On interpretation of the magnetized Kerr–Newman black hole

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(Received 6 July 1990; accepted for publication 3 October 1990)

An electromagnetic field in the vicinity of the magnetized Kerr–Newman black hole is studied. In the extreme case, the field does not thread the horizon of an electrically neutral configuration. Analogically, the magnetic field is expelled out of an extreme black hole with zero angular momentum. Uniqueness of the magnetized Schwarzschild black hole is discussed.

I. INTRODUCTION

The main purpose of this paper is to elucidate some interesting features of the magnetized Kerr–Newman (MKN) metric. The MKN metric[1,2] is an exact stationary solution of the Einstein–Maxwell equations, which reduces to the ordinary Kerr–Newman black hole when the magnetic field parameter B_0 is set equal to zero. Therefore, the MKN metric may be relevant for astrophysical applications and is in this respect intensively investigated.[3,4] For $B_0 \neq 0$, however, the space-time is not asymptotically flat and the remaining three parameters of this solution M, a, and e, have no direct physical interpretation. Particularly, the magnetized Kerr black hole ($e = 0$) possesses a nonzero electric charge, while the magnetized Reissner–Nordstrom black hole ($a = 0$) has a nonzero angular momentum; in these two cases the magnetic flux across the horizon of a magnetized extreme hole does not vanish, as we could expect on the basis of our experience with the test field approximations.[5,6]

To improve our insight, we search, in Sec. II, for configurations with vanishing charge or angular momentum of the hole. In the extreme case, we study electromagnetic horizon fields to show how the field is expelled out of the horizon, in close analogy with the case of an extreme Kerr[7] or extreme Reissner–Nordstrom[8] black hole immersed in a weak, axisymmetric, asymptotically uniform magnetic field. This enables us to extend one of Hiscock's uniqueness theorems on magnetized black holes.[9]

We start out from the MKN metric in spheroidal coordinates $(x^0,x^1,x^2,x^3) = (t,r,\theta,\varphi)$:

$$g = |\Lambda|^2\Sigma[\Delta^{-1}dr^2 + d\theta^2 - \Delta A^{-1}\,dt^2] + |\Lambda|^{-2}A\Sigma^{-1}\sin^2\theta\,[d\varphi - \omega'\,dt\,]^2, \qquad (1)$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2 + e^2,$$
$$A = (r^2+a^2)^2 - \Delta a^2\sin^2\theta, \quad \Lambda = 1 + B_0\Phi - \tfrac{1}{4}B_0^2\mathscr{E}. \qquad (2)$$

Functions $\Phi(r,\theta)$ and $\mathscr{E}(r,\theta)$ are the Ernst potentials of the Kerr–Newman solution,

$$\Phi = ear\Sigma^{-1}\sin^2\theta - ie(r^2+a^2)\Sigma^{-1}\cos\theta,$$
$$\mathscr{E} = -\Sigma^{-1}[A\sin^2\theta + e^2(a^2 + r^2\cos^2\theta)] + 2ia\Sigma^{-1}\cos\theta\,[M\Sigma(3-\cos^2\theta) + Ma^2\sin^4\theta - re^2\sin^2\theta\,], \qquad (3)$$

and ω' is given by the equation

$$d\omega' = |\Lambda|^2\,d\omega + i\Sigma A^{-1}\sin^{-1}\theta\,[\,(\Lambda\overleftrightarrow{\partial}_\theta\Lambda^*)dr - (\Lambda\overleftrightarrow{\partial}_r\Lambda^*)\Delta\,d\theta\,], \qquad (4)$$

with $\omega = (2Mr - e^2)aA^{-1}$. The Ernst potentials $\Phi'(r,\theta)$, $\mathscr{E}'(r,\theta)$ of the MKN solution are

$$\Phi' = \Lambda^{-1}[\Phi - \tfrac{1}{2}B_0\mathscr{E}\,], \quad \mathscr{E}' = \Lambda^{-1}\mathscr{E}, \qquad (5)$$

and corresponding orthonormal components of the electromagnetic field in a locally nonrotating frame are

$$H_{(r)} + iE_{(r)} = A^{-1/2}\sin^{-1}\theta\,\partial_\theta\Phi', \qquad (6)$$
$$H_{(\theta)} + iE_{(\theta)} = -(\Delta/A)^{1/2}\sin^{-1}\theta\,\partial_r\Phi'. \qquad (7)$$

Explicit form of ω' was given in Ref. 2. (Let us note that the parameter E in Ref. 2 corresponds to our $-B_0/2$, so that the asymptotic value of the magnetic field component parallel to the z axis at $r\to\infty$, $\sin\theta\neq 0$ is equal to B_0.)

Denoting canonical Killing vectors $k = \partial_t$, $m = \partial_\phi$ and the horizon Killing vector $l = k + \omega' m$, and putting $l^\mu l_\mu = 0$ we obtain $r \equiv r_+ = M + (M^2 - a^2 - e^2)^{1/2}$, which is where the horizon is located in the Kerr–Newman limit. Since in the magnetized case the only singularity is again hidden below a nonsingular horizon, we name (1) as a black hole. Range of angular coordinates is given by $0 \leqslant \theta < \pi$, $0 \leqslant \varphi < 2\pi|\Lambda_0|^2$, where Λ_0 denotes the value of Λ on the polar axis $\theta = 0$, π.

