

Stellar capture by an accretion disc

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ABSTRACT

The long-term evolution of a stellar orbit captured by a massive galactic centre via successive interactions with an accretion disc has been examined. An analytical solution describing the evolution of the stellar orbital parameters during the initial stage of the capture has been found. Our results are applicable to thin Keplerian discs with an arbitrary radial distribution of density and a rather general prescription for the star–disc interaction. Temporal evolution is given in the form of quadrature which can be carried out numerically.

Key words: accretion, accretion discs – celestial mechanics, stellar dynamics – stars: kinematics – galaxies: nuclei.

1 INTRODUCTION

Evolution of the orbit of a star under the influence of interactions with an accretion disc has been studied by numerous authors because this situation is relevant to the inner regions of active galactic nuclei. The trajectory of an individual star is determined mainly by the gravity of the central mass and the surrounding stars while periodic transitions through the disc act as a tiny perturbation. The final goal is to understand the fate of a star, and the transfer of mass and angular momentum between the star and the disc, and also to determine how star–disc interactions influence the distribution of stellar orbits near a massive central object. An important and difficult task is to estimate the probability that a star is captured from an originally unbound orbit, and to determine whether this probability is significant compared with other mechanisms of capture.

The orbital evolution of a body crossing an accretion disc has been discussed with various approaches, first within the framework of Newtonian gravity, both in the theory of the Solar system (Pollack, Burns & Tauber 1979; McKinnon & Leith 1995) and for active galactic nuclei (Goldreich & Tremaine 1980; Syer, Clarke & Rees 1991; Artymowicz 1994; Podsiadlowski & Rees 1994). These studies have been generalized in order to account for the effects of general relativity (Vokrouhlický & Karas 1993) and to model a dense star cluster in a galactic nucleus (Pineault & Landry 1994; Rauch 1995). It has been recognized that detailed physical description of the star–disc interaction is a difficult task (Zurek, Siemiginowska & Colgate 1994, 1996). In this Letter a simplified analytical treatment of stellar orbital parameters is presented, describing the initial stage of star–disc collisions (when the star crosses the disc once per revolution). A great deal of our calculation is independent of the microphysics of star–disc interaction. We show how our solution matches the corresponding Rauch (1995)

solution which is valid in later stages, when the eccentricity of the orbit becomes small enough.

In the next section our approach to the problem is formulated and an analytical solution is given. Then, in Section 2.2, further details about the derivation of the results are presented, and finally a simple example of the orbital evolution is shown in Section 2.3.

2 STELLAR CAPTURE BY A DISC

2.1 Formulation and results

The Newtonian gravitational law is assumed throughout this paper. Our solution is based on the following assumptions:

- (i) the disc is geometrically thin and its rotation is Keplerian;
- (ii) at the moment of crossing the plane of the disc, the velocity of the star is changed by a tiny amount; this impulse is collinear with the relative velocity of the star with respect to the material forming the disc;
- (iii) the star crosses the disc once per revolution (the model is applicable to the initial phase of the stellar capture).

Condition (i) is a standard simplification in which the disc is treated in terms of vertically integrated quantities, while (ii) can be expressed by the formula for an impulsive change of the star's velocity:

$$\delta\mathbf{v} = \Sigma(\mathbf{r}, \mathbf{v}) \mathbf{v}_{\text{rel}}; \quad (1)$$

Σ is an unconstrained function, given by a detailed model of the star–disc interaction, and \mathbf{v}_{rel} is the relative velocity of the star and the disc material. We stress that our results are uniquely based on this assumption of collinearity, $\delta\mathbf{v} \propto \mathbf{v}_{\text{rel}}$; the coupling factor Σ is arbitrary and it can be as complex as necessary. In particular, Σ contains information about the surface density κ of the disc ($\kappa = 0$ outside the outer edge $r = R_d$ of the disc). The form of Σ must be specified only for examination of the temporal evolution of the orbital parameters. We will assume, in analogy with Rauch (1995),

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a simplified formula for

$$\Sigma(\mathbf{r}, \mathbf{v}) = -\frac{\pi R_\star^2}{m_\star} \kappa(r) \frac{v_{\text{rel}}}{v_\perp} \xi, \quad (2)$$

$$\xi \approx 1 + \left(\frac{v_\star}{v}\right)^4 \ln \Lambda, \quad (3)$$

when it is needed for purpose of an example. In equation (2) R_\star denotes the radius of the star, m_\star is its mass, v_\star is the escape velocity ($v_\star^2 = 2Gm_\star/R_\star$); v_\perp is the normal component of the star's velocity to the disc plane, and $\ln \Lambda$ is the usual long-range interaction factor (Coulomb logarithm).

