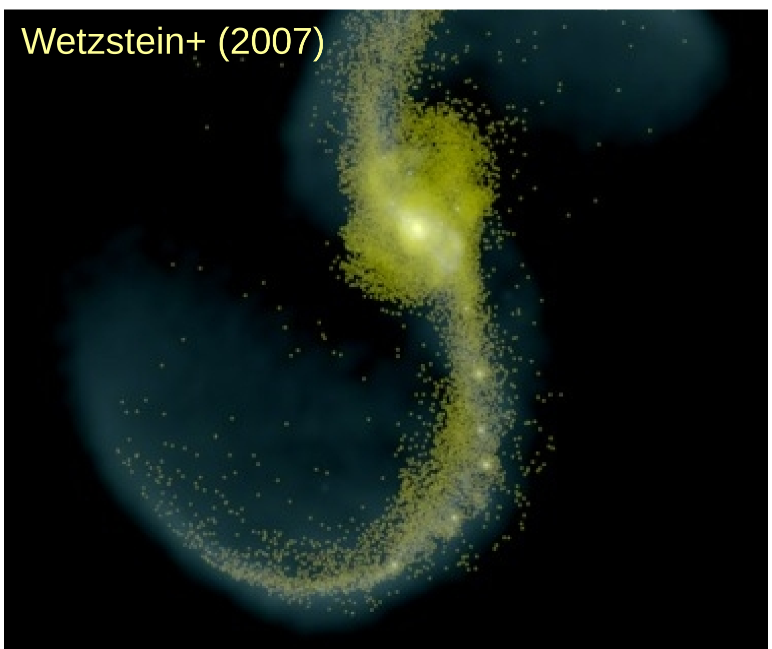
Last time:

- Formation scenarios for dwarf elliptical (galaxy-like) dwarf galaxies, i.e. all galaxies with extensions that place them above the “Gilmore Gap”, i.e. dEs of all kinds.
- Spatial distribution of dwarf galaxies (i.e. UCDs and dEs)
- Are disks of satellites around major galaxies constituted by dEs a common occurrence, and what are the implications?

Formation scenarios for dwarf elliptical galaxies



Dwarf elliptical galaxies may form as primordial galaxies in dark-matter halos of the appropriate size, most of which are bound to larger halos according to the ΛCDM-model.

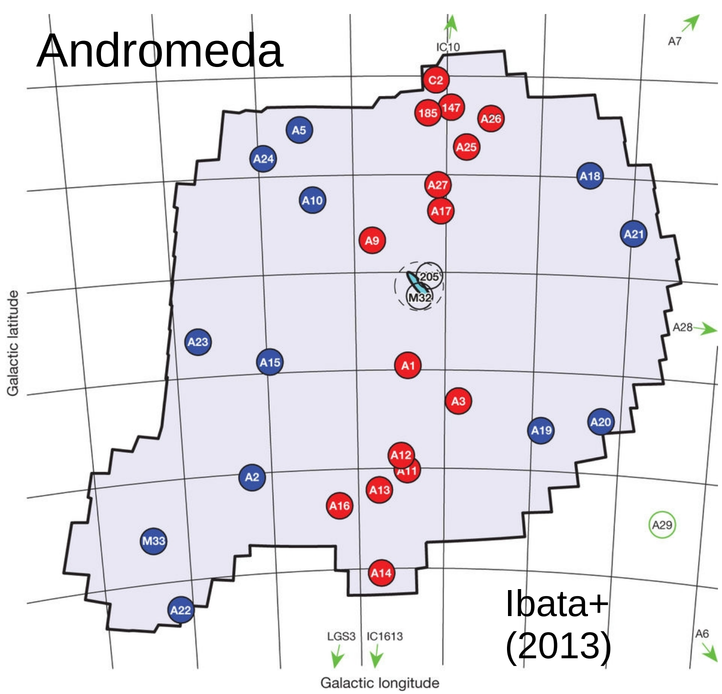


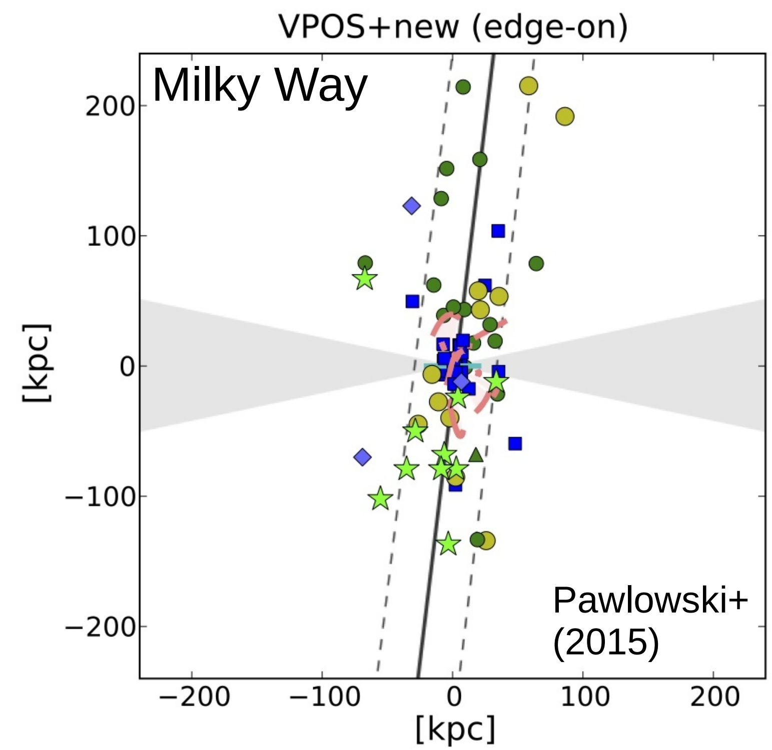
Dwarf elliptical galaxies may form as tidal dwarf galaxies from the matter ejected through tidal forces acting on encountering galaxies.

The formation off such galaxies observed, and also predicted in the ΛCDM-model, but they would be without dark matter according to that model.

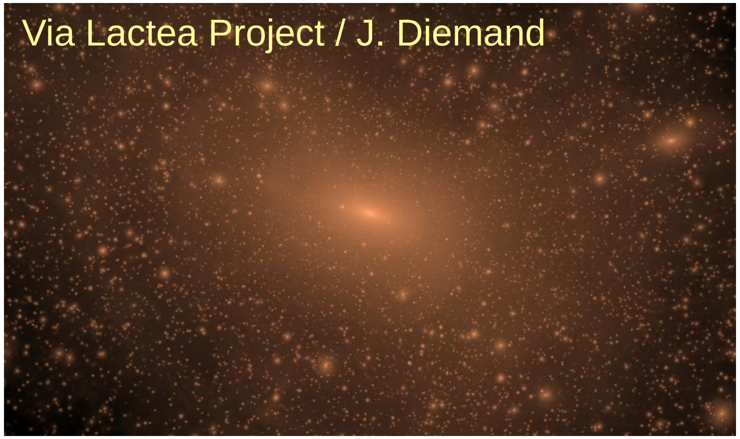
Spatial distribution of dwarf galaxies

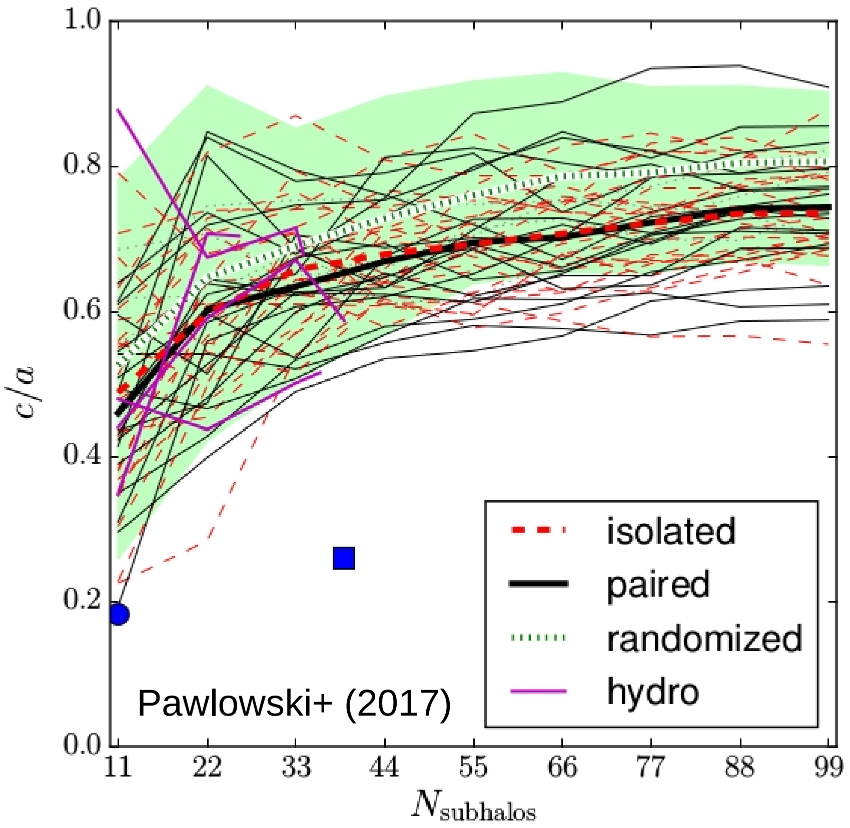
- The spatial distribution UCDs is largely the same as the one of GCs – consistent with the notion that they are very large GCs.
- Rotating disks of satellites around major galaxies constituted by dEs do exist and they are quite common. They are found by applying statistical methods. Looking at plots (or animated figures) usually does not suffice to detect or to exclude them.





Implications of disks of satellites

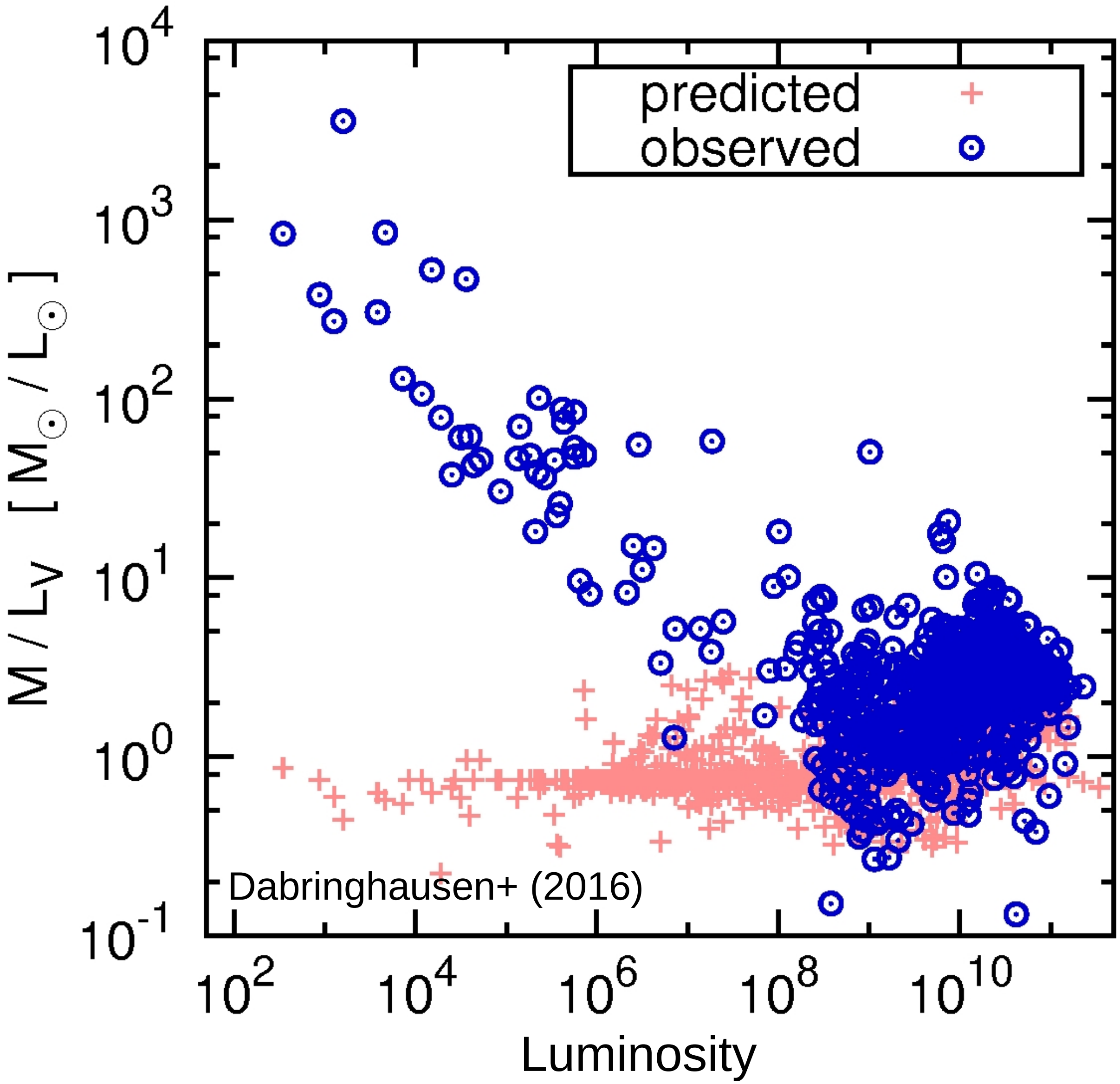




The observed distribution of satellite galaxies is much more anisotropic than simulations of galaxy formation in the ΛCDM-model imply for the distribution of primordial dwarf galaxies around their hosts – this implies that at least the galaxies that constitute the disks of satellites are tidal dwarfs instead of primordial, dark-matter dominated dwarfs.

This is a very serious challenge for the ΛCDM-model! (The **“missing-satellite problem”** reborn.)

Implications of disks of satellites



Tidal dwarf galaxies do not contain significant amounts of dark matter, even if their progenitors did.

If many, if not most or all dwarf elliptical galaxies are in fact of tidal origin, why do they have such high mass-to-light ratios?

An introduction to dwarf galaxies

This time:

- How to estimate the mass of a galaxy, and what can be learned from such an estimate.
- The stellar populations of galaxies and their implications on their mass.

Mass estimates for galaxies

1. **The dynamical mass** – estimated from the observed motion of a suitable tracer population (usually stars or gas).
2. **The mass of the visible matter** – for many galaxies essentially the mass of the stellar population (including stellar remnants).

The dynamical mass

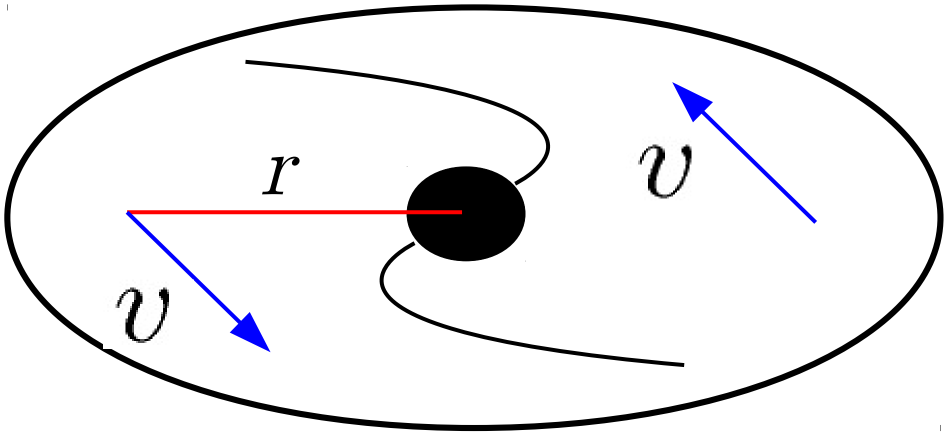
Spiral galaxies

Spiral galaxies rotate, and random motions are very small compared to this ordered motion.

Their mass can be estimated by comparing the gravitational force with the centrifugal force.