II. PARAMETERS OF THE MKN METRIC

The space-time (1) is not asymptotically flat, nevertheless, some results that were derived for ordinary black holes remain valid. The extreme configuration can be defined by vanishing of the surface gravity of the black hole,

$$\kappa_H^2 \equiv \kappa^2(r = r_+) = \tfrac{1}{2}l^{\mu;\nu}l_{\nu;\mu}(r = r_+) = 0. \qquad (8)$$

The expression for κ is rather complicated, but it reduces to its Kerr–Newman value $\kappa_H = (r_+ - M)(r_+^2 + a^2)^{-1}$ at $r = r_+$. [Here, $\omega' = \text{const}$, as can be verified directly from Eq. (4)]. Thus Eq. (8) yields $M^2 = a^2 + e^2$, independently of the value of the magnetic field parameter.

The electric charge of the hole is given by the surface integral of the electromagnetic field:

$$Q_H = \frac{1}{8\pi}\int_{\mathscr{H}} F^{\mu\nu}\,d\Sigma_{\mu\nu} = -|\Lambda_0|^2\,\mathrm{Im}[\Phi'(r_+,0)]. \qquad (9)$$

The uncharged configuration $Q_H = 0$ is then given by

$$B_0 \equiv B_\pm = 2e^{-3}[2Ma \pm (4M^2a^2 + e^4)^{1/2}]. \qquad (10)$$

0022-2488/91/030714-03$03.00

Analogically, the magnetic flux across an upper hemisphere $\mathscr{S}$ located axisymmetrically on the horizon is

$$F_H = \frac{1}{2}\int_{\mathscr{S}} {}^*F^{\mu\nu}\, d\Sigma_{\mu\nu}$$
$$= 2\pi|\Lambda_0|^2 \mathrm{Re}[\Phi'(r_+,\theta)]_{\theta=0}^{\theta=\pi/2}. \qquad (11)$$

In the extreme case with $B_0 = B_\pm$, we have $\Phi'(r_+,\theta) = \pm\frac{1}{2}e^3/(2a^2+e^2) = \mathrm{const}$ and the magnetic flux (11) is zero. Then it follows from Eq. (6) that *the electromagnetic field does not thread the horizon of an electrically neutral, extreme, magnetized Kerr–Newman black hole.*

Let us remark that this result is valid only for extreme configurations of the MKN metric. It cannot be considered as a general feature of the solutions obtained by the two-parameter Ernst transformation technique (when both magnetic and electric parameters B_0 and E_0 are nonzero). For example, the extreme, electromagnetized Reissner–Nordstrom ($a=0$) configurations with $Q_H=0$ are characterized by $E_0^2 + B_0^2 = 4e^{-2}$; however, the condition $E_{(r)}(r_+)=0$, $H_{(r)}(r_+)=0$ then requires $E_0=0$, $B_0 = \pm 2e^{-1}$, which is a special case of the previous MKN solution [Eq. (10) with $a=0$].

The magnetic flux (11) across an extreme hole also vanishes for

$$B_0 \equiv B'_\pm = 2a^{-1}(3e^2+4a^2)^{-1}[\pm(2a^2+e^2) - e(a^2+e^2)^{1/2}], \qquad (12)$$

but in this case $Q_H \neq 0$. This solution will be discussed later.

It is instructive to consider the limit of a weak magnetic field ($B_0^2=0$) in Eq. (11). With the uncharged solution $Q_H=0$ we obtain

$$F \cong \pi B_0[\Delta + 2Mr\Sigma^{-1}(r^2-a^2)]\sin^2\theta, \qquad (13)$$

in perfect analogy with the approximation of asymptotically uniform, aligned, magnetic test field on the Kerr background.[6] The extreme configuration is now given by $a^2 = M^2$, in which case the flux across the horizon vanishes. However, as noted above, the test field approximation does not correspond to the linearization of the magnetized Kerr solution: Curiously, assuming $e=0$ in (11) we obtain

$$F \cong \pi B_0[\Delta + 2Mr\Sigma^{-1}(r^2+a^2)]\sin^2\theta. \qquad (14)$$

Now we turn to Komar's integral relations for the angular momentum and mass contained within a black hole:

$$J_H = -\frac{1}{16\pi}\int_{\mathscr{H}} m^{\mu;\nu}\, d\Sigma_{\mu\nu}$$
$$= -\tfrac{1}{8}|\Lambda_0|^2(r_+^2+a^2)^4\int_0^\pi \Sigma^{-2}|\Lambda|^{-4}\,\partial_r\omega'\sin^3\theta\, d\theta, \qquad (15)$$

$$M_H = \frac{1}{8\pi}\int_{\mathscr{H}} k^{\mu;\nu}\, d\Sigma_{\mu\nu} = 2\omega'_H J_H + \frac{1}{4\pi}\kappa_H \mathscr{A}, \qquad (16)$$

where $\mathscr{A} = 4\pi(r_+^2+a^2)|\Lambda_0|^2$ is the surface of the horizon and $\omega'_H = \omega'(r_+)$. Equation (16) acquires the form of the well-known Smarr formula, which has been, however, usually considered only for asymptotically flat space-times. Evaluating integral (15) for the angular momentum at $r\to\infty$ and assuming B_0 nonzero, we obtain $J_\infty = 0$; on the other hand, M_∞, which is given by (16) at $r\to\infty$, diverges whenever $B_0 \neq 0$, because the amount of energy in the magnetic field is infinite.