The star's orbit is traditionally characterized by the Keplerian osculating elements: semimajor axis a , eccentricity e , inclination I to the accretion disc plane, and longitude of pericentre ω (measured from the ascending node). A derived set of parameters turns out to be better suited to our problem: $\alpha = 1/a$, $\eta = \sqrt{1 - e^2}$, $\mu = \cos I$, and $k = e \cos \omega$. We will show (Section 2.2) that the evolution of a stellar orbit following the capture by a disc can be written in a parametrical form:

$$\alpha(\zeta) = \phi(\zeta) \{ \alpha_0 \phi^{-1}(\zeta_0) + \sigma^2 [\psi(\zeta) - \psi(\zeta_0)] \}, \quad (4)$$

$$\eta^2(\zeta) = \zeta \phi(\zeta) \{ \alpha_0 \phi^{-1}(\zeta_0) \sigma^{-2} + [\psi(\zeta) - \psi(\zeta_0)] \}, \quad (5)$$

$$\mu(\zeta) = \sqrt{\zeta} + \theta(\zeta), \quad (6)$$

$$|k(\zeta)| = \zeta - 1, \quad (7)$$

where the auxiliary functions $\phi(\zeta)$, $\theta(\zeta)$ and $\psi(\zeta)$ read

$$\phi(\zeta) = \frac{1}{C} \left(1 \pm \sqrt{1 - C + C\zeta} \right)^2, \quad (8)$$

$$\theta(\zeta) = \mp \frac{1}{C} \sqrt{\frac{1 - C + C\zeta}{\zeta}} \left(1 \pm \sqrt{1 - C + C\zeta} \right), \quad (9)$$

$$\psi(\zeta) = \frac{1}{1 \pm \sqrt{1 - C + C\zeta}} \left(2 + \frac{C}{1 \pm \sqrt{1 - C + C\zeta}} \right). \quad (10)$$

The formal parameter ζ of the solution decreases from its initial value $\zeta_0 = 1 + e_0 |\cos \omega_0|$ to the final value ζ_f , given by

$$\zeta_f = \frac{2R_d \sigma^2}{1 + R_d \sigma^2}. \quad (11)$$

At this instant, the orbit starts crossing the disc twice per revolution and our solution ceases to be applicable. Obviously, the integration constants in (4)–(10) are determined in terms of the initial Keplerian orbital elements (a_0, e_0, I_0, ω_0) by

$$\alpha_0 = \frac{1}{a_0}, \quad (12)$$

$$\zeta_0 = 1 + e_0 |\cos \omega_0|, \quad (13)$$

$$\sigma^{-2} = \frac{a_0 \eta_0^2}{\zeta_0}, \quad (14)$$

$$C = \frac{z_0(z_0 + 2)}{(z_0 + 1)^2} \frac{1}{1 - \zeta_0}, \quad (15)$$

where

$$z_0 = -\frac{1 - \zeta_0}{\sqrt{\zeta_0}(\cos I_0 - \sqrt{\zeta_0})}. \quad (16)$$

The upper signs in (8)–(10) are for the initial inclination I_0 greater than a critical value I_\star given by

$$\cos I_\star = \frac{1}{\sqrt{\zeta_0}}; \quad (17)$$

the lower signs apply otherwise. The integration constant C is singular ($C \rightarrow \infty$) for $I_0 = I_\star$ ($z_0 = -1$), and the solution can be simplified further. For instance, $\mu(\zeta) = 1/\sqrt{\zeta}$ for all values of ζ down to ζ_f .

Solution (4)–(7) can be extended easily to the case of initially parabolic orbits by setting $\alpha_0 = 0$, $e_0 = 1$ and $\sigma^2 = (\zeta_0/2R_p)$. Here, R_p denotes the pericentre distance of the initial parabolic orbit.