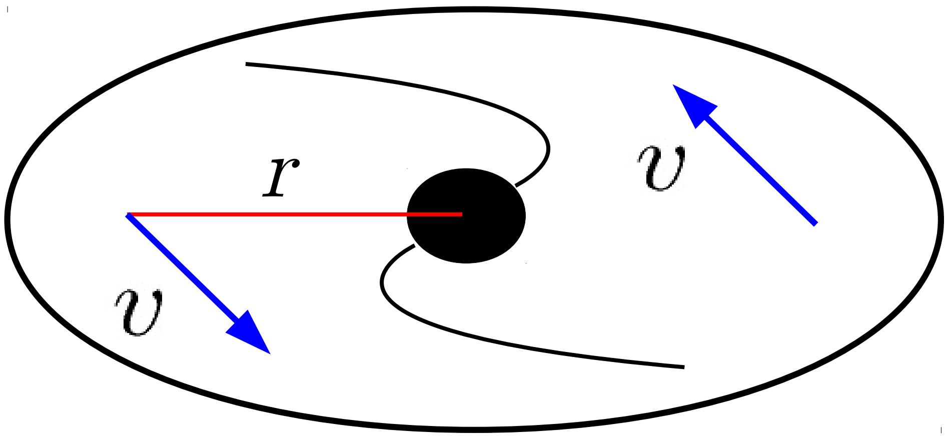
$$F_z = -F_g$$

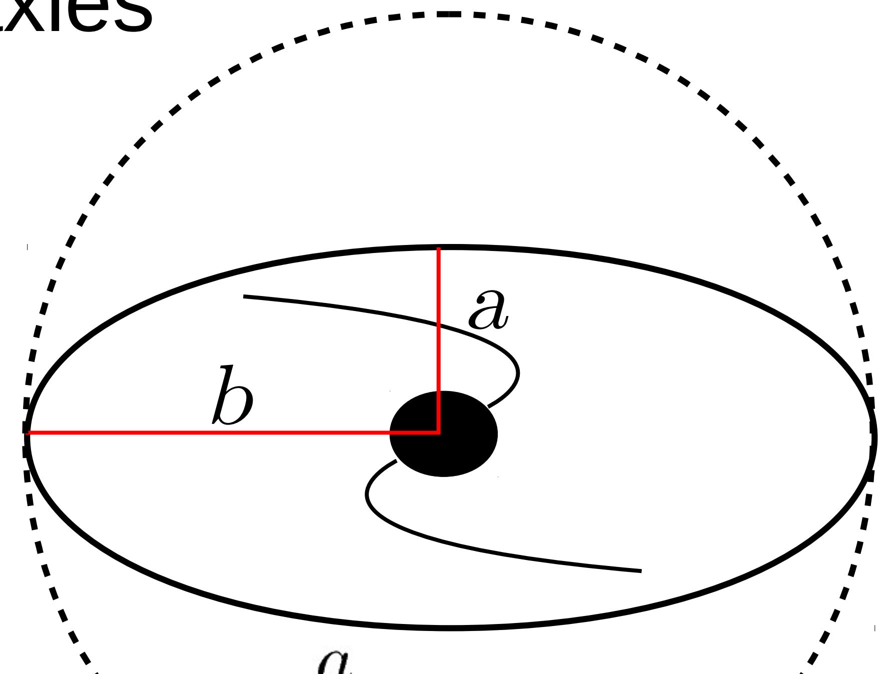
$$\frac{mv^2}{r} = G\frac{mM}{r^2}$$



The dynamical mass

Spiral galaxies

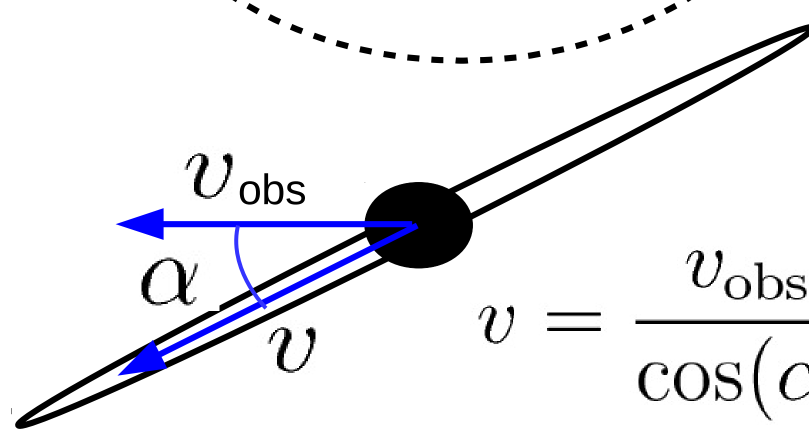




$$\frac{a}{b} = \sin(\alpha)$$

$$F_z = -F_g \longrightarrow \frac{mv^2}{r} = G\frac{mM}{r^2}$$

$$\longrightarrow M = \frac{r}{G}\left[\frac{v_{\text{obs}}}{\cos(\alpha)}\right]^2$$



$$v = \frac{v_{\text{obs}}}{\cos(\alpha)}$$

The dynamical mass

Elliptical galaxies

Estimating the dynamical mass of Spiral galaxies is pretty easy once their rotation has been measured – thus mainly an observational problem.

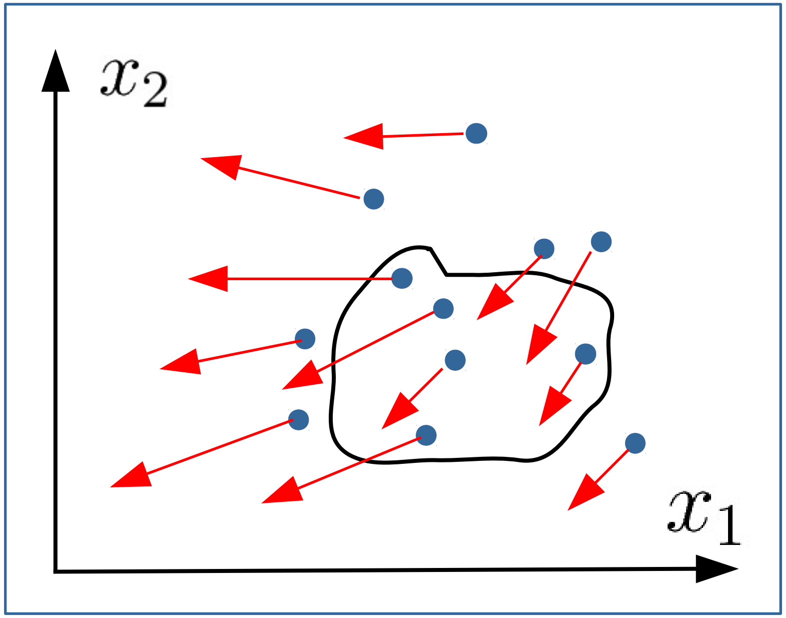
However, dwarf galaxies are usually elliptical, where most kinetic energy is in random motion instead of rotation.

How to deal with them? – Lets start with something rather simple: **The continuity equation**.

The continuity equation

The continuity equation for an n-dimensional volume without sinks or sources:

$$-\int_V \frac{\partial \rho}{\partial t}\, d^n\mathbf{x} = \int_S \rho\, \dot{\mathbf{x}} \cdot \mathbf{n}_S\, d^{n-1}\mathbf{x}$$

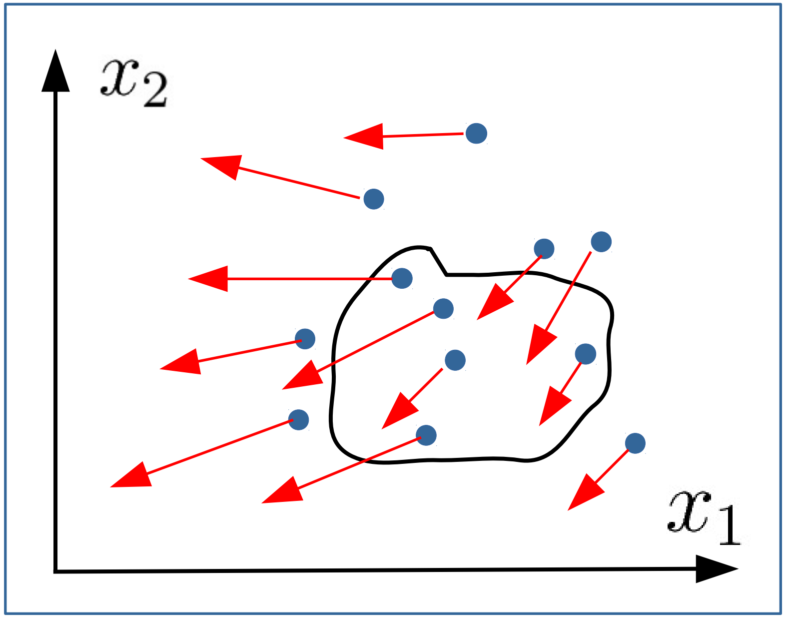


The continuity equation

The continuity equation for an n-dimensional volume without sinks or sources:

$$-\int_V \frac{\partial \rho}{\partial t}\, d^n\mathbf{x} = \int_S \rho\, \dot{\mathbf{x}} \cdot \mathbf{n}_S\, d^{n-1}\mathbf{x} = \int_V \nabla \cdot (\rho\, \dot{\mathbf{x}})\, d^n\mathbf{x}$$

Divergence Theorem: $\displaystyle \int_V \nabla \cdot \mathbf{F}\, d^n\mathbf{x} = \int_S \mathbf{F} \cdot \mathbf{n}_S\, d^{n-1}\mathbf{x}$



$$\nabla = \left(\frac{\partial}{\partial x_1} \quad \cdots \quad \frac{\partial}{\partial x_n} \right)$$

The continuity equation

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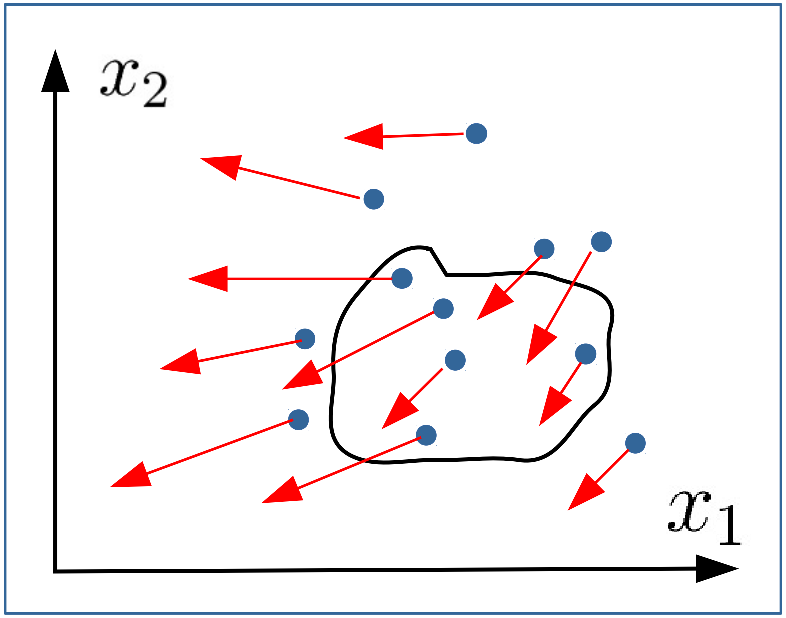


$$\rightarrow \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \dot{\mathbf{x}}) \right] d^n\mathbf{x} = 0$$

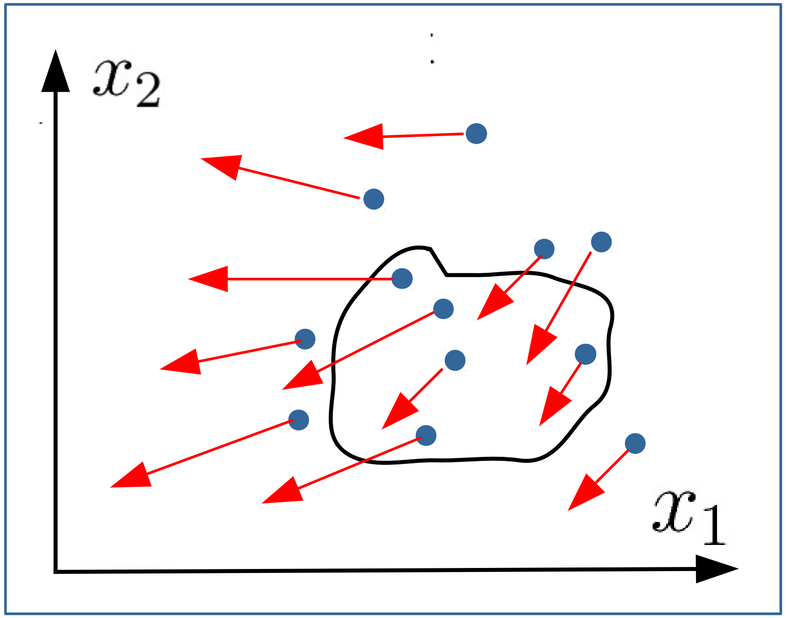
Since the volume is arbitrary:

$$\rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \dot{\mathbf{x}}) = 0$$

The continuity equation

The continuity equation:

$$-\int_V \frac{\partial \rho}{\partial t}\, d^n\mathbf{x} = \int_S \rho\, \dot{\mathbf{x}} \cdot \mathbf{n}_S\, d^{n-1}\mathbf{x}$$



Using the divergence theorem:

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \dot{\mathbf{x}}) \right] d^n\mathbf{x} = 0$$

Since the volume is arbitrary:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \dot{\mathbf{x}}) = 0$$

A different formulation:

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\rho \dot{x}_i) = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial \rho}{\partial x_i} \dot{x}_i + \rho \frac{\partial \dot{x}_i}{\partial x_i} \right) = 0$$

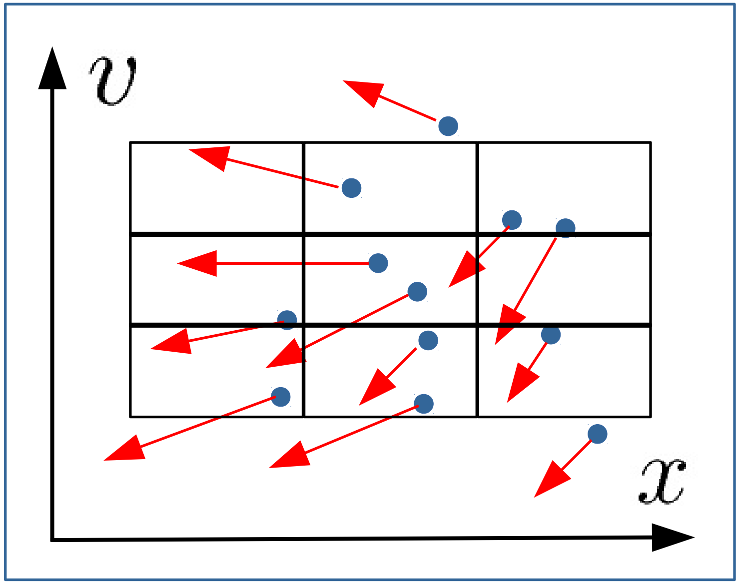
The distribution function or phase space density

The phase-space density of stars (or matter in general) is given by the distribution function: $f = f(\mathbf{x}(t), \mathbf{v}(t), t) = f(\mathbf{w}(t), t)$

$f(\mathbf{x}(t), \mathbf{v}(t), t)$ fulfills the continuity equation:

$$\rightarrow \quad \frac{\partial f}{\partial t} + \nabla \cdot (f\dot{\mathbf{w}}) = \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \left(\frac{\partial f}{\partial w_\alpha} \dot{w}_\alpha + f \frac{\partial \dot{w}_\alpha}{\partial w_\alpha} \right) = 0$$

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$



The distribution function or phase space density

$f(\mathbf{x}(t), \mathbf{v}(t), t)$ fulfills the continuity equation:

$$\rightarrow \quad \frac{\partial f}{\partial t} + \nabla \cdot (f\dot{\mathbf{w}}) = \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \left(\frac{\partial f}{\partial w_\alpha} \dot{w}_\alpha + f \frac{\partial \dot{w}_\alpha}{\partial w_\alpha} \right) = 0$$

Let's have closer look at the last term:

$$\rightarrow \quad \sum_{\alpha=1}^{6} \frac{\partial \dot{w}_\alpha}{\partial w_\alpha} = \sum_{i=1}^{3} \left(\frac{\partial v_i}{\partial x_i} + \frac{\partial \dot{v}_i}{\partial v_i} \right) = \sum_{i=1}^{3} -\frac{\partial}{\partial v_i} \left(\frac{\partial \Phi}{\partial x_i} \right) = 0$$

use the definition of the vector w

= 0 because independent variables in phase space

$$\frac{dv_i}{dt} = \dot{v}_i = \qquad a_i = -\frac{\partial \Phi}{\partial x_i}$$

since force is independent of velocity: $\frac{\partial a_i}{\partial v_i} = 0$

$$\rightarrow \quad \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{\partial f}{\partial w_\alpha} \dot{w}_\alpha = 0$$

The collisionless Boltzmann equation

The collisionless Boltzmann equation (CBE)

Using the definition of w in order to express the CBE in terms of actual positions and velocities:

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{\partial f}{\partial w_\alpha}\dot{w}_\alpha = \sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i}\dot{x}_i + \frac{\partial f}{\partial v_i}\dot{v}_i\right) + \frac{\partial f}{\partial t} = 0$$

The CBE is the total time derivative of the phase space density:

$$\sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial f}{\partial v_i}\frac{dv_i}{dt}\right) + \frac{\partial f}{\partial t} = 0 = \frac{d}{dt}f(\mathbf{x}(t), \mathbf{v}(t), t)$$

A very popular formulation:

$$\frac{df}{dt} = \sum_{i=1}^{3}\left(\frac{\partial f}{\partial x_i}v_i - \frac{\partial f}{\partial v_i}\frac{\partial \Phi}{\partial x_i}\right) + \frac{\partial f}{\partial t} = 0$$

The CBE is one of the central equations in galactic dynamics.