We verified numerically that no combination of the parameters a, e satisfies both conditions $J_H=0$, $Q_H=0$ simultaneously, except for the magnetized Schwarzschild black hole ($e=a=0$). By applying theorems proved in Ref. 9 we can state that *the only static magnetized Kerr–Newman black hole is given by (1) with $J_H=0$ and $Q_H=0$.* Comparing with a nonmagnetized case, staticity requires an additional condition $Q_H=0$ when $B_0\neq 0$.

In the extreme case, Eq. (4) yields $\partial_r\omega' = \mathrm{const}$ on the horizon. The solution to the equation $J_H=0$ is then given by the real roots of

$$\partial_r\omega - 2\Sigma A^{-1}|\Lambda|^{-2}\sin^{-1}\theta\,[\mathrm{Im}(\Lambda)\overleftrightarrow{\partial}_\theta\,\mathrm{Re}(\Lambda)] = 0 \qquad (17)$$

at $r=r_+$, which, interestingly, coincide with $B'_\pm$, the second pair of roots of $F_H=0$ (Fig. 1). In this case we have $\mathrm{Re}[\Phi'(r_+,\theta)] = \pm\frac{1}{2}a(4a^2+3e^2)(2a^2+e^2)^{-1} = \mathrm{const}$, so that the radial tetrad component of the magnetic field on the horizon again vanishes. (On the other hand, $\mathrm{Im}[\Phi'(r_+,\theta)]$ is now a complicated function of θ; the electric field on the horizon does not vanish.)

The weak-field limit of the magnetic flux (11) with $J_H=0$ is

$$F \cong \pi B_0[r^2 - 3e^2 + 2e^2r_+^3\, r^{-1}(3Mr_+ - 2e^2)^{-1}]\sin^2\theta,$$
$$a = B_0 e r_+^3\,(3Mr_+ - 2e^2)^{-1}. \qquad (18)$$

It is tempting to compare this result with the case of asymptotically uniform magnetic test field on the Reissner–Nordstrom background. One can verify that taking the integration constant in Eq. (4.2) of Ref. 8 as

$$a = B_0 e r_+^3\,(3Mr_+ - 2e^2)^{-1} - 2B_0 e^3 M^{-1} \qquad (19)$$

results of both approaches coincide. (In Sec. V of Ref. 8 the authors assumed $a=0$; in this case, however, the angular

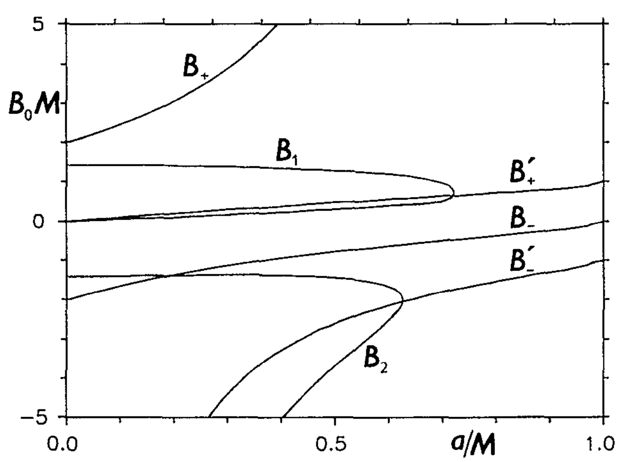

FIG. 1. Dimensionless magnetic parameter B_0M as a function of a/M for several extreme MKN solutions. The curves $B_\pm$ correspond to electrically neutral configurations [$Q_H=0$, Eq. (10)]; $B'_\pm$ denotes configurations with zero angular momentum J_H and mass M_H [Eq. (12)]; the curves B_1, B_2 have $M_H=0$ but $J_H\neq 0$ [Eq. (16) with $\kappa_H=0$, $\omega'_H=0$].

momentum of the hole is nonzero and this configuration should not be interpreted as a nonrotating charged black hole.) A more intuitive picture can be obtained by constructing the lines of constant magnetic flux, $F=$ const. In the extreme case the field lines are expelled out of the horizon and their shape is very similar to the lines of the test field when constructed in Boyer–Lindquist coordinates.[6,8]

We conclude by emphasizing the following point. The behavior of the horizon electromagnetic field and magnetic fluxes allows us to consider the MKN solution with $Q_H = 0$ or $J_H = 0$ as magnetized generalizations of the Kerr black hole or the Reissner–Nordstrom black hole, respectively. These solutions have both parameters a, e nonzero, except for a trivial case of the magnetized Schwarzschild solution.

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