It is worth mentioning that the parameter ζ does not determine the time-scale on which the evolution takes place. Indeed, equations (4)–(7) do not provide temporal information because it depends on the precise form of Σ in equation (1). On the other hand, the strength and the beauty of the parametric solution (4)–(7) lie in its independence of a particular model for Σ . We will also illustrate an example of temporal evolution later in the text, and only for this purpose will the form of Σ be needed. Assuming relation (2)–(3) one obtains

$$t - t_0 = \frac{\pi}{\sigma^3 \sqrt{GM\Sigma_c}} \int_{\zeta}^{\zeta_0} \frac{\sigma^3 dz}{z^{1/2} \alpha^{3/2}(z) \nu(z) \theta(z)} \quad (18)$$

where t_0 is the initial time, M is the central mass, and $\Sigma_c = (\pi R_\star^2/m_\star) \kappa(r_c) \xi$ with $r_c = \sigma^{-2}$ being the radial distance of the point of intersection with the disc. Function $\nu = v_{\text{rel}}/v_\perp$ is determined by orbital parameters which themselves depend on ζ according to equations (4)–(7).

We note that Rauch (1995) conjectured that the function $R = a\eta^2 \cos^4(I/2)$ remains nearly conserved along the evolving stellar orbit, and he used this function for estimates of the radius of the final, circularized orbit in the disc plane. In our notation,

$$R(\zeta) = \frac{\zeta}{4\sigma^2} \left[1 + \sqrt{\zeta} + \theta(\zeta) \right]^2. \quad (19)$$

Hereinafter, we show that $R(\zeta)$ is a well-conserved quantity at later stages of the orbital evolution (when eccentricity is sufficiently small), but it fails to be conserved at the very beginning of the capture when the orbit is still nearly parabolic, close to an unbound trajectory.

2.2 Details of the solution

In this section, we present more details of the derivation of the solution given above.

Taking into account the fundamental assumption (1), one can easily demonstrate that the Keplerian orbital elements are perturbed at each transition (due to interaction with the disc material) by quantities

$$\delta(\sqrt{a}) = \Sigma \sqrt{a} \eta f_1, \quad (20)$$

$$\delta(\sqrt{a} \eta) = \Sigma \sqrt{a} \eta f_2, \quad (21)$$

$$\delta(\sqrt{a} \eta \mu) = \Sigma \sqrt{a} \eta f_3, \quad (22)$$

$$\delta(k) = 2\Sigma(1 + k) f_2. \quad (23)$$

Here, we introduced auxiliary functions

$$f_1 = \eta^{-3} \left\{ (1+k) \left[1 - \mu \sqrt{1+k} \right] + e^2 + k \right\}, \quad (24)$$

$$f_2 = 1 - \frac{\mu}{\sqrt{1+k}}, \quad (25)$$

$$f_3 = \mu - \frac{1}{\sqrt{1+k}}. \quad (26)$$

Combining equations (21) and (23) we find that

$$\frac{\sqrt{1+k}}{\sqrt{a\eta}} = \sigma \quad (27)$$

is conserved at the star–disc interaction. Hence, σ is constant whatever the evolution of elements a , e and k . In fact, condition (27) states that the initial Keplerian orbit has the same radius of intersection as the final orbit, after successive interactions with the disc. The longitude of the node is also conserved and can be set to zero without loss of generality. The above-given formulae (20)–(27) correspond to $k > 0$ (i.e. $|\omega| < \pi/2$); for $k < 0$ one should replace $k \rightarrow -k$. We note that all these relations can be easily reparametrized in terms of the binding energy $E = GM/(2a)$, the angular momentum $L = \sqrt{GMa\eta}$, and the component of the angular momentum with respect to the axis, $L_z = \sqrt{GMa\eta}\mu$.

Combining equations (21) and (22) with the help of (27), and introducing auxiliary variables $y = \sqrt{a\eta}\mu$ and $x = \sqrt{a\eta}$, we obtain the differential equation

$$\frac{dy}{dx} = \frac{x(\sigma y - 1)}{\sigma x^2 - y}. \quad (28)$$

(We were allowed to change variations, δ , to differentials, d , assuming an infinitesimally small perturbation of the stellar orbit at each intersection with the disc.) The Abel-type differential equation (28) can be solved beautifully by standard methods of mathematical analysis (see, e.g., Kamke 1959). An appropriate change of variables gives directly a solution for the evolution of inclination, equation (6).