The Jeans equations

The Jeans equations are velocity moments of the CBE. Thus, lets integrate the CBE over all velocities and see where this gets us:

$$\sum_{i=1}^{3}\int \frac{\partial f}{\partial x_i} v_i \, d^3\mathbf{v} - \sum_{i=1}^{3}\int \frac{\partial f}{\partial v_i}\frac{\partial \Phi}{\partial x_i}\, d^3\mathbf{v} + \int \frac{\partial f}{\partial t}\, d^3\mathbf{v} = 0$$

Note that:

$$\int \frac{\partial f}{\partial t}\, d^3\mathbf{v} = \frac{\partial}{\partial t}\int f\, d^3\mathbf{v}$$

because the range of velocities over which is integrated does not change time. Note also:

$$\int \frac{\partial f}{\partial x_i} v_i \, d^3\mathbf{v} = \frac{\partial}{\partial x_i}\int f v_i \, d^3\mathbf{v}$$

because v and x are independent variables in phase space.

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$$\sum_{i=1}^{3}\int \frac{\partial f}{\partial x_i} v_i \, d^3\mathbf{v} - \sum_{i=1}^{3}\int \frac{\partial f}{\partial v_i}\frac{\partial \Phi}{\partial x_i}\, d^3\mathbf{v} + \int \frac{\partial f}{\partial t}\, d^3\mathbf{v} = 0$$

Note that:

$$\boxed{a_i = -\frac{\partial \Phi}{\partial x_i}}$$

$$-\sum_{i=1}^{3}\int \frac{\partial f}{\partial v_i}\frac{\partial \Phi}{\partial x_i}\, d^3\mathbf{v} = \int \sum_{i=1}^{3} \frac{\partial f}{\partial v_i} a_i \, d^3\mathbf{v}$$

$$= \int \nabla(f\mathbf{a})\, d^3\mathbf{v} = \int f\mathbf{a}\cdot\mathbf{n}_S \, d^2\mathbf{v} = 0$$

$$\boxed{\nabla = \left(\frac{\partial}{\partial v_1} \quad \frac{\partial}{\partial v_2} \quad \frac{\partial}{\partial v_3}\right)}$$

Divergence theorem

$f(\mathbf{x}(t), \mathbf{v}(t), t) \rightarrow 0$
for $|\mathbf{v}(t)| \rightarrow \infty$

The Jeans equations

The Jeans equations are velocity moments of the CBE. Thus, lets integrate the CBE over all velocities and see where this gets us:

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \int f v_i \, d^3\mathbf{v} + \frac{\partial}{\partial t} \int f \, d^3\mathbf{v} = 0$$

Note that:

$$\rho = \int f \, d^3\mathbf{v}$$

This is just the matter density in space.

We define the velocity moment of a quantity as its integral over all velocities, normalized with the matter density. Thus for v_i :

$$\overline{v}_i \equiv \frac{1}{\rho} \int f v_i \, d^3\mathbf{v}$$

This can be understood as the average velocity in i-direction.

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$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \int f v_i \, d^3\mathbf{v} + \frac{\partial}{\partial t} \int f \, d^3\mathbf{v} = 0$$

With $\rho = \int f \, d^3\mathbf{v}$ and $\overline{v}_i \equiv \frac{1}{\rho} \int f v_i \, d^3\mathbf{v}$:

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\rho \overline{v}_i) = 0$$

First Jeans Equation

This is again a continuity equation. Next step will be to multiply the CBE with v_j and do the integration over all velocities then.

The Jeans equations

The Jeans equations are velocity moments of the CBE. Thus, lets multiply the CBE with v_j and then integrate over all velocities and see where this gets us:

$$\sum_{i=1}^{3} \int \frac{\partial f}{\partial x_i} v_i v_j \, d^3\mathbf{v} - \sum_{i=1}^{3} \int \frac{\partial f}{\partial v_i} \frac{\partial \Phi}{\partial x_i} v_j \, d^3\mathbf{v} + \int \frac{\partial f}{\partial t} v_j \, d^3\mathbf{v} = 0$$

Note that:

$$\int \frac{\partial f}{\partial t} v_j \, d^3\mathbf{v} = \frac{\partial}{\partial t} \int f v_j \, d^3\mathbf{v} = \frac{\partial}{\partial t} (\rho \overline{v}_j)$$

because the range of velocities over which is integrated does not change time. Note also:

$$\int \frac{\partial f}{\partial x_i} v_i v_j \, d^3\mathbf{v} = \frac{\partial}{\partial x_i} \int f v_i v_j \, d^3\mathbf{v} = \frac{\partial}{\partial x_i} (\rho \overline{v_i v_j})$$

because v and x are independent variables in phase space.

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$$\sum_{i=1}^{3} \int \frac{\partial f}{\partial x_i} v_i v_j \, d^3\mathbf{v} - \sum_{i=1}^{3} \int \frac{\partial f}{\partial v_i} \frac{\partial \Phi}{\partial x_i} v_j \, d^3\mathbf{v} + \int \frac{\partial f}{\partial t} v_j \, d^3\mathbf{v} = 0$$

Note that using the product rule $(g \cdot h)' = g' \cdot h + g \cdot h'$:

$$\sum_{i=1}^{3} \int \frac{\partial f}{\partial v_i} \frac{\partial \Phi}{\partial x_i} v_j \, d^3\mathbf{v} = \sum_{i=1}^{3} \int \frac{\partial}{\partial v_i} \left(f \frac{\partial \Phi}{\partial x_i} v_j \right) d^3\mathbf{v}$$

$$- \sum_{i=1}^{3} \int f \frac{\partial \Phi}{\partial x_i} \frac{\partial v_j}{\partial v_i} \, d^3\mathbf{v}$$

This term is 0, as can be shown with the divergence theorem

This term is the interesting one

The Jeans equations

The Jeans equations are velocity moments of the CBE. Thus, lets multiply the CBE with v_j and then integrate over all velocities and see where this gets us:

$$\sum_{i=1}^{3}\int \frac{\partial f}{\partial x_i} v_i v_j \, d^3\mathbf{v} - \sum_{i=1}^{3}\int \frac{\partial f}{\partial v_i}\frac{\partial \Phi}{\partial x_i} v_j \, d^3\mathbf{v} + \int \frac{\partial f}{\partial t} v_j \, d^3\mathbf{v} = 0$$

Note that:

$$a_i = -\frac{\partial \Phi}{\partial x_i}$$

$$\sum_{i=1}^{3}\int \frac{\partial}{\partial v_i}\left(f \frac{\partial \Phi}{\partial x_i} v_j\right) d^3\mathbf{v} = \int \sum_{i=1}^{3} \frac{\partial}{\partial v_i}\left(f a_i v_j\right) d^3\mathbf{v}$$

$$= \int \nabla \cdot (f v_j \, \mathbf{a}) \, d^3\mathbf{v} = \int f v_j \, \mathbf{a} \cdot \mathbf{n}_S \, d^2\mathbf{v} = 0$$

$$\nabla = \left(\frac{\partial}{\partial v_1} \quad \frac{\partial}{\partial v_2} \quad \frac{\partial}{\partial v_3}\right)$$

Divergence theorem

$f(\mathbf{x}(t), \mathbf{v}(t), t) \rightarrow 0$
for $|\mathbf{v}(t)| \rightarrow \infty$

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The Jeans equations are velocity moments of the CBE. Thus, lets multiply the CBE with v_j and then integrate over all velocities and see where this gets us:

$$\sum_{i=1}^{3}\int \frac{\partial f}{\partial x_i} v_i v_j \, d^3\mathbf{v} - \sum_{i=1}^{3}\int \frac{\partial f}{\partial v_i}\frac{\partial \Phi}{\partial x_i} v_j \, d^3\mathbf{v} + \int \frac{\partial f}{\partial t} v_j \, d^3\mathbf{v} = 0$$

Note that:

$$-\sum_{i=1}^{3}\int f \frac{\partial \Phi}{\partial x_i}\frac{\partial v_j}{\partial v_i} \, d^3\mathbf{v} = -\sum_{i=1}^{3}\int f \frac{\partial \Phi}{\partial x_i}\delta_{ij} \, d^3\mathbf{v}$$

$$= -\frac{\partial \Phi}{\partial x_j}\int f \, d^3\mathbf{v} = -\frac{\partial \Phi}{\partial x_j}\rho$$

The density

$$\delta_{ij} = \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases}$$

The Jeans equations

The Jeans equations are velocity moments of the CBE. Thus, lets multiply the CBE with v_j and then integrate over all velocities and see where this gets us:

$$\frac{\partial}{\partial t}(\rho\overline{v}_j) + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\rho\overline{v_i v_j}) + \frac{\partial\Phi}{\partial x_j}\rho = 0$$

Second Jeans Equation

$\overline{v_i v_j}$ can be splitted in a component due to ordered motion and a component due to random motion:

$$\sigma_{ij}^2 \equiv \overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)} \equiv \frac{1}{\rho}\int (v_i - \overline{v}_i)(v_j - \overline{v}_j) f\, d^3\mathbf{v}$$

$$= \frac{1}{\rho}\int v_i v_j f\, d^3\mathbf{v} - \frac{1}{\rho}\int \overline{v}_i v_j f\, d^3\mathbf{v} - \frac{1}{\rho}\int v_i \overline{v}_j f\, d^3\mathbf{v} + \frac{1}{\rho}\int \overline{v}_i\overline{v}_j f\, d^3\mathbf{v}$$

$$= \frac{1}{\rho}\int v_i v_j f\, d^3\mathbf{v} - \frac{\overline{v}_i}{\rho}\int v_j f\, d^3\mathbf{v} - \frac{\overline{v}_j}{\rho}\int v_i f\, d^3\mathbf{v} + \frac{\overline{v}_i\overline{v}_j}{\rho}\int f\, d^3\mathbf{v}$$

$$= \overline{v_i v_j} - \overline{v}_i\overline{v}_j - \overline{v}_i\overline{v}_j + \overline{v}_i\overline{v}_j = \overline{v_i v_j} - \overline{v}_i\overline{v}_j$$

The Jeans equations

The Jeans equations are velocity moments of the CBE. Thus, lets multiply the CBE with v_j and then integrate over all velocities and see where this gets us:

$$\frac{\partial}{\partial t}(\rho\overline{v}_j) + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\rho\overline{v_i v_j}) + \frac{\partial\Phi}{\partial x_j}\rho = 0$$

Second Jeans Equation

$\overline{v_i v_j}$ can be splitted in a component due to ordered motion and a component due to random motion:

$$\frac{\partial}{\partial t}(\rho\overline{v}_j) + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\rho\overline{v}_i\overline{v}_j) + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\rho\sigma_{ij}^2) + \frac{\partial\Phi}{\partial x_j}\rho = 0$$

Third Jeans Equation

The virial equations

Multiply the second Jeans equation with x_k and integrate over all positions:

$$\int x_k \frac{\partial}{\partial t}(\rho \overline{v}_j)\, d^3\mathbf{x} + \int \sum_{i=1}^{3} x_k \frac{\partial}{\partial x_i}(\rho \overline{v_i v_j})\, d^3\mathbf{x} + \int x_k \frac{\partial \Phi}{\partial x_j} \rho\, d^3\mathbf{x} = 0$$

Note that using the product rule $(g \cdot h)' = g' \cdot h + g \cdot h'$:

$$\int \sum_{i=1}^{3} x_k \frac{\partial}{\partial x_i}(\rho \overline{v_i v_j})\, d^3\mathbf{x} = \int \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(x_k \rho \overline{v_i v_j})\, d^3\mathbf{x}$$

$$- \int \sum_{i=1}^{3} \frac{\partial x_k}{\partial x_i} \rho \overline{v_i v_j}\, d^3\mathbf{x}$$

with the divergence theorem and since $\rho(\mathbf{x}) \to 0$ for $|\mathbf{x}| \to \infty$:

$$\int \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(x_k \rho \overline{v_i v_j})\, d^3\mathbf{x} = \int x_k \rho v_j\, \overline{\mathbf{v}} \cdot \mathbf{n}_S\, d^2\mathbf{v} = 0$$

The virial equations

Multiply the second Jeans equation with x_k and integrate over all positions:

$$\int x_k \frac{\partial}{\partial t}(\rho \overline{v}_j)\, d^3\mathbf{x} + \int \sum_{i=1}^{3} x_k \frac{\partial}{\partial x_i}(\rho \overline{v_i v_j})\, d^3\mathbf{x} + \int x_k \frac{\partial \Phi}{\partial x_j} \rho\, d^3\mathbf{x} = 0$$

Note that using the product rule $(g \cdot h)' = g' \cdot h + g \cdot h'$:

$$\int \sum_{i=1}^{3} x_k \frac{\partial}{\partial x_i}(\rho \overline{v_i v_j})\, d^3\mathbf{x} = \int \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(x_k \rho \overline{v_i v_j})\, d^3\mathbf{x}$$

$$- \int \sum_{i=1}^{3} \frac{\partial x_k}{\partial x_i} \rho \overline{v_i v_j}\, d^3\mathbf{x}$$

The last term is more interesting because it is not zero:

$$- \int \sum_{i=1}^{3} \frac{\partial x_k}{\partial x_i} \rho \overline{v_i v_j}\, d^3\mathbf{x} = - \int \sum_{i=1}^{3} \delta_{ki} \rho \overline{v_i v_j}\, d^3\mathbf{x} = - \int \rho \overline{v_k v_j}\, d^3\mathbf{x}$$

The virial equations

Multiply the second Jeans equation with x_k and integrate over all positions:

$$\int x_k \frac{\partial}{\partial t}(\rho \overline{v}_j)\, d^3\mathbf{x} + \int \sum_{i=1}^{3} x_k \frac{\partial}{\partial x_i}(\rho \overline{v_i v_j})\, d^3\mathbf{x} + \int x_k \frac{\partial \Phi}{\partial x_j} \rho\, d^3\mathbf{x} = 0$$

Thus, the second term is the kinetic energy tensor:

$$K_{kj} \equiv \frac{1}{2} \int \rho \overline{v_k v_j}\, d^3\mathbf{x} = \frac{1}{2} \int \rho(\overline{v}_k \overline{v}_j + \sigma^2_{kj})\, d^3\mathbf{x}$$

This tensor is symmetric: $K_{kj} = K_{jk}$

The third term is the potential energy tensor:

$$W_{kj} \equiv -\int \rho(\mathbf{x}) x_k \frac{\partial \Phi(\mathbf{x})}{\partial x_j}\, d^3\mathbf{x}$$

This tensor, too, is symmetric, as we will see

The potential energy tensor

The gravitational potential is

$$\Phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \, d^3\mathbf{x}'$$

$$W_{kj} \equiv - \int \rho(\mathbf{x}) x_k \frac{\partial \Phi(\mathbf{x})}{\partial x_j} \, d^3\mathbf{x}$$

$$= G \int \rho(\mathbf{x}) x_k \frac{\partial}{\partial x_j} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \, d^3\mathbf{x}' d^3\mathbf{x}$$

x and x' are independent variables

$$= G \int \int \rho(\mathbf{x}) x_k \frac{\partial}{\partial x_j} \left(\frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \right) \, d^3\mathbf{x}' d^3\mathbf{x}$$

Do the differentiation

$$= G \int \int \rho(\mathbf{x}) x_k \frac{\rho(\mathbf{x}')(x'_j - x_j)}{|\mathbf{x}' - \mathbf{x}|^3} \, d^3\mathbf{x}' d^3\mathbf{x}$$

Rearrange

$$= G \int \int \rho(\mathbf{x}) \rho(\mathbf{x}') \frac{x_k (x'_j - x_j)}{|\mathbf{x}' - \mathbf{x}|^3} \, d^3\mathbf{x}' d^3\mathbf{x}$$

The potential energy tensor

$$G \int \int \rho(\mathbf{x})\rho(\mathbf{x}') \frac{x_k(x'_j - x_j)}{|\mathbf{x}' - \mathbf{x}|^3}\, d^3\mathbf{x}' d^3\mathbf{x}$$

$$= G \int \int \rho(\mathbf{x}')\rho(\mathbf{x}) \frac{x'_k(x_j - x'_j)}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3\mathbf{x} d^3\mathbf{x}'$$

x and x' are interchangeable dummy variables

Add the two terms with the interchanged dummy variables:

$$W_{kj} = -\frac{1}{2} G \int \int \rho(\mathbf{x})\rho(\mathbf{x}') \frac{(x_k - x'_k)(x_j - x'_j)}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3\mathbf{x}' d^3\mathbf{x}$$

Thus, this tensor is symmetric: $W_{kj} = W_{jk}$

The virial equations

Multiply the second Jeans equation with x_k and integrate over all positions:

$$\int x_k \frac{\partial}{\partial t}(\rho \overline{v}_j)\, d^3\mathbf{x} + \int \sum_{i=1}^{3} x_k \frac{\partial}{\partial x_i}(\rho \overline{v_i v_j})\, d^3\mathbf{x} + \int x_k \frac{\partial \Phi}{\partial x_j} \rho\, d^3\mathbf{x} = 0$$

Using the definitions of the kinetic energy tensor and the potential energy tensor and their symmetry, it follows for the first term in the equation above:

$$\int x_k \frac{\partial}{\partial t}(\rho \overline{v}_j)\, d^3\mathbf{x} = \frac{\partial}{\partial t} \int \rho x_k \overline{v}_j\, d^3\mathbf{x}$$

The integration range does not depend on time

$$= 2K_{kj} + W_{kj} = \frac{\partial}{\partial t} \int \rho x_j \overline{v}_x\, d^3\mathbf{x}$$

Thus: $\displaystyle \frac{\partial}{\partial t} \int \rho x_k \overline{v}_j\, d^3\mathbf{x} = \frac{1}{2} \frac{\partial}{\partial t} \int \rho (x_k \overline{v}_j + x_j \overline{v}_k)\, d^3\mathbf{x}$

The virial equations

Multiply the second Jeans equation with x_k and integrate over all positions:

$$\int x_k \frac{\partial}{\partial t}(\rho \overline{v}_j)\, d^3\mathbf{x} + \int \sum_{i=1}^{3} x_k \frac{\partial}{\partial x_i}(\rho \overline{v_i v_j})\, d^3\mathbf{x} + \int x_k \frac{\partial \Phi}{\partial x_j} \rho\, d^3\mathbf{x} = 0$$

The first term in the equation above is:

$$\frac{\partial}{\partial t} \int \rho x_k \overline{v}_j\, d^3\mathbf{x} = \frac{1}{2}\frac{\partial}{\partial t} \int \rho(x_k \overline{v}_j + x_j \overline{v}_k)\, d^3\mathbf{x}$$

By choosing an arbitrary, but then fixed position, the partial time derivative can be replaced with the total one:

$$\frac{d}{dt} = \frac{\partial}{\partial t} \longrightarrow \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial \rho}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial \rho}{\partial t} + \nabla\rho \cdot \dot{\mathbf{x}} = \frac{\partial \rho}{\partial t}$$

$$\frac{1}{2}\frac{\partial}{\partial t} \int \rho(x_k \overline{v}_j + x_j \overline{v}_k)\, d^3\mathbf{x} = \frac{1}{2}\frac{d}{dt} \int \rho(x_k \overline{v}_j + x_j \overline{v}_k)\, d^3\mathbf{x}$$

The virial equations

Thus, with all the definitions and calculations so far, the virial equation reads:

$$\frac{1}{2}\frac{d}{dt}\int \rho(x_k\overline{v}_j + x_j\overline{v}_k)\, d^3\mathbf{x} = 2K_{jk} + W_{jk}$$

This can be rewritten by defining the moment of inertia tensor (again replacing the partial time derivative with the total one):

$$I_{jk} \equiv \int \rho x_j x_k\, d^3\mathbf{x} \qquad \frac{1}{2}\frac{dI_{jk}}{dt} = \frac{1}{2}\int \frac{\partial \rho}{\partial t} x_j x_k\, d^3\mathbf{x}$$

Recall the first Jeans equation (the continuity equation):

$$\frac{1}{2}\frac{dI_{jk}}{dt} = \frac{1}{2}\int \frac{\partial \rho}{\partial t} x_j x_k\, d^3\mathbf{x} = -\frac{1}{2}\int \sum_{i=1}^{3} \frac{\partial(\rho\overline{v}_i)}{\partial x_i} x_j x_k\, d^3\mathbf{x}$$

The virial equations

Thus, with all the definitions and calculations so far, the virial equation reads:

$$\frac{1}{2}\frac{d}{dt}\int \rho(x_k\overline{v}_j + x_j\overline{v}_k)\, d^3\mathbf{x} = 2K_{jk} + W_{jk}$$

This can be rewritten by defining the moment of inertia tensor (again replacing the partial time derivative with the total one):

$$\frac{1}{2}\frac{dI_{jk}}{dt} = \frac{1}{2}\int \frac{\partial \rho}{\partial t} x_j x_k\, d^3\mathbf{x} = -\frac{1}{2}\int \sum_{i=1}^{3} \frac{\partial(\rho\overline{v}_i)}{\partial x_i} x_j x_k\, d^3\mathbf{x}$$

$$-\frac{1}{2}\int \sum_{i=1}^{3} \frac{\partial(\rho\overline{v}_i)}{\partial x_i} x_j x_k\, d^3\mathbf{x} = -\frac{1}{2}\int \sum_{i=1}^{3} \frac{\partial(\rho\overline{v}_i x_j x_k)}{\partial x_i}\, d^3\mathbf{x}$$

$$+\frac{1}{2}\int \sum_{i=1}^{3} \rho\overline{v}_i \frac{\partial(x_j x_k)}{\partial x_i}\, d^3\mathbf{x}$$

Once again the product rule.

The virial equations

Thus, with all the definitions and calculations so far, the virial equation reads:

$$\frac{1}{2}\frac{d}{dt}\int \rho(x_k\overline{v}_j + x_j\overline{v}_k)\, d^3\mathbf{x} = 2K_{jk} + W_{jk}$$

This can be rewritten by defining the moment of inertia tensor (again replacing the partial time derivative with the total one):

$$\frac{1}{2}\frac{dI_{jk}}{dt} = \frac{1}{2}\int \frac{\partial \rho}{\partial t} x_j x_k\, d^3\mathbf{x} = -\frac{1}{2}\int \sum_{i=1}^{3} \frac{\partial(\rho\overline{v}_i)}{\partial x_i} x_j x_k\, d^3\mathbf{x}$$

$$-\frac{1}{2}\int \sum_{i=1}^{3} \frac{\partial(\rho\overline{v}_i x_j x_k)}{\partial x_i}\, d^3\mathbf{x} = -\frac{1}{2}\int \nabla\cdot(\rho x_j x_k \overline{\mathbf{v}})\, d^3\mathbf{x}$$

$$= -\frac{1}{2}\int \rho x_j x_k \overline{\mathbf{v}}\cdot\mathbf{n}_S\, d^2\mathbf{x} = 0$$

Once again the divergence theorem rule and $\rho(\mathbf{x}) \to 0$ for $|\mathbf{x}| \to \infty$

The virial equations

Thus, with all the definitions and calculations so far, the virial equation reads:

$$\frac{1}{2}\frac{d}{dt}\int \rho(x_k\overline{v}_j + x_j\overline{v}_k)\, d^3\mathbf{x} = 2K_{jk} + W_{jk}$$

This can be rewritten by defining the moment of inertia tensor (again replacing the partial time derivative with the total one):

$$\frac{1}{2}\frac{dI_{jk}}{dt} = \frac{1}{2}\int \frac{\partial \rho}{\partial t} x_j x_k\, d^3\mathbf{x} = -\frac{1}{2}\int \sum_{i=1}^{3} \frac{\partial(\rho\overline{v}_i)}{\partial x_i} x_j x_k\, d^3\mathbf{x}$$

$$\frac{1}{2}\int \sum_{i=1}^{3} \rho\overline{v}_i \frac{\partial(x_j x_k)}{\partial x_i}\, d^3\mathbf{x} = \frac{1}{2}\int \sum_{i=1}^{3} \rho\overline{v}_i \left(\frac{\partial x_j}{\partial x_i} x_k + x_j \frac{\partial x_k}{\partial x_i}\right)\, d^3\mathbf{x}$$

$$= \frac{1}{2}\int \sum_{i=1}^{3} \rho\overline{v}_i(\delta_{ji}x_k + \delta_{ki}x_j)\, d^3\mathbf{x} = \frac{1}{2}\int \sum_{i=1}^{3} \rho(\overline{v}_j x_k + \overline{v}_k x_j)\, d^3\mathbf{x}$$

The virial equations

Thus, substituting the result from the time-derivation of the moment of inertia tensor, the virial equation

$$\frac{1}{2}\frac{d}{dt}\int \rho(x_k\overline{v}_j + x_j\overline{v}_k)\, d^3\mathbf{x} = 2K_{jk} + W_{jk}$$

can also be expressed as

$$\frac{1}{2}\frac{d^2 I_{jk}}{dt^2} = 2K_{jk} + W_{jk}$$

If the system is stationary (i.e. matter may move, but the matter density does not change with time):

$$\frac{1}{2}\frac{d^2 I_{jk}}{dt^2} = 0$$

The virial equations

The trace of the virial tensor is particularly interesting.

$$\begin{aligned}
\mathrm{trace}(\mathbf{W}) &\equiv \sum_{j=1}^{3} W_{jj} \\
&= -\frac{1}{2}G\int\int \rho(\mathbf{x})\rho(\mathbf{x}')\frac{\sum_{j=1}^{3}(x_j - x'_j)^2}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3\mathbf{x}'d^3\mathbf{x} \\
&= -\frac{1}{2}G\int\int \rho(\mathbf{x})\rho(\mathbf{x}')\frac{|\mathbf{x}-\mathbf{x}'|^2}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3\mathbf{x}'d^3\mathbf{x} \\
&= -\frac{1}{2}G\int \rho(\mathbf{x})\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'d^3\mathbf{x} \\
&= \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})\,d^3\mathbf{x} \\
&= W
\end{aligned}$$

i.e. the total potential energy.

The virial equations

The trace of the virial tensor is particularly interesting.

$$\mathrm{trace}(\mathbf{K}) \equiv \sum_{j=1}^{3} K_{jj} = \frac{1}{2}\int \sum_{j=1}^{3} \rho \overline{v_j v_j}\, d^3\mathbf{x}$$

$$= \frac{1}{2}\int \sum_{j=1}^{3} \rho(\overline{v}_j^2 + \sigma_{jj}^2)\, d^3\mathbf{x}$$

$= K$ i.e. the total kinetic energy.

Thus, for a stationary system: $\boxed{2K + W = 0}$

The virial equations

For a stationary system: $2K + W = 0$ or $Q \equiv \frac{K}{W} = 0.5$

More specifically, with $\langle v^2 \rangle \equiv \sum_{j=1}^{3} (\overline{v}_j^2 + \sigma_{jj}^2)$:

$$M \langle v^2 \rangle = \frac{GM^2}{r_g}$$

with r_g being the density-profile dependent gravitational radius

The dynamical mass

The dynamical mass is given through: $M\left\langle v^2\right\rangle = \frac{GM^2}{r_g}$

Special cases:

Spiral galaxies: $M = \frac{1}{G} r_c v_c^2$

where r_c is the distance to the center v_c and the circular velocity.

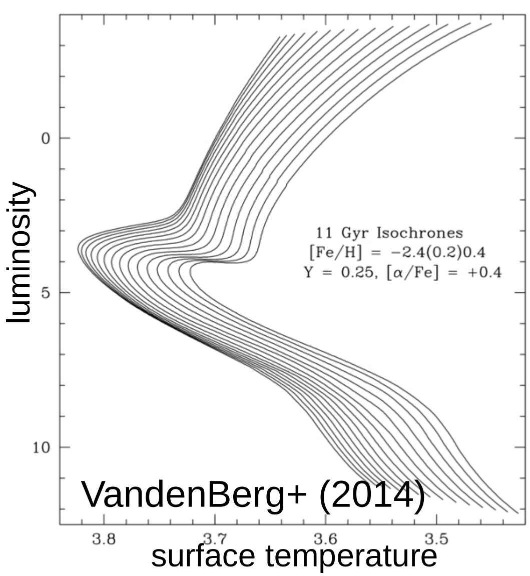
Elliptical galaxies: $M = \frac{K_{\mathrm{V}}}{G} r_e \sigma_0^2$

where r_e is the projected half-mass radius, σ_0 is the line-of-sight velocity dispersion and K_{V} is a factor roughly between 1 and 10 which depends on the density profile.

The mass of the baryonic matter

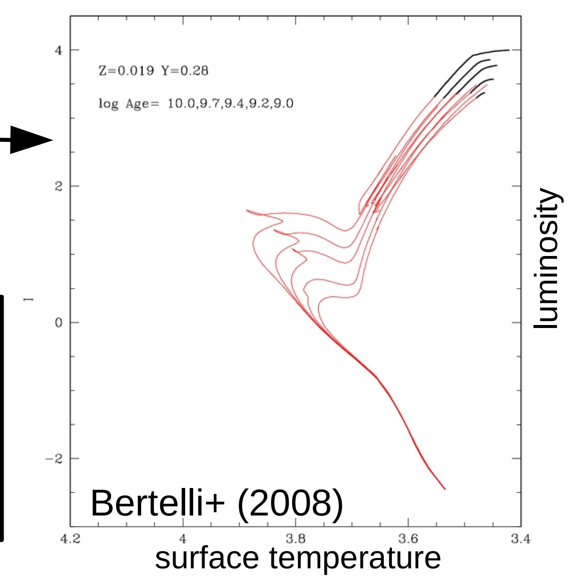
For many (especially elliptical) galaxies, the mass of their baryonic matter is essentially the mass of their stellar population, including stellar remnants.