Similarly, considering (20) and (21) in terms of $\alpha = 1/a$, we obtain, after brief algebraic transformations,

$$-\sqrt{\xi}\theta(\xi) \frac{d\alpha}{d\xi} + \alpha(\xi) = \sigma^2 \left[2 - \xi + \sqrt{\xi}\theta(\xi) \right] \quad (29)$$

with $\theta(\xi)$ given by equation (9). This is a linear differential equation, integration of which yields $\alpha(\xi)$ and then, by equations (4)–(5), also $\eta(\xi)$.

At this point, one can see that Rauch’s (1995) ‘quasi-integral’ R is changed at each passage across the disc according to

$$\delta(\ln R) = 2\Sigma \left(1 - \frac{1}{\sqrt{1+k}} \right). \quad (30)$$

Realizing that $k \approx e$ we conclude that $\ln R$ indeed stays nearly constant at later stages of the orbit evolution, when eccentricity has decreased enough. On the other hand, at the very beginning of the capture process, when eccentricity is still high, R fails to serve as a quasi-integral of the problem. Instead, its evolution is given by equation (19).

For temporal evolution, equations (20)–(27) must be supplemented by an additional relation,

$$\delta(t) = \frac{2\pi}{\sqrt{GM}} a^{3/2}, \quad (31)$$

which determines the interval between successive intersections with the disc. Combining equation (31) with (27) one obtains a

separated differential equation which yields formula (18). Recall that this last step requires assumption (2) about the form of Σ . In the present case,

$$\nu(\xi) = \frac{1}{\xi} \sqrt{\frac{2\xi(1-\xi-\sqrt{\xi}\theta) + \xi - \eta^2}{1-\xi-2\sqrt{\xi}\theta-\theta^2}}. \quad (32)$$

The relation for time is apparently too complicated to be integrated analytically, but numerical evaluation is straightforward.

2.3 Example

We shall briefly demonstrate some properties of the analytical solution from Section 2.1.

We examine parabolic orbits ($\alpha_0 = 0$, $e_0 = 1$) with the pericentre in the disc plane ($\omega_0 = 0$), and the pericentre distance R_p equal to the disc radius ($R_p = R_d = r_c$). The initial inclination I_0 of the orbit to the disc plane is a free parameter in this example. Evolution of this set of orbits is split into two phases.

First, we let the orbits evolve according to the solution of equations (4)–(7) from the initial value $\xi_0 = 2$ of the formal parameter ξ to its final value $\xi_f = 1$. Fig. 1 illustrates the evolution of the inclination $I(\xi)$, measured in terms of the initial value I_0 . Notice that the critical inclination I_* of equation (17) is 45° . The eccentricity of the orbits under consideration decreases according to a simple formula $e(\xi) = \xi - 1$ (independently of I_0), leading eventually to circularized orbits at $\xi = \xi_f$. We observe that orbits with $I_0 < I_*$ terminate at the final state $I_f = 0$, suggesting that the circularization time-scale is comparable to that necessary for tilting the orbit into the disc plane. On the other hand, when $I_0 > I_*$ the final circular orbits remain inclined significantly to the disc plane. ($I > 90^\circ$ corresponds to retrograde orbits.) Hence, for those orbits the circularization time-scale is shorter than the time necessary for tilting the orbit to the disc plane. Additional time to