Thus, knowing the composition of the stellar population (mass spectrum, age spectrum, metallicity spectrum) would be sufficient to estimate the mass of the galaxy.



Stellar isochrones – same ages, different metallicities

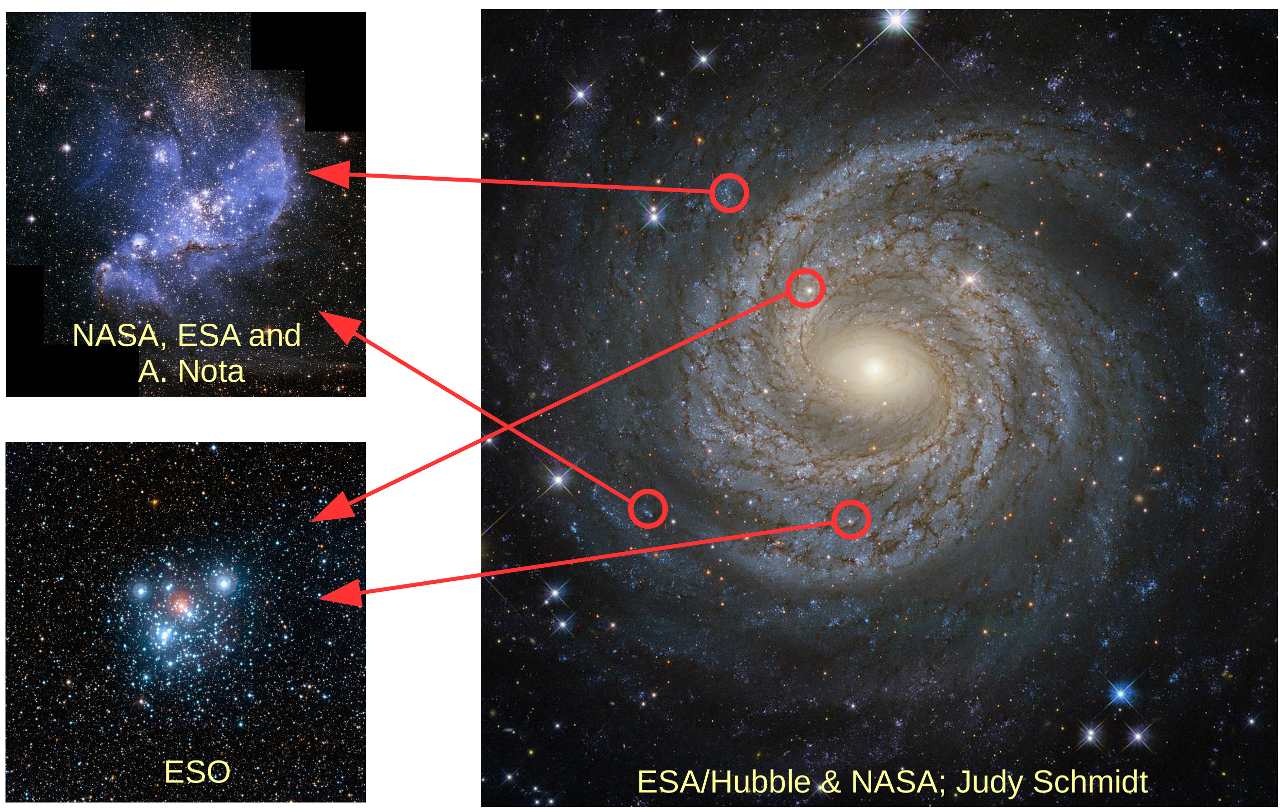
Stellar isochrones – same metallicity, different ages



The mass of the baryonic matter

Knowing the composition of the stellar population (mass spectrum, age spectrum, metallicity spectrum) would be sufficient to estimate the mass of the galaxy.

Stars form in star clusters. Thus, the mass spectrum of stars in a galaxy is the combined mass spectrum of the stars in all its star clusters.



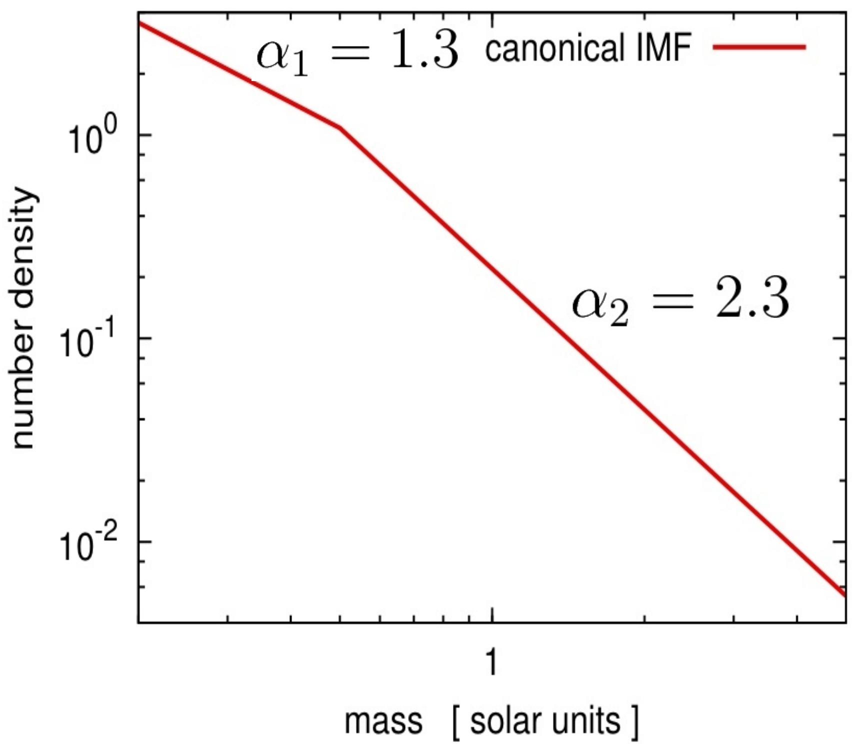
The mass of the baryonic matter

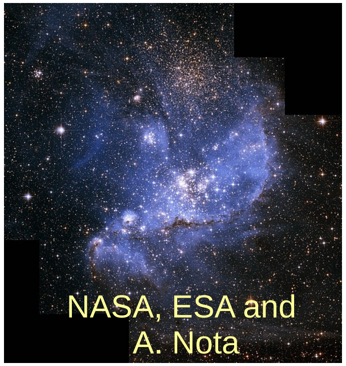
Knowing the composition of the stellar population (mass spectrum, age spectrum, metallicity spectrum) would be sufficient to estimate the mass of the galaxy.

Stars form in star clusters. Thus, the mass spectrum of stars in a galaxy is the combined mass spectrum of the stars in all its star clusters.

The mass spectrum of forming stars was long found to be remarkably invariant in star clusters.

It is quantified with the canonical **stellar initial mass function (IMF)**

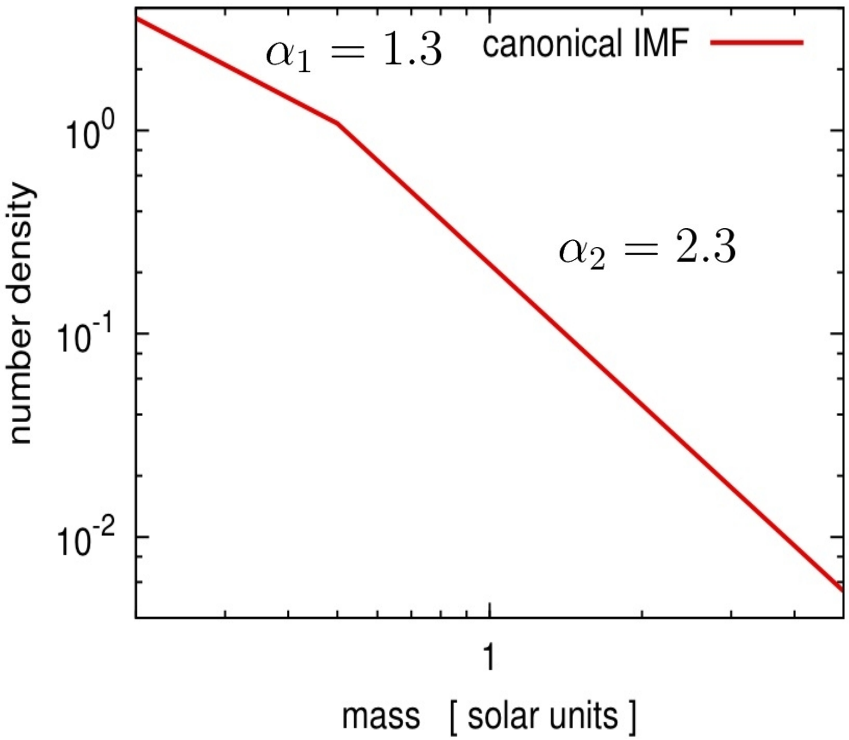


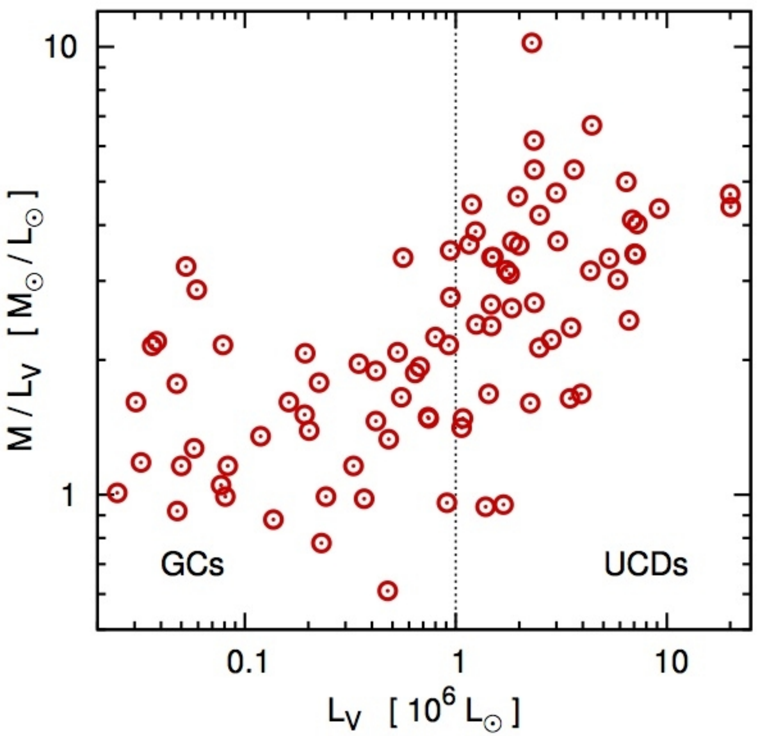


The mass of UCDs

The canonical IMF is the standard assumption when estimating the **stellar mass** of a star cluster or a galaxy based on their photometry and / or spectroscopy and comparing it to its **dynamical mass.**

How does this work for UCDs?

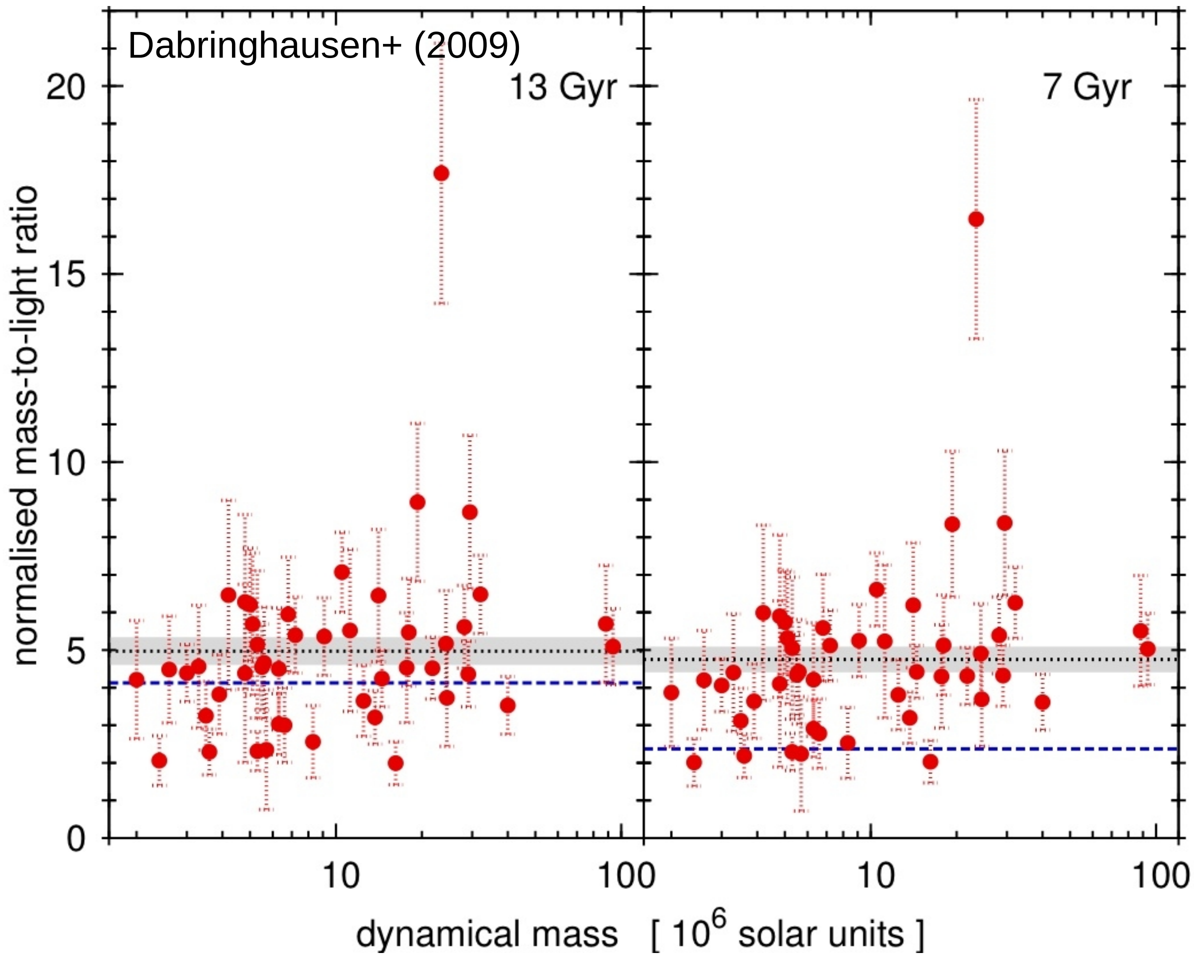




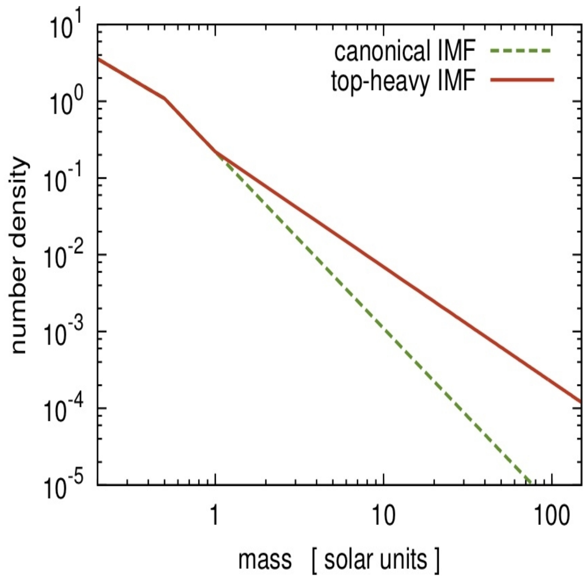
The mass of UCDs

The canonical IMF is the standard assumption when estimating the **stellar mass** of a star cluster or a galaxy based on their photometry and / or spectroscopy and comparing it to its **dynamical mass.**

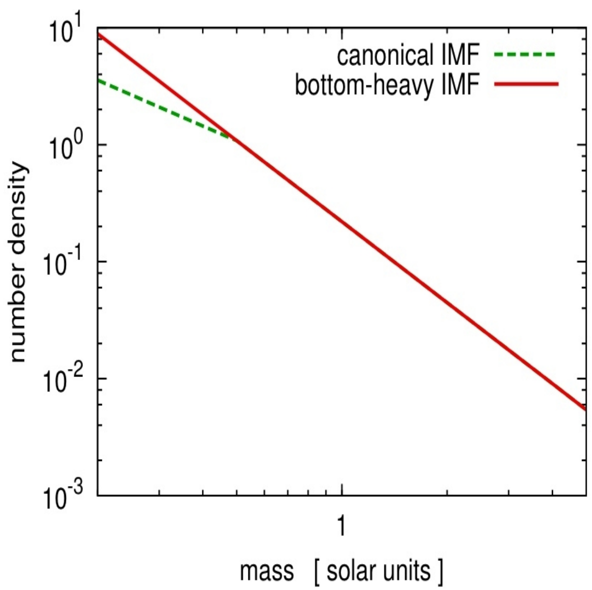
The dynamical mass-to-light ratios of UCDs are inconsistent with the mass-to-light ratios implied by any realistic stellar population, as long as the canonical IMF is assumed.