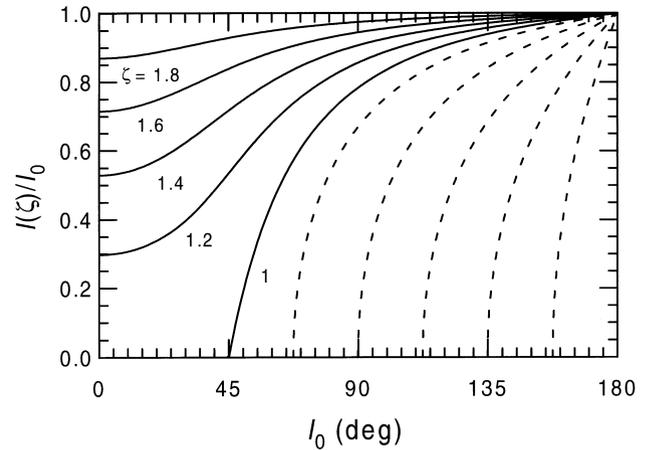


Figure 1. Inclination $I(\xi)$ of a captured stellar orbit (measured in units of the initial inclination I_0) versus I_0 . This graph corresponds to initially parabolic orbits with pericentre distance equal to the disc radius. The curves are parametrized by ξ . Temporal evolution of some particular orbit starts with $I = I_0$, $\xi = 2$, and goes down along the vertical line, across $\xi = \text{constant}$ curves. Our analytic solution (solid lines) is valid in the region $2 \geq \xi \geq 1$, and it corresponds to eccentric orbits. The circularization time-scale is equal to the time-scale necessary for tilting the orbit to the disc plane if $I_0 < I_* = 45^\circ$, while the former is shorter than the latter for orbits with $I_0 > I_*$. At $\xi = 1$, i.e. zero eccentricity, our solution matches Rauch’s $R = \text{constant}$ solution (dashed lines).

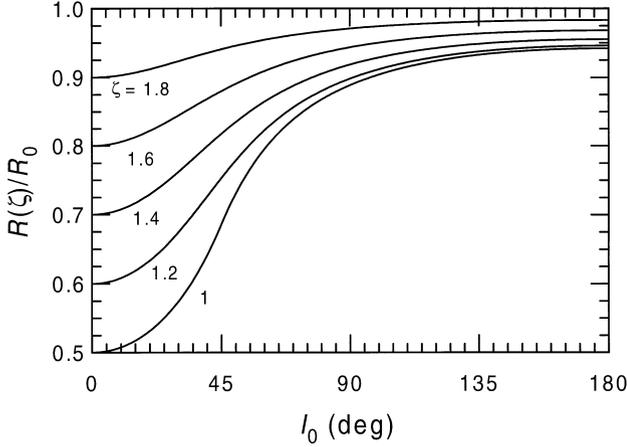


Figure 2. Function R (normalized to its initial value R_0) versus the initial inclination I_0 of the orbit for different values of parameter ζ . The ζ -dependence is stronger for orbits with smaller values of I_0 . Once the orbit becomes circular with $\zeta = 1$, R reaches its terminal value and does not change any more. Therefore, R does not acquire values below the bottom curve.

incline a circularized orbit is not much longer than the circularization time, however.¹ The difference is typically a factor of 10 for highly retrograde orbits.

Secondly, we examine the evolution of circularized orbits which started with $I_0 > I_*$ and have settled to non-zero $I(\zeta = 1)$. Because these orbits have zero eccentricity, there exists Rauch's integral in the form $R_1 = \sigma^2 a \cos^4(I/2) = \zeta \cos^4(I/2)$. Here, we adopt a formal continuation of the ζ -parameter to values smaller than unity (in this phase, $\zeta = a/R_d$). For each orbit, we calculate the value of $R_1 \equiv R(\zeta = 1)$, so that the inclination is given by

$$\mu(\zeta) = \sqrt{\frac{4R_1}{\zeta}} - 1. \quad (33)$$

Obviously, a given orbit terminates its evolution at $\zeta = 4R_1$ when it is pushed completely into the disc plane. Dashed curves in Fig. 1 correspond to constant values of $\zeta < 1$.

Fig. 2 illustrates how function $R(\zeta)$ changes during the first circularization phase of the evolution. For each orbit we have chosen the same steps in ζ (0.2) in the range $1.8 \geq \zeta \geq 1.0$, as in Fig. 1, and we have computed the corresponding values of $R(\zeta)$ from equation (19). Our results agree with Rauch's (1995) finding, namely that $R(\zeta)$ is conserved up to a factor of ≈ 2 for orbits with large eccentricity. During the second phase of the evolution the R -function is constant.

Fig. 3 shows time intervals $t_e(\zeta)$ which elapse in the course of gradual circularization when the eccentricity decreases from $e = \zeta - 1$ to some terminal value (here, the terminal eccentricity has been fixed to $e = 10^{-3}$; notice that t_e goes to infinity for terminal eccentricity $e \rightarrow 0$). We have verified the graph also by direct numerical integration of the corresponding orbits. The numerical factor in front of the integral on the right-hand side of equation (18) can be written in physical units in the form

$$10^7 \left(\frac{r_c}{10^3 R_g} \right)^{9/4} \left(\frac{R_g}{10^5 R_\star} \right) \left(\frac{10^3 R_{g\star}}{R_\star} \right) \left(\frac{10^3}{\xi} \right) \text{yr}, \quad (34)$$

¹We thank the referee for pointing out this fact, confirmed also by other estimates (Syer et al. 1991; McKinnon & Leith 1995).

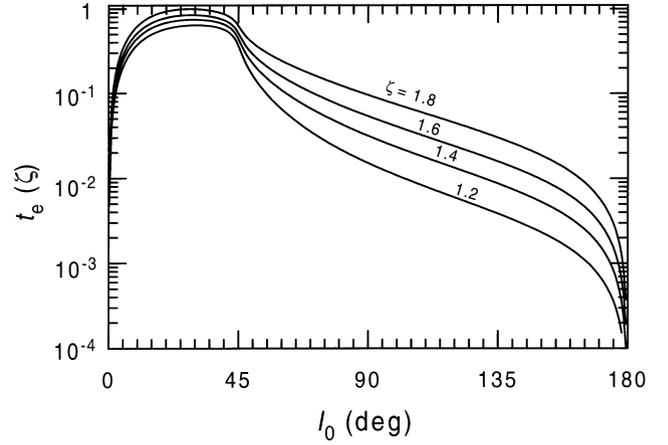


Figure 3. Time $t_e(\zeta)$ of orbital circularization of parabolic orbits with initial inclination I_0 , as in Fig. 1. Here, time (arbitrary units on the vertical axis) is recorded starting from eccentricity $e = \zeta - 1$ (given with each curve) down to $e = 10^{-3}$ (nearly circular orbit). Notice the change in form of the curves at the critical inclination $I_* = 45^\circ$.

$R_g = 2GM/c^2$ and $R_{g\star} = 2Gm_\star/c^2$ are the gravitational radii of the central mass and the star, respectively. A typical surface density profile of the disc has been assumed, as in equation (1) of Rauch (1995). The value of $\xi \approx 10^3$ corresponds to the estimate in addendum to Zurek et al. (1996).

3 CONCLUSION

We have found a solution describing the evolution of orbital parameters of a star orbiting around a massive central body in a galactic nucleus and interacting with a thin Keplerian disc. The solution is in a parametrical form valid for an *arbitrary radial distribution of density and a very broad range of models of the star-disc interaction*. Temporal evolution can be given in terms of quadrature provided that the star-disc interaction is specified completely (in terms of the function Σ). Our approach can be applied to other situations but the form of equation (28) is linked to the assumption about interactions, equation (1). Also the situation when the orbit intersects the disc twice per revolution requires a specific form of Σ to be given and, most likely, it does not allow a complete analytic solution.

Our solution thus describes the initial phases of the stellar capture (large eccentricity) and it matches smoothly the low-eccentricity approximation. Apart from an interesting form of the above-given analytical expressions, our approach is useful as a part of more elaborate calculations. In an accompanying detailed paper, additional effects are taken into account (e.g., gravity of the disc) and the distribution of a large number of stars is investigated (Vokrouhlický & Karas 1997).

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REFERENCES

- Artymowicz P., 1994, ApJ, 423, 581
Goldreich P., Tremaine S., 1980, ApJ, 241, 425

Kamke E., 1959, *Differentialgleichungen: Lösungsmethoden und Lösungen*. Akademische Verlagsgesellschaft, Leipzig
McKinnon W. B., Leith A. C., 1995, *Icarus*, 118, 392
Pineault S., Landry S., 1994, *MNRAS*, 267, 557
Podsiadlowski P., Rees M. J., 1994, in Holt S. S., Day C. S., eds, *The Evolution of X-ray Binaries*. AIP Press, New York, p. 71
Pollack J. B., Burns J. A., Tauber M. E., 1979, *Icarus*, 37, 587
Rauch K. P., 1995, *MNRAS*, 275, 628

Syer D., Clarke C. J., Rees M. J., 1991, *MNRAS*, 250, 505
Vokrouhlický D., Karas V., 1993, *MNRAS*, 265, 365
Vokrouhlický D., Karas V., 1997, *MNRAS*, submitted
Zurek W. H., Siemiginowska A., Colgate S. A., 1994, *ApJ*, 434, 46
Zurek W. H., Siemiginowska A., Colgate S. A., 1996, *ApJ*, 470, 652

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