Explanations for the masses of UCDs

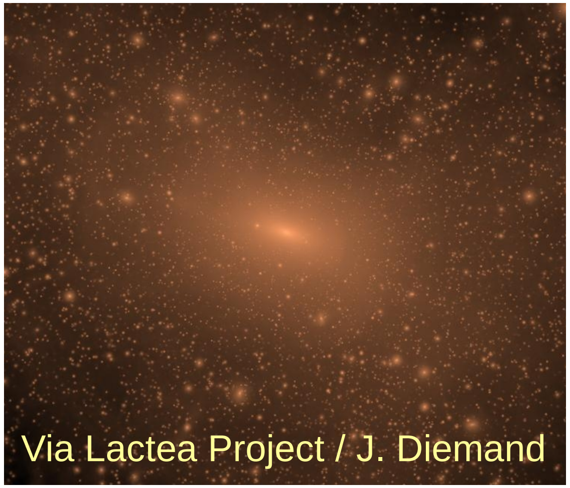


A top-heavy IMF – in old stellar populations, massive stars have turned into neutron stars (NSs) that add nothing to the luminosity. M/L-ratios and NS-frequencies may agree. (Dabringhausen+ 2009, 2012)

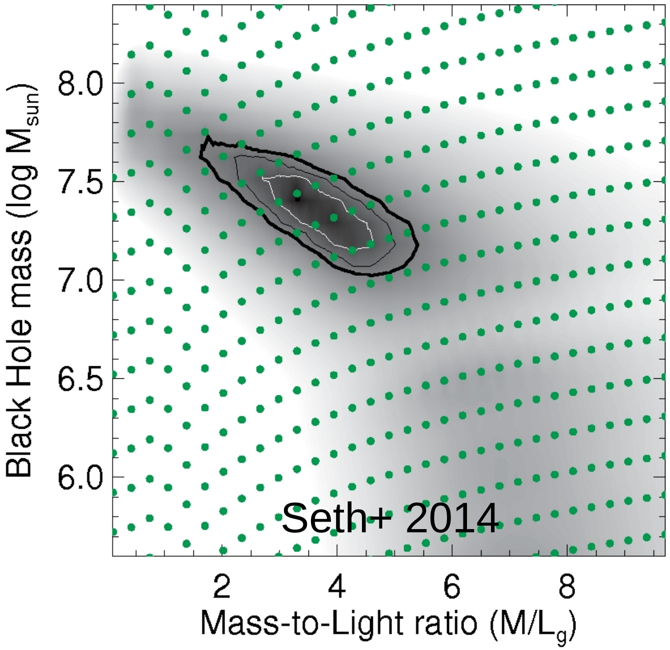


A bottom-heavy IMF – low-mass stars have high mass-to-light ratios. A search of spectral features characteristic for low-mass stars were inconclusive so far (Mieske & Kroupa 2008)

Explanations for the masses of UCDs



Non-baryonic dark matter (DM) – UCDs have been speculated to be small, primordial galaxies in DM halos, but DM-halos are not compact enough to have an impact on dynamics inside UCDs (Murray 2009)

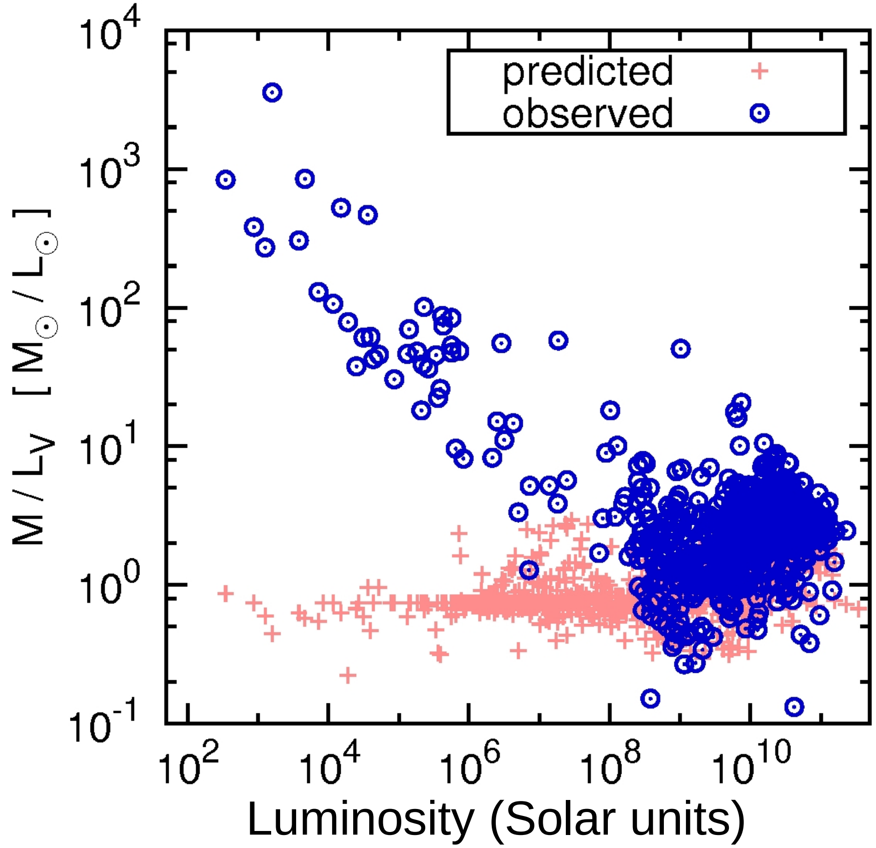


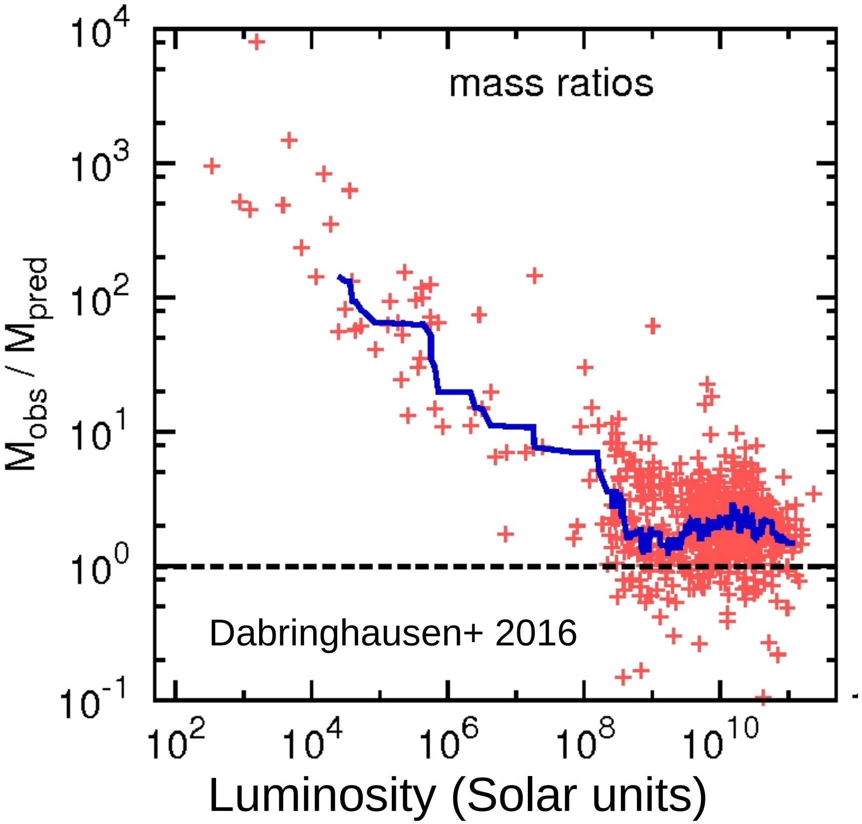
Super-massive black holes – would be expected if UCDs are remants of larger galaxies, and the best explanation for the UCDs with the most extreme M/L-ratios (Mieske+ 2013, Seth+ 2014, Janz+ 2015)

The mass of elliptical galaxies

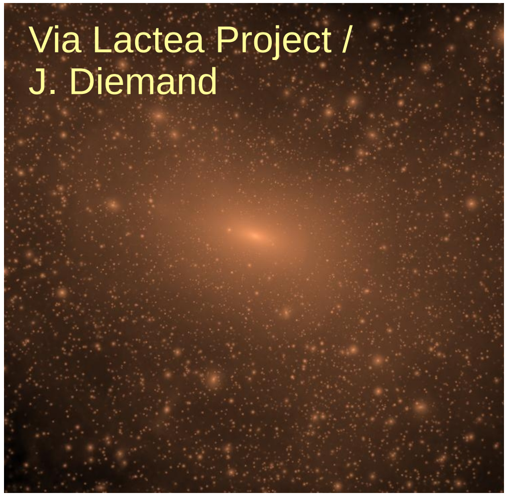
The canonical IMF is the standard assumption when estimating the **stellar mass** of a star cluster or a galaxy based on their photometry and / or spectroscopy and comparing it to its **dynamical mass.**

This does not work well for elliptical galaxies, dwarf or giant.

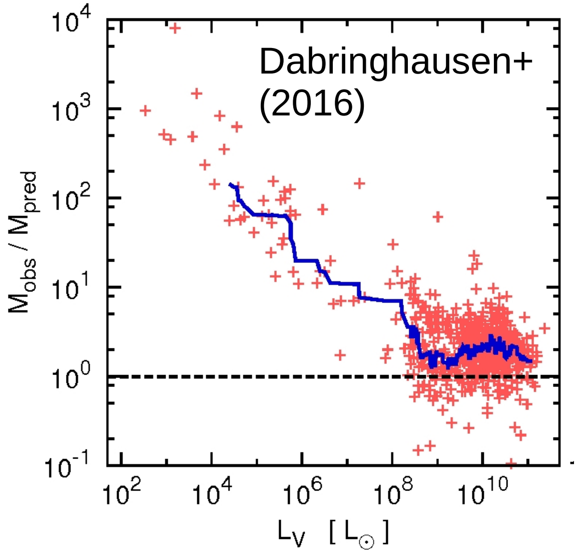


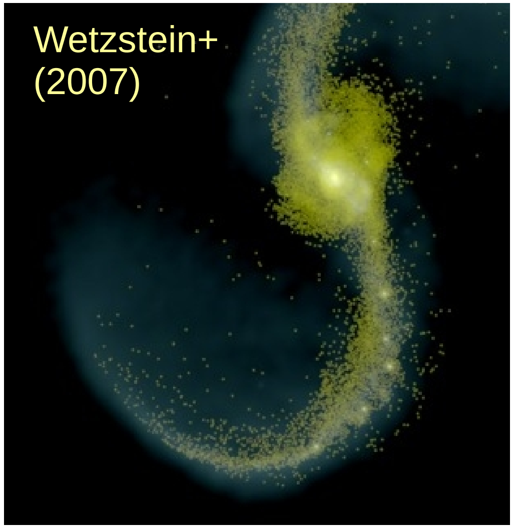


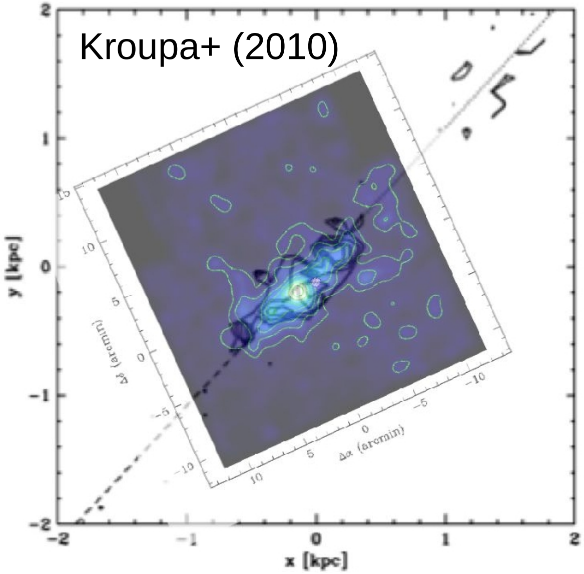
Explanations for the masses of dEs



Non-baryonic dark matter – according to the Λ CDM-model, galaxies form in DM-haloes, which would explain the high M/L-ratios (e.g. Mateo 1998, Strigari+ 2008)







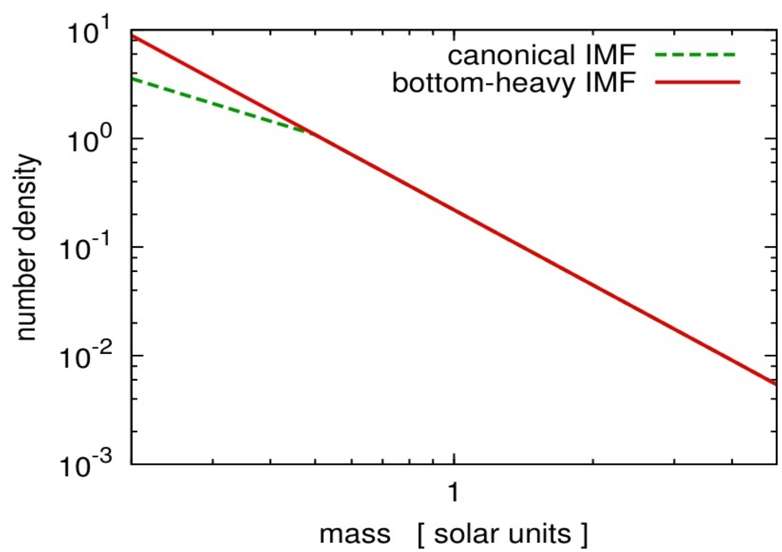
Tidal fields – non-equilibrium dynamics leads to higher velocity dispersions, which seem like high M/L-ratios if systems are assumed in equilibrium (e.g. Kroupa 1997, Dominguez+ 2016)

Explanations for the masses of dEs

Unidentified binaries – they increase the observed velocity dispersion, but their contribution is not linked to the potential, and thus the mass of a galaxy. Doesn't play a big role in practize. (McConnachie+ 2010, Dabringhausen+ 2016).

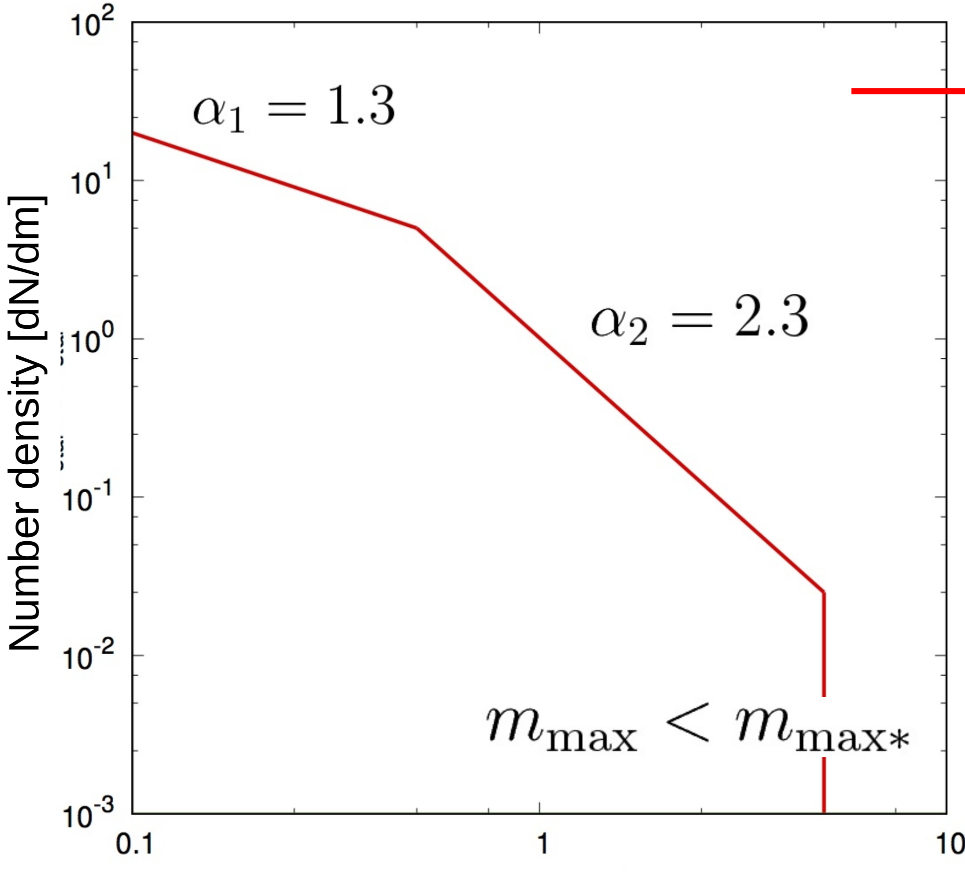
Modified gravitational dynamics – Modified Newtonian Dynamics (MOND, Milgrom 1983) increases the gravitational forces in the limit of very small space-time curvature. Nicely explains the dynamics of spiral galaxies, but insufficient for dEs (Dabringhausen+ 2016).

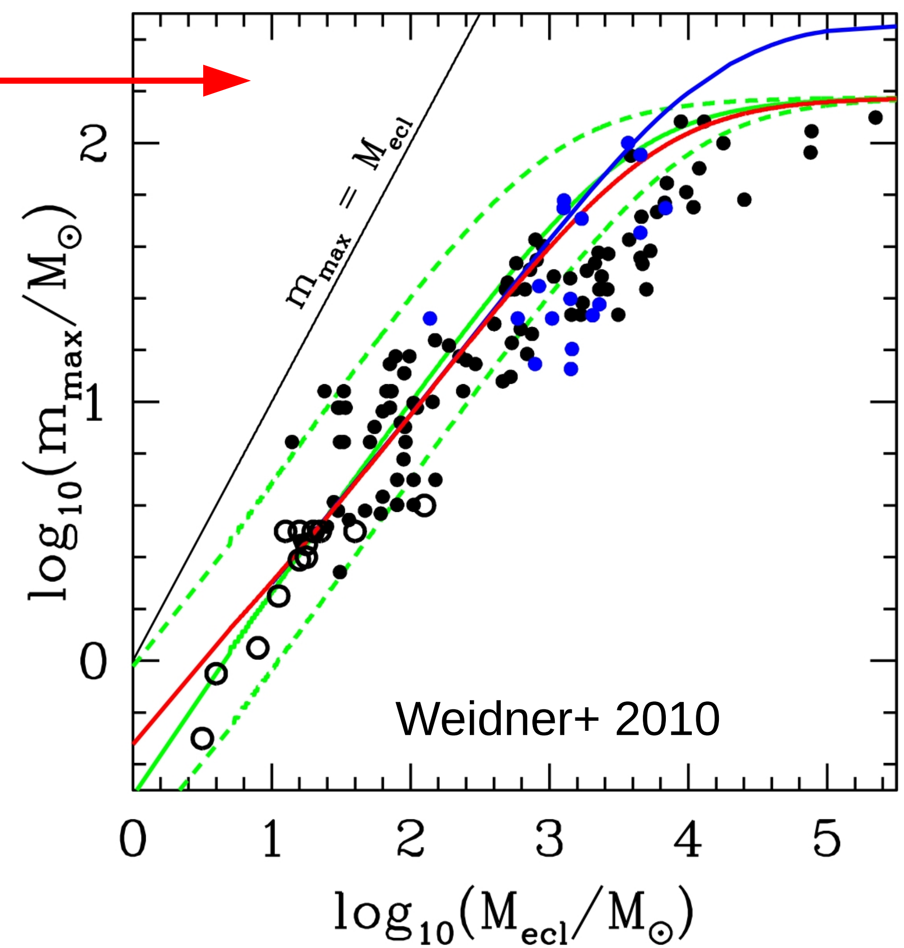
Non-canonical IMF – Are there (early-type) galaxies, whose M/L-ratios could be explained with a variation of the IMF?



Variation of the IMF

1. A variation of the upper mass limit

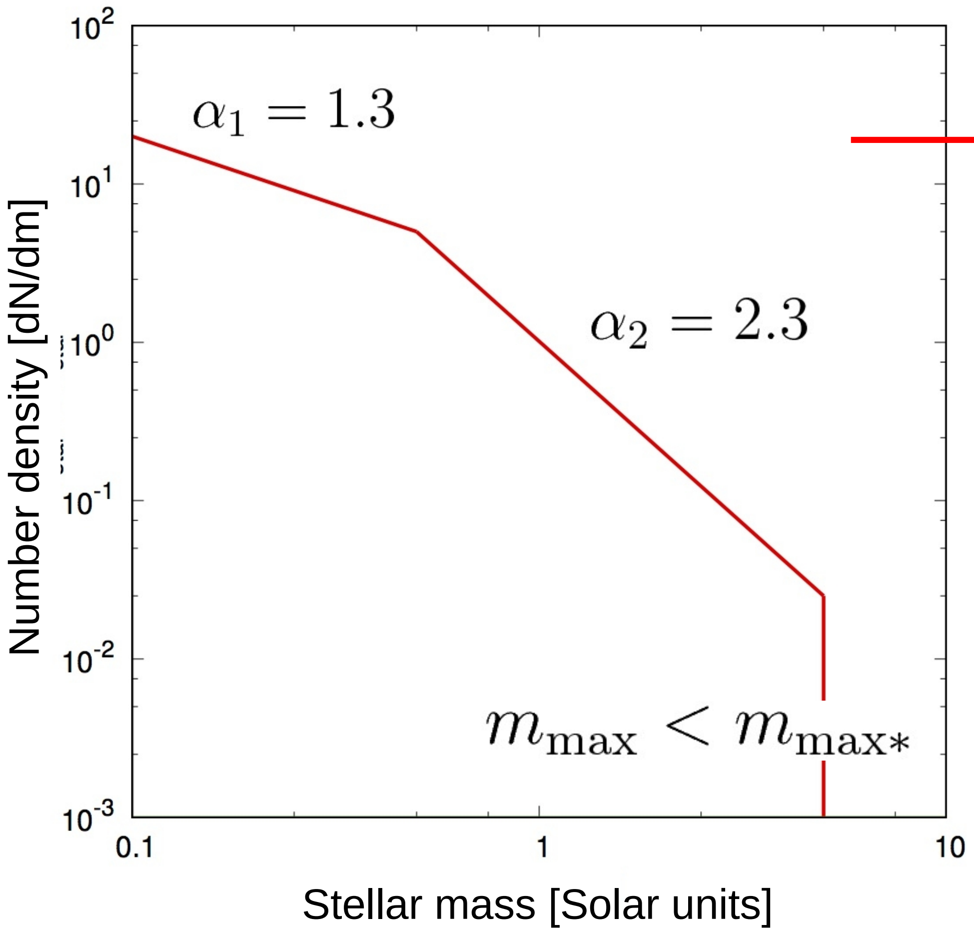


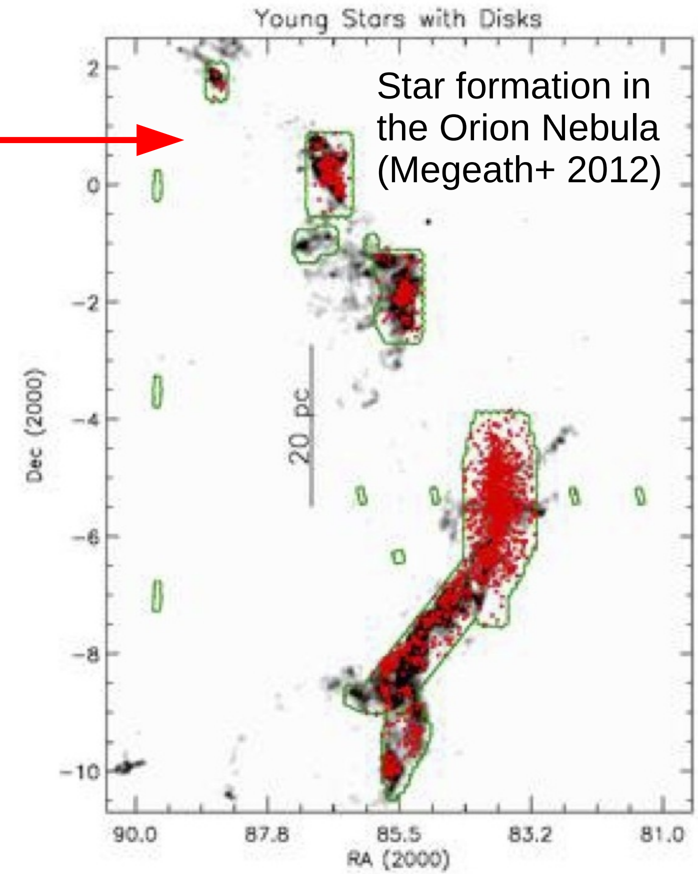


The mass of the most massive star in a star cluster depends on the total mass of the star cluster where it formed.

Variation of the IMF

1. A variation of the upper mass limit

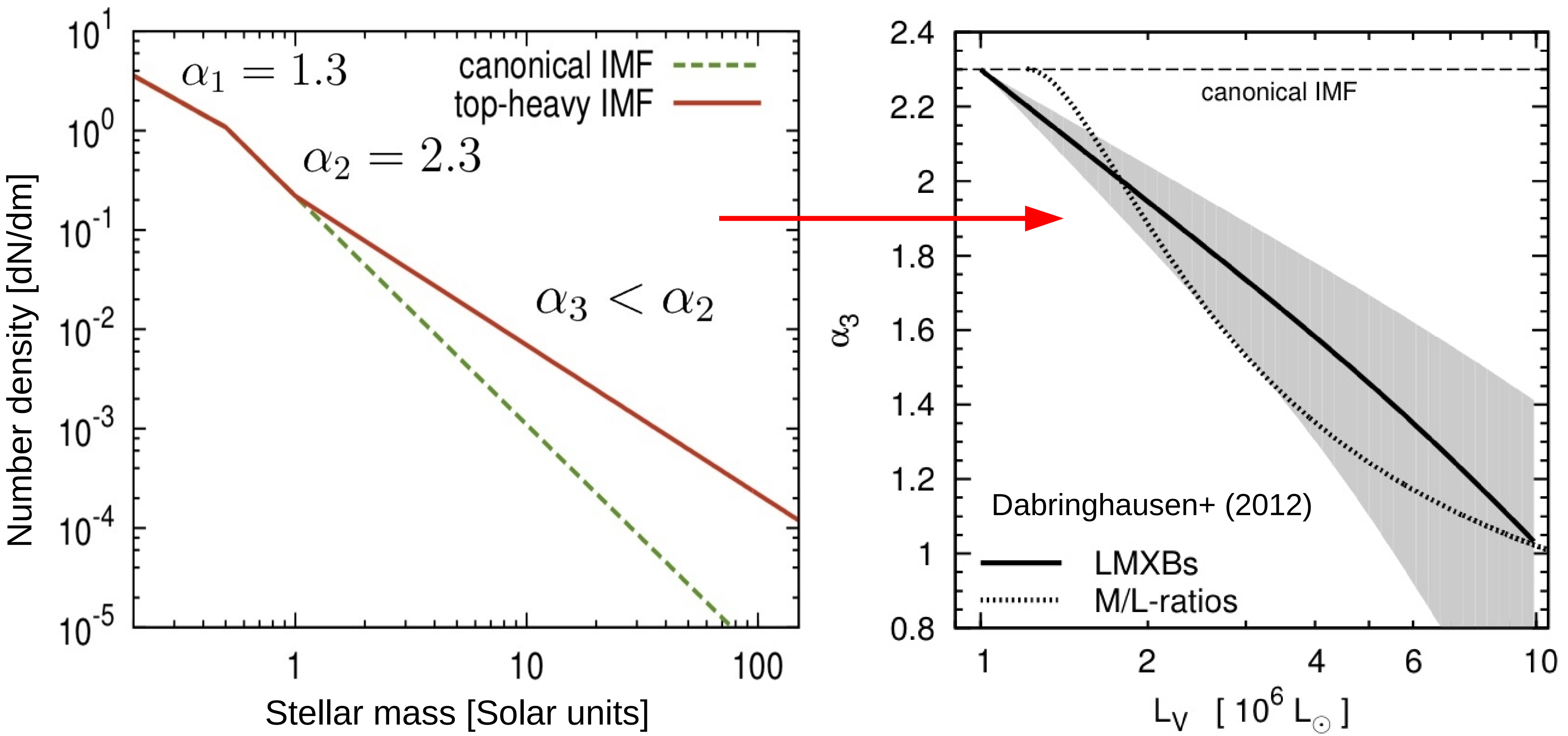




The mass of the most massive star in a star cluster depends on the total mass of the star cluster where it formed.

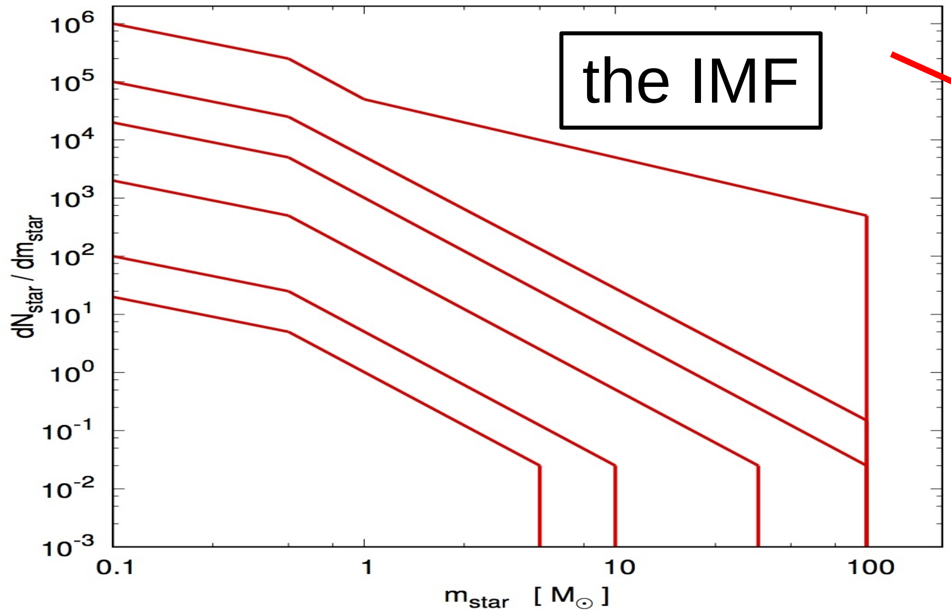
Variation of the IMF

2. A variation of the high-mass IMF slope

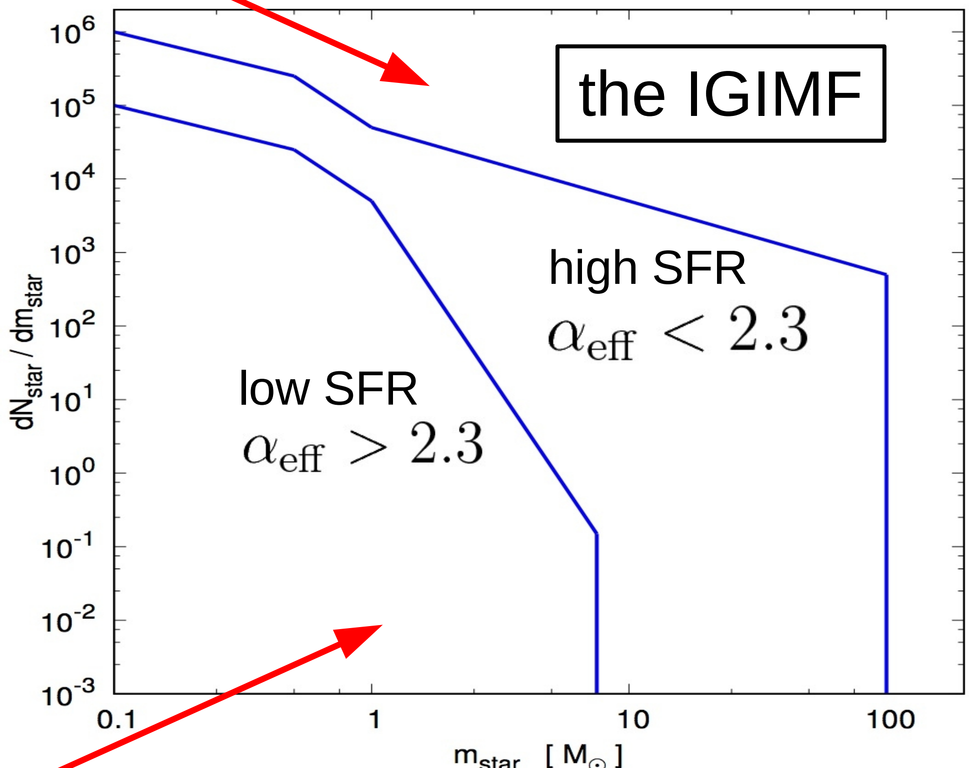


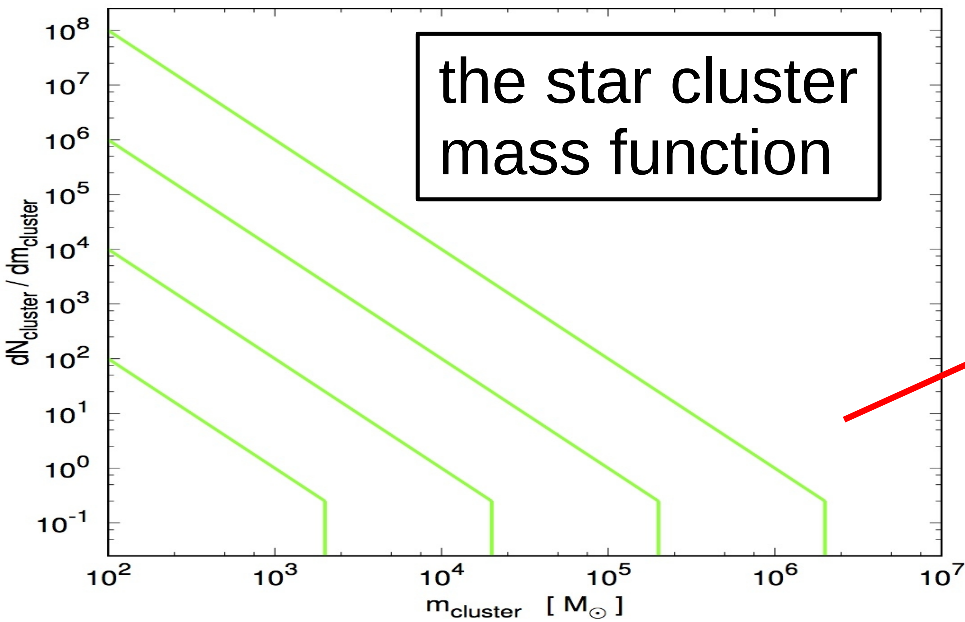
The mass-to-light ratios and the number of neutron star detections suggest consistently that massive star clusters (i.e. UCDs) have over-proportionally many massive stars.

The galaxy-wide stellar initial mass function (IGIMF)



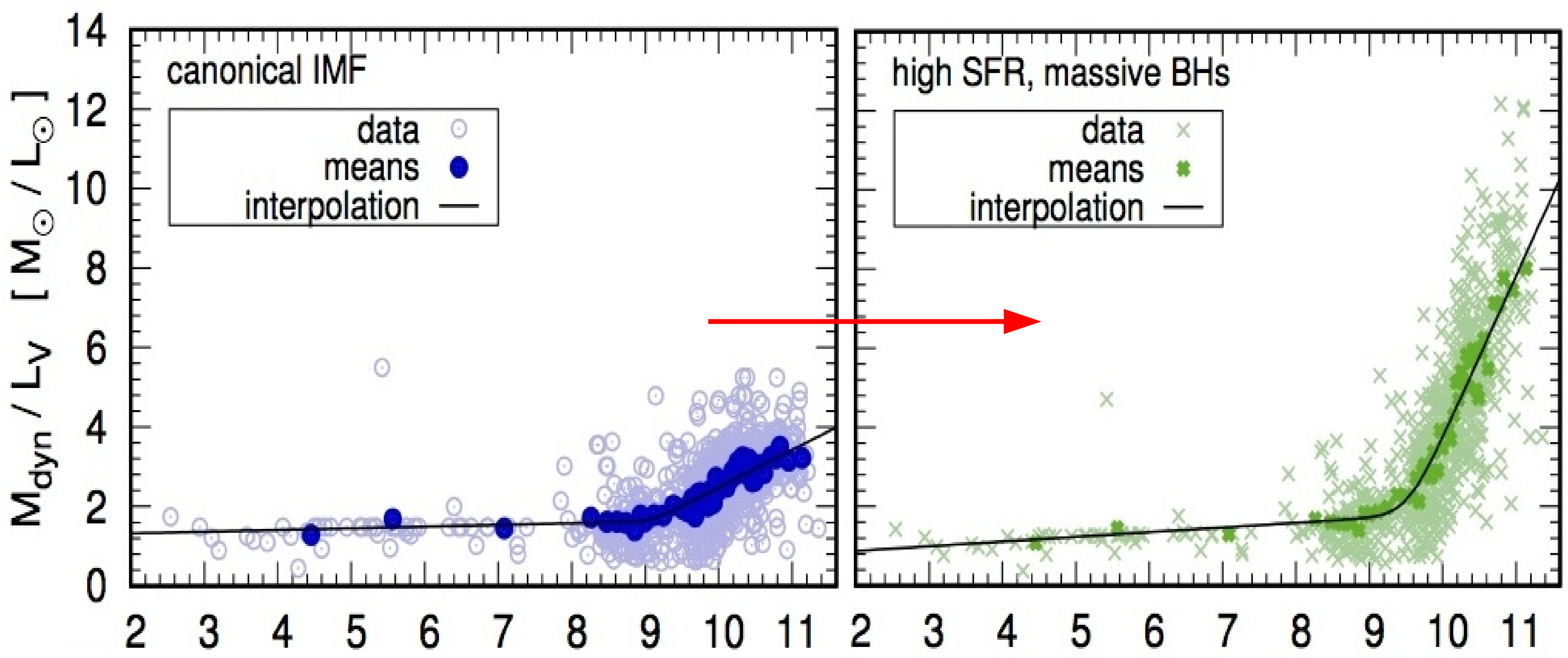
The IMF depends on star-cluster mass.





The star formation rate of a galaxy determines the mass of the most massive star cluster.

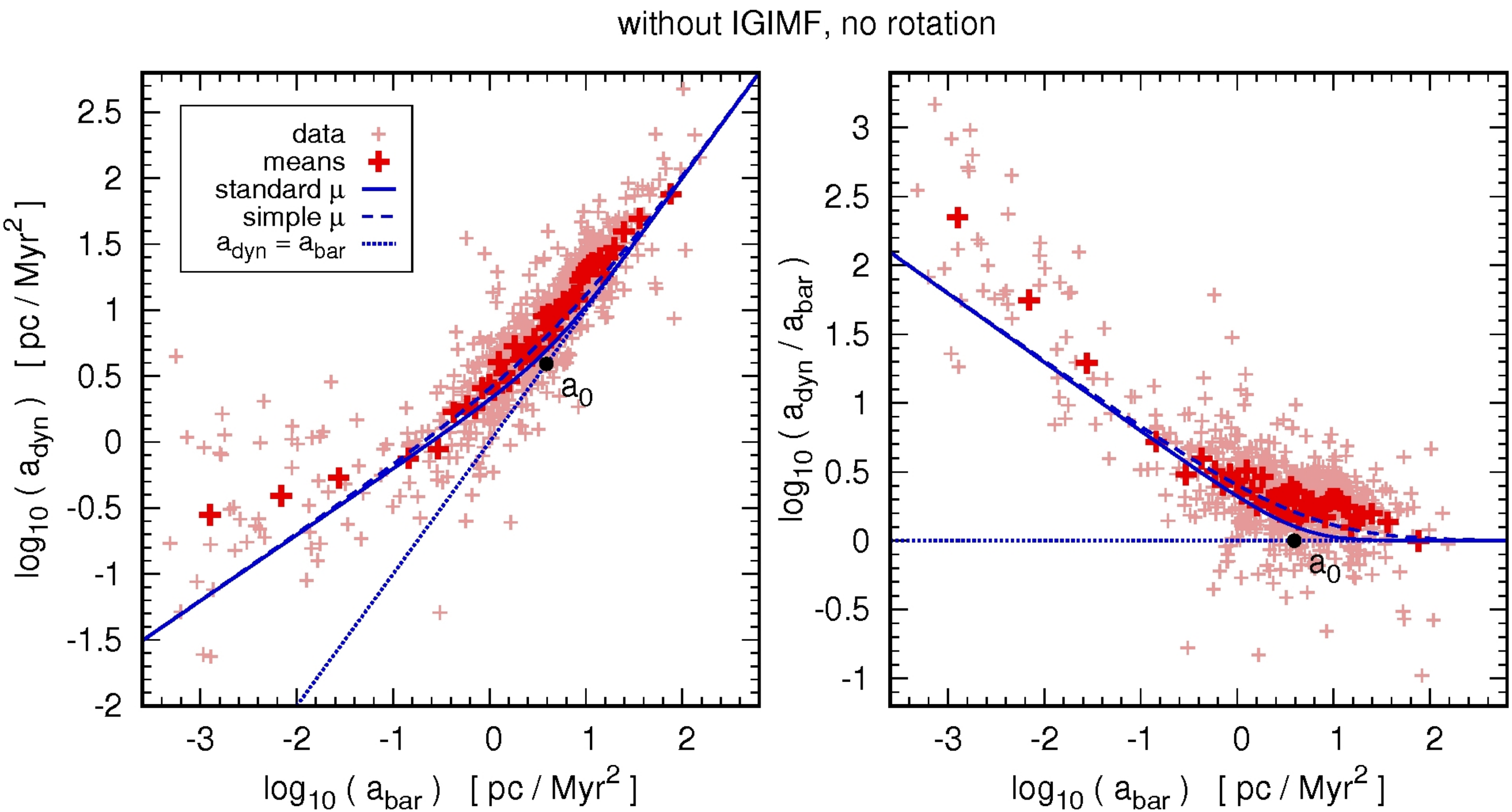
The galaxy-wide stellar initial mass function (IGIMF)



Luminosity [Solar units]

Luminosity [Solar units]

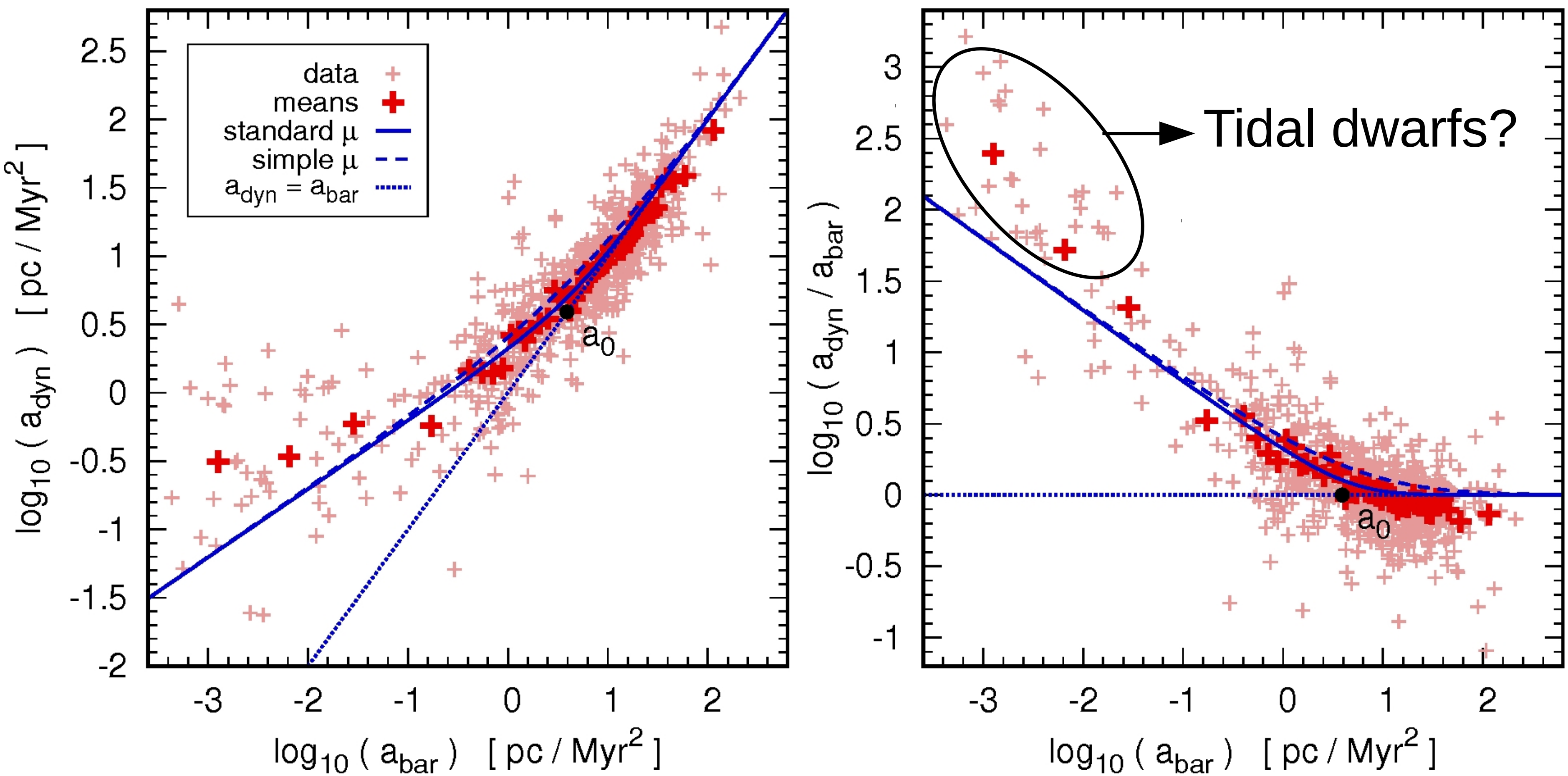
The mass of elliptical galaxies



With the canonical IMF, the accelerations inferred from the internal dynamics of early-type galaxies are systematically higher than the ones predicted from the mass of their stars.

The mass of elliptical galaxies

with IGIMF, high SFR, massive BHs, no rotation



By considering the effect of the IGIMF, the masses of the more massive early-type galaxies can easily be explained with stellar remnants. Are tidal fields bringing the low-mass galaxies out of equilibrium?